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Hamilton-connected properties of 3-connected {claw, hourglass, bull}-free graphs

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ABSTRACT

An *hourglass* Γ_0 is the graph with degree sequence $\{4, 2, 2, 2, 2\}$. In this paper, for integers $j \geq i \geq 1$, the *bull* $B_{i,j}$ is the graph obtained by attaching endvertices of two disjoint paths of lengths i, j to two vertices of a triangle. We show that every 3-connected $\{K_{1,3}, \Gamma_0, X\}$-free graph, where $X \in \{B_{2,12}, B_{4,10}, B_{6,8}\}$, is Hamilton-connected. Moreover, we give an example to show the sharpness of our result, and complete the characterization of forbidden induced bulls implying Hamilton-connectedness of a 3-connected {claw, hourglass, bull}-free graph.

Keywords: Hamilton-connected, Forbidden subgraph, Claw, Hourglass, Bull

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1. Introduction

In this paper, we basically follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer the reader to [1]. By a graph we always mean a simple finite undirected graph $G = (V(G), E(G))$; whenever we admit multiple edges, we always speak about a *multigraph*. For a set X, the cardinality of X is denoted

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by $|X|$. We write $|G|$ for $|V(G)|$. For a family of graphs $\mathcal{F}$, we say that G is $\mathcal{F}$-free if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the graphs in $\mathcal{F}$ are referred to in this context as *forbidden induced subgraphs*. If $\mathcal{F} = \{F\}$, then we simply say that G is F-free. Here, the *claw* is the graph $K_{1,3}$.

Several further graphs that will be used as forbidden subgraphs are shown in Figure 1 (specifically, the vertex of degree 2 in the triangle of the bull $B_{i,j}$ will be called its *mouth* and denoted $\mu(B_{i,j})$). When listing vertices of an induced subgraph $F \cong B_{i,j}$, we will always list first $\mu(F)$, and then vertices of the two paths, starting (if possible) with the shorter one. In addition, let P_i and C_i denote the path and cycle with i vertices.

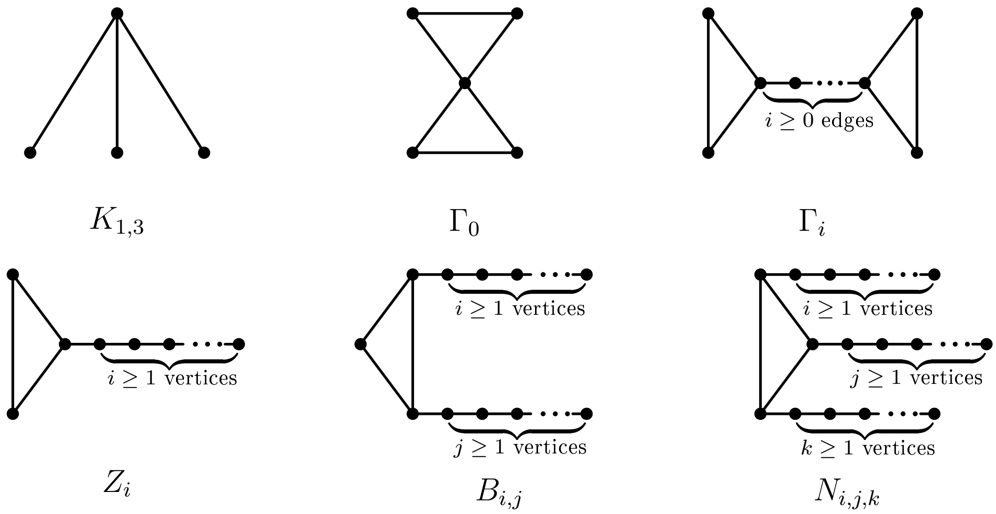

Fig. 1. The graphs $K_{1,3}$, Γ_0, Γ_i, Z_i, $B_{i,j}$ and $N_{i,j,k}$

In this paper, we will consider these questions in 3-connected and claw-free graphs. A graph G is *hamiltonian* if G has a spanning cycle. The hamiltonian problem is generally considered to be determining conditions under which a graph contains a spanning cycle. To determine whether a graph is hamiltonian is very basic and popular problem. There are many results on hamiltonian properties of graphs in classes defined in terms of forbidden induced subgraphs. We first summarize some known results.

Theorem 1.1. *Let G be a 3-connected $K_{1,3}$-free graph. Then*

(1) (Fujisawa [7]) *if G is Z_9-free, then either G is hamiltonian, or G is isomorphic to the line graph of the graph obtained from the Petersen graph by adding one pendant edge to each vertex.*

(2) (Hu and Lin [8], Xiong et al. [23]) *if G is $N_{i,j,k}$-free with positive integers $i+j+k \leq 9$, then G is hamiltonian.*

(3) (Du and Xiong [6]) *if G is $B_{i,j}$-free with positive integers $i + j \leq 9$, then G is hamiltonian.*

In 2002, Brousek [3] start to consider a triples of forbidden subgraphs for a graph to be Hamiltonian. Ryjáček et al. [19] and Du and Xiong [6] continue in this direction by showing that Theorem 1.1 can be substantially strengthened under an additional assumption that G is Γ_0-free, and it shows that these results of Hamiltonicity are sharp.

Theorem 1.2. *Let G be a 3-connected $\{K_{1,3},\Gamma_0\}$-free graph. Then if G is*

(1) (Ryjáček et al. [19]) *Z_{18}-free, or*

(2) (Ryjáček et al. [19]) *$N_{2i,2j,2k}$-free with positive integers $i+j+k \leq 9$, or*

(3) (Du and Xiong [6]) *$B_{2i,2j}$-free with positive integers $i+j \leq 9$,*

then G is hamiltonian.

Theorem 1.2 adds the condition that G is hourglass-free on the basis of Theorem 1.1. Ryjáček and Vrána [17] give the following result.

Theorem 1.3. (Ryjáček and Vrána [17]) *Let G be a 3-connected $\{K_{1,3},Z_7\}$-free graph of order $n \geq 21$. Then G is Hamilton-connected.*

In 2018, Ryjáček et al. [19] start to consider a triples of forbidden subgraphs for a graph to be Hamilton-connected. Recently, Liu and Xiong [12] also considered a triples of forbidden subgraphs for a 3-connected graph to be Hamilton-connected, this result on Hamilton-connectedness are sharp.

Theorem 1.4. *Let G be a 3-connected $\{K_{1,3},\Gamma_0,X\}$-free graph, where*

(1) (Ryjáček, Vrána and Xiong [19]) *$X = P_{12}$, or*

(2) (Liu and Xiong [12]) *$X = P_{16}$.*

Then G is Hamilton-connected.

Theorem 1.5 lists known result on pairs of forbidden subgraphs implying Hamilton-connectedness of a 3-connected graph.

Theorem 1.5. (Ryjáček and Vrána [18]) *Let $X \in \{B_{1,6},B_{2,5},B_{3,4}\}$, and let G be a 3-connected $\{K_{1,3},X\}$-free graph. Then G is Hamilton-connected.*

By adding the condition "Γ_0-free" to Theorem 1.5, we further prove that every 3-connected, $\{\text{claw},\text{hourglass},\text{bull}\}$-free graph is Hamilton-connected.

Theorem 1.6. *Let $X \in \{B_{2,12},B_{4,10},B_{6,8}\}$, and let G be a 3-connected $\{K_{1,3},\Gamma_0,X\}$-free graph. Then G is Hamilton-connected.*

Proof of Theorem 1.6, consisting in direct case-distinguishing, is postponed to Section 3. In Section 2, we collect necessary known results on line graphs and on closure operations.

2. Preliminaries

In order to state results clearly, we further introduce the following notation. We denote by $N_G(v)$ (or simply $N(v)$) and $d_G(v)$ (or simply $d(v)$) the neighborhood and the degree of a vertex v in G, respectively. For each integer $i \geq 0$, define $V_i(G) = \{v \in V(G) : d(v) = i\}$. Let $N[v] = N(v) \cup \{v\}$. Let $S \subseteq V(G)$, the subgraph with S as the vertex set and all the edges with both end-vertices in S as the edge set is called the subgraph induced from the vertex set S (or simply *induced subgraph*), denoted by $G[S]$. Let $S' \subseteq E(G)$, the subgraph with S' as edge set and all the end-vertices of S' as vertex set is called the subgraph induced from edge set S' (or simply *edge induced subgraph*), denoted as $G[S']$.

A vertex-cut (edge-cut, respectively) X of a multigraph G is *essential* if $G - X$ has at least two nontrivial components, and G is *essentially k-connected* (*essentially k-edge-connected*, respectively) if every essential vertex-cut (essential edge-cut, respectively) of H is of size at least k. Let $\kappa'(G)$, $c(G)$ denote the *edge connectivity* and the *circumference* of G, respectively.

In Subsections 2.1-2.3, we summarize some facts that will be need in our proof of Theorem 1.6.

2.1. Line graphs of multigraphs and their preimages

The *line graph* of a given G, denoted by $L(G)$, is a graph with vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex in G. The induced sub(multi)graph on a set $M \subset V(G)$, denoted by $G[M]$.

The multigraph H will be called the *preimage* of a line graph G and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

A vertex $x \in V(G)$ is *eligible* if $G[N(x)]$ is a connected noncomplete graph, and we use $V_{EL}(G)$ to denote the set of all eligible vertices of G. The *local completion* of G at a vertex x is the graph G^*_x obtained from G by adding all edges with both vertices in $N(x)$ (note that the local completion at x turns x into a simplicial vertex, and preserves the $K_{1,3}$-free property of G). The *closure* $cl(G)$ of a $K_{1,3}$-free graph G was defined as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible(more precisely: $cl(G) = G_k$, where $G_1, \ldots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_x$ for some $x \in V_{EL}(G_i), i \in 1, \ldots, k-1$, and $V_{EL}(G_k) = \emptyset$). We say that G is *closed* if $G = cl(G)$. The closure $cl(G)$ of a $K_{1,3}$-free graph G is uniquely determined, is the line graph of a triangle-free graph, and is Hamiltonian if and only if so is G. However, as observed in [2], the closure operation does not preserve (non-)Hamilton-connectedness of G. It is a well-known fact that

Fact 2.1. *A line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected.*

We recall that if $G = L(H)$, then a graph F is an induced subgraph of G if and only if $L^{-1}(F)$ is a subgraph (not necessarily induced) of H.

The *core* of H is the multigraph H_0 obtained from H by deleting all the vertices of degree 1, and replacing the path xyz by the edge xz for each y of degree 2, and denoted $co(H)$.

Obviously, if G is $K_{1,3}$-free, then so is G_x^*. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, G_x^* is the line graph of the graph obtained from H by contracting the edge $L^{-1}(x)$ into a vertex and replacing the created loop(s) by pendant edge(s). The following results show some properties of eligible vertexs.

Lemma 2.2. (Ryjáček et al. [19]) *Let G be a $K_{1,3}$-free graph such that every induced hourglass in G is centered at an eligible vertex, and let $x \in V_{EL}(G)$. Then every induced hourglass in G_x^* is centered at an eligible vertex.*

The following theorem was proved in [4], [5].

Theorem 2.3. (Brousek and Ryjáček [4, 5]) *Let G be a $\{K_{1,3}, \Gamma_0\}$-free graph, and let $x \in V_{EL}(G)$. Then G_x^* is $\{K_{1,3}, \Gamma_0\}$-free.*

A multigraph H is *strongly spanning trailable* if for any edge $e_1, e_2 \in E(H)$ (possibly $e_1 = e_2$), the multigraph $H(e_1, e_2)$, which is obtained from H by replacing the edge e_1 by a path $u_1 v_{e_1} v_1$ and the edge e_2 by a path $u_2 v_{e_2} v_2$, has a spanning (v_{e_1}, v_{e_2})-trail. The following theorem establishes a correspondence between a IDT in H and a hamiltonian path in $L(H)$.

Theorem 2.4. (Li et al. [10]) *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

$\mathcal{W}_0$ is the family of multigraphs obtained from the Wagner graph W_8 by subdividing one of its edges and adding at least one edge between the new vertex and exactly one of its neighbors (see Figure 2).

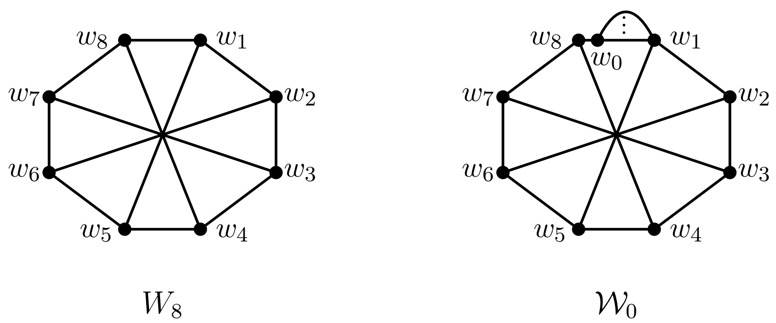

Fig. 2. The graphs W_8 and $\mathcal{W}_0$

Theorem 2.5. (Liu et al. [13]) *The following statements should be true.*

(1) Every 2-connected 3-edge-connected multigraph H with $c(H) \leq 8$ other than W_8 is strongly spanning trailable.

(2) Every 3-edge-connected multigraph H with $|V(H)| \leq 9$ other than a member of $W_8 \cup \mathcal{W}_0$ is strongly spanning trailable.

Theorem 2.6. (Shao [20]) *Let H be an essentially 3-edge-connected multigraph. Then the core H_0 of H satisfies the following.*

(1) H_0 is uniquely defined and $\kappa'(H_0) \geq 3$,

(2) $V(H_0)$ dominates all edges of H,

(3) if H_0 has a spanning closed trail, then H has a DCT,

(4) if H_0 is strongly spanning trailable, then $L(H)$ is Hamilton-connected.

2.2. SM-closure

For a given $K_{1,3}$-free graph G, a graph G^M, as introduced in [9], is defined by the following construction.

(a) If G is Hamilton-connected, we set $G^M = cl(G)$.

(b) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs G_1, ..., G_k such that

(1) $G_1 = G$,

(2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, ..., k-1$,

(3) G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,

(4) for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and set $G^M = G_k$.

A resulting G^M is called a *strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure is said to be SM-closed. Note that for a given graph G, its SM-closure is not uniquely determined. As shown in [15] and [9], if G is SM-closed, then $G = L(H)$, where H does not contain a subgraph(not necessarily induced) isomorphic to any of the graphs in Figure 3.

For $x, y \in V(G)$, a path (trail) with endvertices x, y is referred to as an (x, y)-path ((x, y)-trail), a trail with terminal edges $e, f \in E(G)$ is called an (e, f)-trail, and $\mathrm{Int}(T)$ denotes the set of interior vertices of a trail T. A set of vertices $M \subset V(G)$ dominates an edge e, if e has at least one vertex in M, and a subgraph $F \subset G$ dominates e if $V(F)$ dominates e. A closed trail T is a dominating closed trail (abbreviated DCT) if T dominates all edges of G, and an (e, f)-trail is an internally dominating (e, f)-trail (abbreviated (e,f)-IDT) if $\mathrm{Int}(T)$ dominates all edges of G.

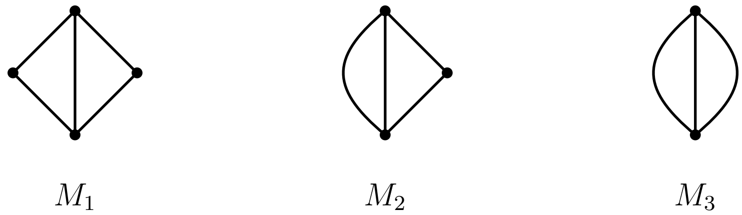

Fig. 3. The diamond M_1, the multitriangle M_2 and the triple edge M_3

The following results show some properties of the SM-closure.

Theorem 2.7. *(Kužel et al., [9]) Let G be a $K_{1,3}$-free graph and G^M be the SM-closure. Then*

(1) $V(G)=V(G^M)$ and $E(G)\subset E(G^M)$.

(2) G^M is obtained from G by a sequence of local completions at eligible vertices.

(3) G is Hamilton-connected if and only if G^M is Hamilton-connected.

(4) if G is Hamilton-connected, then $G^M=cl(G)$.

(5) if G is not Hamilton-connected, then either

(A) $V_{EL}(G^M)=\emptyset$ and $G^M=cl(G)$, or

(B) $V_{EL}(G^M)\neq\emptyset$ and $(G^M)^*_x$ is Hamilton-connected for any $x\in V_{EL}(G^M)$.

(6) $G^M=L(H)$, where H contains either

(A) at most 2 triangles and no multiedge, or

(B) no triangle, at most one double edge and no other multiedge.

(7) If G^M contains no hamiltonian (a,b)-path for some $a,b\in V(G^M)$ and

(A) X is a triangle in H, then $E(X)\cap\{L^{-1}_{G^M}(a),L^{-1}_{G^M}(b)\}\neq\emptyset$.

(B) X is a multiedge in H, then $E(X)=\{L^{-1}_{G^M}(a),L^{-1}_{G^M}(b)\}$.

We will also need the following lemma on SM-closed graphs proved in [16].

Lemma 2.8. (Ryjáček and Vrána [16]) *Let G be an SM-closed graph and let $H=L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2 in H.*

Lemma 2.9. (Ryjáček et,al. [19]) *Let G be a $K_{1,3}$-free graph and let G^M be its SM-closure, and let $H=L^{-1}(G^M)$. Then $v\in V_{EL}(G^M)$ if and only if $e=L^{-1}(v)$ is in a triangle or a multiedge in H.*

Theorem 2.10. (Li et al. [10]) *Let H be a multigraph with $|E(H)|\geq 3$. Then $G=L(H)$ is Hamilton-connected if and only if for any pair of edges $e_1,e_2\in E(G)$, H has an (e_1,e_2)-IDT.*

2.3. Closure operations and bull-free graphs

The concept of SM-closure can be further strengthened by omitting the eligibility assumption in the local completion operation. Specifically, for a given $K_{1,3}$-free graph G, Liu et al. [11] constructed a graph G^U by the following construction.

(a) If G is Hamilton-connected, we set $G^U = K_{|V(G)|}$.

(b) If G is not Hamilton-connected, we recursively perform the local completion operation at such vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

(1) $G_1 = G$,

(2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V(G_i)$, $i = 1, ..., k-1$,

(3) G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,

(4) for any $x \in V(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and set $G^M = G_k$.

A resulting G^U is called a *ultimate M-closure* (or briefly an *UM-closure*) of the graph G, and a graph G equal to its UM-closure is said to be UM-closed. When applying closure techniques to $\{claw, \Gamma_0, bull\}$-free graphs, the main problem is that a closure of a $\{K_{1,3}, \Gamma_0, B_{i,j}\}$-free graph is not necessarily $\{K_{1,3}, \Gamma_0, B_{i,j}\}$-free (i.e., in the terminology of [14], the class of $\{K_{1,3}, \Gamma_0, B_{i,j}\}$-free graphs is not stable under the closure operation). Unfortunately, this is the case with all the closure operations mentioned in the previous subsections.

We say that a vertex $x \in V(G)$ is *simplicial* if the subgraph induced by $G[N(x)]$ is complete graph, and we use $V_{SI}(G)$ to denote the set of all simplicial vertices of G.

It turns out that this difficulty can be overcome by working in a slightly larger class of graphs which contains all the requested $\{K_{1,3}, \Gamma_0, B_{i,j}\}$-free graphs but is stable under the closure. Ryjáček and Vrána [18] defined the class $\mathcal{B}_{i,j}$ as follows, and they proved the following properties.

- For any positive integers i, j, $\mathcal{B}_{i,j}$ is the class of all $K_{1,3}$-free graphs G such that every induced subgraph $F \subset G$, $F \simeq \mathcal{B}_{i,j}$, satisfies $\mu(F) \in V_{SI}(G)$.

Clearly, every $\{K_{1,3}, B_{i,j}\}$-free graph is in $\mathcal{B}_{i,j}$.

Theorem 2.11. (Ryjáček and Vrána [18]) *Let G be a $\{K_{1,3}, B_{i,j}\}$-free graph for some $i, j \geq 1$, and let G^U be a UM-closure of G. Then $G^U \in \mathcal{B}_{i,j}$.*

3. Proof of Theorem 1.6

We will always write the list such that integers $1 \leq i \leq j$ and $i + j = 7$, we use $S_{1,2i+1,2j+1}$ to denote the graph obtained from $K_{1,3}$ by subdividing two of its edges $2i$ and $2j$ times, respectively, where the labeling of vertices as in Figure 4, and the vertex o will be called the center vertex. It is easy to observe that $L^{-1}(\Gamma_0)$ is the unique graph with degree sequence $3, 3, 1, 1, 1, 1$ and $L^{-1}(B_{2i,2j}) = S_{1,2i+1,2j+1}$. We will use the notation $S_{1,i,j}(o, a_1, b_1b_2 \ldots b_i, c_1 \ldots c_j)(S_{1,i,j} \subseteq S(o, a_1, b_1b_2 \ldots b_{i'}, c_1 \ldots c_{j'})$ with integers $i' \geq i$ and

$j' \geq j$) to denote the subgraph $S_{1,i,j}$. Now, we present the proof of Theorem 1.6.

$L^{-1}(\Gamma_0)$ $\qquad$ $S_{1,2i+1,2j+1} = L^{-1}(B_{2i,2j})$

Fig. 4. The graphs $L^{-1}(\Gamma_0)$ and $L^{-1}(B_{2i,2j})$

Proof of Theorem 1.6. Let G be a 3-connected $\{K_{1,3}, \Gamma_0, X\}$-free graph, where $X \in \{B_{2,12}, B_{4,10}, B_{6,8}\}$, and suppose, to the contrary, that G is not Hamilton-connected. By Theorems 2.3 and 2.11, we can assume that G is UM-closed and $G \in \mathcal{B}_{2,12} \cup \mathcal{B}_{4,10} \cup \mathcal{B}_{6,8}$. Obviously, G is also SM-closed, implying that G is a line graph and $H = L^{-1}(G)$ has special structure (contains no diamond, no multitriangle and triple edge), and let H_0 be the core of H. By Theorem 2.6 (4), H_0 is not strongly spanning trailable. By Lemma 2.2, every induced hourglass in G is centered at an eligible vertex. By Theorem 2.7, H has at most two triangles or an multiedge. Hence, we may let

E_0 be the edge set of two triangles or the multiedge in H.

By Theorem 2.6 (1),

$$\kappa'(H_0) \geq 3. \tag{1}$$

For any edge $e \in E(H_0) \backslash E_0$, $L(e)$ is not an eligible vertex in G by Lemma 2.9, i.e., the edge e cannot be a central edge of an $L^{-1}(\Gamma_0)$ for some induced hourglass Γ_0 of G. Thus we have

$$\text{each edge of } E(H_0)\backslash E_0 \text{ should be subdivided by a vertex of degree 2 in } H. \tag{2}$$

It suffices to show that H contains all possible subgraphs $S_{1,2i+1,2j+1} \in \{S_{1,3,13}, S_{1,5,11}, S_{1,7,9}\}$, where positive integers $i + j = 7$.

Claim 3.1. $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$.

Proof. Assume, to the contrary, that $c(H_0) \leq 8$ or $|V(H_0)| \leq 9$. By Theorem 2.5, $H_0 \in \{W_8\} \cup \mathcal{W}_0$. Then H_0 has a 8-cycle $w_1w_2\ldots w_8w_1$ or 9-cycle $w_1w_2\ldots w_8w_0w_1$ with $\{w_1w_5, w_2w_6, w_3w_7, w_4w_8\} \subseteq E(H_0)$ and w_0w_1 is multiple edge. By (2), each edge w_mw_n of H_0 should be subdivided by a vertex $w_{m,n}$ of degree 2 with integers $0 \leq m < n \leq 8$. Then H contains subgraphs

$$S_{1,3,13}(w_1, w_{1,5}, w_{1,8}(w_0)w_8w_{4,8}, w_{1,2}w_2w_{2,3}w_3w_{3,4}w_4w_{4,5}w_5w_{5,6}w_6w_{6,7}w_7w_{7,8}),$$
$$S_{1,5,11}(w_1, w_{1,5}, w_{1,8}(w_0)w_8w_{7,8}w_7w_{3,7}, w_{1,2}w_2w_{2,3}w_3w_{3,4}w_4w_{4,5}w_5w_{5,6}w_6w_{6,7})$$

and

$$S_{1,7,9}(w_1, w_{1,5}, w_{1,8}(w_0)w_8w_{7,8}w_7w_{6,7}w_6w_{2,6}, w_{1,2}w_2w_{2,3}w_3w_{3,4}w_4w_{4,5}w_5w_{5,6}),$$

a contradiction. This proves Claim 3.1. □

Therefore $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$. Throughout the proof, we use the following notation:

- $C_{c(H_0)} = v_1 v_2 \dots v_{c(H_0)} v_1$ always denotes a longest cycle of H_0, and $C = PI_H(C_{c(H_0)})$;
- Set $m^{H_0} = |E_0 \cap E(C_{c(H_0)})|$;
- Set $\mathcal{D}_{H_0} = V(H_0) \setminus V(C_{c(H_0)})$;
- Let $E^1_{H_0}$ be the set of all edges between $C_{c(H_0)}$ and $\mathcal{D}_{H_0}$. Then $|E^1_{H_0}| \geq 3$;

By (2), $C_{c(H_0)}$ has at least $c(H_0) - m^{H_0}$ edges that should be subdivided by $c(H_0) - m^{H_0}$ vertices of degree 2 in H_0, then $|V(C)| = 2c(H_0) - m^{H_0}$. For integers $1 \leq r < s \leq c(H_0)$, we use $v_{r,s}$ to denote the vertex subdivide edge $v_r v_s$ in $C_{c(H_0)}$. An edge $v_r v_s \in E(C_{c(H_0)})$ is a l-chord if the shortest one of the two subpaths of $C_{c(H_0)}$ determined by v_r and v_s has l internal vertices.

Claim 3.2. For a pair of vertices x and y in $C_{c(H_0)}$, and a path $P_{H_0 \setminus C_{c(H_0)}}(x, y)$ with x, y as its end vertices and their internal vertices are not in $C_{c(H_0)}$. Let $P_{C_{c(H_0)}}(x, y)$ be the subpath of $C_{c(H_0)}$. Then $|P_{C_{c(H_0)}}(x, y)| \geq |P_{H_0 \setminus C_{c(H_0)}}(x, y)|$.

Proof. Suppose Claim 3.2 false, $|P_{C_{c(H_0)}}(x_0, y_0)| < |P_{H_0 \setminus C_{c(H_0)}}(x_0, y_0)|$ for some x_0, y_0 satisfying the hypothesis Claim 3.2. Then

$$C' = H_0[(E(C_{c(H_0)}) \cup E(P_{H_0 \setminus C_{c(H_0)}}(x_0, y_0)) \setminus E(P_{C_{c(H_0)}}(x_0, y_0)]$$

is a cycle of length at least $c(H_0) + 1$, which contradicts the choice of $C_{c(H_0)}$. This proves Claim 3.2. □

Claim 3.3. $V(E_0) \cap V(C_{c(H_0)}) \neq \emptyset$.

Proof. Assume, to the contrary, that $V(E_0) \cap V(C_{c(H_0)}) = \emptyset$. Then $|V(\mathcal{D}_{H_0})| \geq 2$ and $E^1_{H_0} \cap E_0 = \emptyset$. By (2), $|V(C)| = 2c(H_0) \geq 18$. Moreover, there is at least one edge in $E^1_{H_0}$ with v_{i_0} as its end-vertex should be subdivided by a vertex x_0 of degree 2 in H_0. Then H contains all subgraphs $S_{1,2i+1,2j+1} \subseteq H[V(C) \cup \{x_0\}]$ with its center vertex v_{i_0}, for positive integers $i + j = 7$, a contradiction. This proves Claim 3.3. □

Claim 3.4. H_0 has no multiple edges.

Proof. Assume, to the contrary, that H_0 contains multiple edges. By Theorem 2.7 (6), H_0 contains at most two mutiple edges and no other multiple edge. Let $\{e'_1, e'_2\} \subseteq E(H_0)$ be a pair of multiple edges, with u_1, u_2 as their end-vertices.

Suppose first that $|V(H_0)| = c(H_0)$. Then $T = v_{i_0} e'_1 v_{i_0+1} \dots v_{i_0} e'_2 v_{i_0+1}$ is an (e'_1, e'_2)-IDT in H with $\{u_1, u_2\} = \{v_{i_0}, v_{i_0+1}\}$, contradicting Theorem 2.7 (7). Now suppose that $|V(H_0)| > c(H_0)$. Firstly, suppose that $m^{H_0} = 2$, but then $c(H_0) = 2$, contradicting

$c(H_0) \geq 9$. Then, suppose that $m^{H_0} = 0$. By Claim 3.3, $\{u_1, u_2\} \cap V(C_{c(H_0)}) \neq \emptyset$. If $|\{u_1, u_2\} \cap V(C_{c(H_0)})| = 2$ and $u_1u_2 \notin E(C_{c(H_0)})$, since H is triangle-free, e_1' is a k-chord in $C_{c(H_0)}$ with $k \geq 2$, then $\Gamma_0 \subseteq L^{-1}(H[u_1, u_1^+, u_1^-, u_2, u_2^+, u_2^-])$, a contradiction; otherwise, $\{u_1, u_2\} \cap V(C_{c(H_0)}) = \{u_1\} = \{v_{i_0}\}$(say $v_{i_0} \in V(C_{c(H_0)})$) and u_2 is not in $C_{c(H_0)}$, by (2), $|V(C)| = 2c(H_0) \geq 18$. Then H contains all subgraphs $S_{1,2i+1,2j+1} \subseteq H[V(C) \cup \{u_2\}]$ with its center vertex u_1, and positive integers $i + j = 7$, a contradiction. Finally suppose that $m^{H_0} = 1$, say $e_1' = v_{i_0}v_{i_0+1} \in E(C_{c(H_0)})$. By (2), $|V(C)| = 2c(H_0) - 1 \geq 17$. Moreover, there is at least one edge in $E^1_{H_0}$ with v_{j_0} as its end-vertex that should be subdivided a vertex x_0 of degree 2 in H. Then H contains all subgraphs $S_{1,2i+1,2j+1} \subseteq H[V(C) \cup \{x_0\}]$ with its center vertex v_{j_0}, for positive integers $i + j = 7$, a contradiction. Thus H_0 has no multiple edges. This proves Claim 3.4. □

By Claims 3.3 and 3.4, we can get that H_0 is simple graph.

Claim 3.5. H_0 is not triangle-free simple graph.

Proof. Assume, to the contrary, that H_0 is a triangle-free simple graph. By (2), $|V(C)| = 2c(H_0) \geq 18$. Since $\kappa'(H_0) \geq 3$, there is a vertex $x_0 \in N_H(v_{i_0})$ and $x_0 \notin V(C_{c(H_0)})$. Then H contains all subgraphs $S_{1,2i+1,2j+1} \subseteq H[V(C) \cup \{x_0\}]$ with center vertex at $V(C)$, for positive integers $i + j = 7$, a contradiction. This proves Claim 3.5. □

By Claims 3.4 and 3.5, H_0 contains at least one triangle.

Claim 3.6. Let $u_1u_2u_3u_1$ be a triangle of H_0. Then

(1) $d_{H_0}(u_1) = d_{H_0}(u_2) = d_{H_0}(u_3) = 3$.

(2) $m^{H_0} \in \{2, 4\}$ if H_0 contains at least one triangle.

Proof. (1). By Lemma 2.8, suppose to the contrary that $d_{H_0}(u_1) \geq 4$. Since $d_{H_0}(u_3) \geq 3$, $\Gamma_0 \subseteq L^{-1}(H[u_1, u_1^+, u_1^-, u_3, u_3^+, u_3^-])$, a contradiction. Then $d_{H_0}(u_1) = 3$. Similarly, $d_{H_0}(u_3) = 3$. If $d_{H_0}(u_2) \geq 4$, then $\Gamma_0 \subseteq L^{-1}(H[u_1, u_1^+, u_1^-, u_2, u_2^+, u_2^-])$, a contradiction. We have that $d_{H_0}(u_2) = 3$.

(2). Firstly, suppose that $m^{H_0} = 3$, but then $c(H_0) = 3$, contradicting $c(H_0) \geq 9$. We can easily get that $m \notin \{3, 5, 6\}$. Then, suppose that $m^{H_0} = 0$. By Claim 3.3, $\{u_1, u_2, u_3\} \cap V(C_{c(H_0)}) \neq \emptyset$. Then for some vertex $u_0 \in \{u_1, u_2, u_3\}$, $d_{H_0}(u_0) \geq 4$, which contradicts $d_{H_0}(u_0) = 3$, a contradiction. Finally, suppose $m^{H_0} = 1$, and $u_1u_2 \in E(C_{c(H_0)})$. By Claim 3.2, $u_3 \in V(C_{c(H_0)})$. Then $d_{H_0}(u_3) \geq 4$, a contradiction. Therefore $m^{H_0} \in \{2, 4\}$. This proves Claim 3.6. □

If $m^{H_0} = 2$, then H_0 has exactly one triangle and $C_{c(H_0)}$ contains exactly three vertices of the triangle, where the vertices of the triangle are pairwise adjacent in H_0. Without loss of generally, we denote the triangle by $v_1v_2v_3v_1$. Choose a shortest path $P^{r,s} \subseteq H[E(\mathcal{D}_{H_0}) \cup E^1_{H_0}]$ with two vertices $v_r, v_s \in V(C_{c(H_0)})$ as its end-vertices, respectively, where the edges incident with vertices v_r and v_s in $P^{r,s}$ are denoted by $e^r_{r,s}$ and $e^s_{r,s}$, respectively. For edge $e^r_{r,s}, e^s_{r,s} \notin E_0$ and $e^r_{r,s}, e^s_{r,s} \in E^1_{H_0}$, by (2), they should be subdivided by two vertices of degree 2 in H, say $x^r_{r,s}, x^s_{r,s} \in V(H)$, respectively. Let $\mathcal{P}'$ be the set of all path $P^{r,s}$ satisfying integer $1 \leq r, s \leq c(H_0)$.

Claim 3.7. If $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$, then $m^{H_0} \neq 2$.

Proof. Assume, to the contrary, that $m^{H_0} = 2$. Then $E^1_{H_0} \cap E_0 = \emptyset$. By (2), $|V(C)| = 2c(H_0) - 2 \geq 16$. For any path $P^{r,s} \in \mathcal{P}'$, if $r = s$, then H contains a subgraph $S_{1,2i+1,2j+1} \subseteq S(v_r, v_{r+1}, P^{r,s}v_s \backslash v_s, v_{r-1,r}v_{r-1} \dots v_{r+1,r+2})$, a contradiction. Therefore integer $1 \leq r < s \leq c(H_0)$ in $P^{r,s}$. Since $\kappa'(H_0) \geq 3$, there is at least a vertex $x^0_r \in N_{H_0}(v_r) \backslash \{v_{r-1}, v_{r+1}\}$ for all $v_r \in V(C_{c(H_0)}) \backslash \{v_1, v_3\}$.

Case 1. $G \in \mathcal{B}_{2,12}$.

Suppose that $x^0_r \notin V(C_{c(H_0)})$ and $x^0_r \in V(\mathcal{P}')$. Then H contains a subgraph $S_{1,3,13} \subseteq S(v_2, v_1, P^{2,s}_H x^s_{2,s}, v_{2,3}v_3 \dots v_9)$ or $S_{1,3,13} \subseteq S(v_r, v_{r-1,r}, P^{r,s}x^s_{r,s}, v_{r,r+1}v_{r+1} \dots v_{r-1})$ with $v_r \in \{v_4, v_5, \dots v_{c(H_0)}\}$, a contradiction. Now suppose that $x^0_r \in V(C_{c(H_0)})$ and $|V(H_0)| = c(H_0) \geq 10$. Then $v_2v_s \in E(H_0)$, where $v_s \in \{v_5, \dots, v_{c(H_0)-1}\}$, and H contains a subgraph $S_{1,3,13}(v_2, v_{2,s}, v_1 \dots v_{c(H_0)}, v_3v_{3,4} \dots v_9)$, a contradiction. Therefore $d_{H_0}(v_2) = 2$, contradicting (1). This proves Case 1.

Case 2. $G \in \mathcal{B}_{4,10}$.

Suppose that $x^0_r \notin V(C_{c(H_0)})$ and $x^0_r \in V(\mathcal{P}')$ and $|P^{r,s}| \geq 4$. Then H contains a subgraph $S_{1,5,11} \subseteq S(v_r, v_{r+1}, P^{r,s}_H x^s_{r,s}, v_{r-1,r}v_{r-1} \dots v_{r+2})$, where $v_r \in \{v_2, v_4, v_5, \dots v_{c(H_0)}\}$, a contradiction. Then, suppose that $|P^{r,s}| = 3$ with $P^{r,s}_H = v_r x^r_{r,s} x_{r,s}(x^0_r) x^s_{r,s} v_s$ and $c(H_0) \geq 10$. By (2), $|V(C)| = 2c(H_0) - 2 \geq 18$. Then H contains a subgraph $S_{1,5,11} \subseteq H[V(C) \cup \{x^r_{r,s}\}]$ with its center vertex v_r, a contradiction. We have that $c(H_0) = 9$, say $C_{c(H_0)} = v_1v_2 \dots v_9v_1$. Firstly, suppose $N_{H_0}(v_6) \backslash \{v_7, v_5\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{6,r}$ for any possibility $r \in \{2, 4, 8, 9\}$. Then H contains a subgraph

$$S_{1,5,11}(v_1, v_{1,9}, v_3v_{3,4} \dots v_5, v_2P^{2,6}_H v_6v_{6,7} \dots v_9), S_{1,5,11}(v_6, v_{5,6}, P^{4,6}_H v_4v_{4,5}, v_{6,7}v_7 \dots v_{3,4}),$$
$$S_{1,5,11}(v_6, v_{6,7}, P^{6,8}_H v_8v_{7,8}, v_{5,6}v_5 \dots v_{8,9}) \text{ or } S_{1,5,11}(v_9, v_{1,9}, v_{8,9}v_8 \dots v_{6,7}, P^{6,9}_H v_6v_{5,6} \dots v_3v_2),$$

a contradiction. Hence v_6 is on a chord of $C_{c(H_0)}$ ($v_2v_6 \in E(H_0)$ or $v_6v_9 \in E(H_0)$). Similarly, v_7 is on a chord of $C_{c(H_0)}$ ($v_2v_7 \in E(H_0)$ or $v_4v_7 \in E(H_0)$). Then, suppose $N_{H_0}(v_2) \backslash \{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$, $\{v_2, v_s\} \subseteq N(\mathcal{D}_{H_0}) \cap V(C_{c(H_0)})$, by symmetry, $s \in \{4, 5\}$. Then H contains a subgraph

$$S_{1,5,11}(v_4, v_{3,4}, P^{2,4}_H v_2v_3, v_5v_{5,6} \dots v_{1,9})$$

or

$$S_{1,5,11}(v_7, v_{4,7}, v_{7,8}v_8 \dots v_{1,9}, v_{6,7} \dots v_5P^{2,5}_H v_2v_3v_{3,4}v_4),$$

a contradiction. v_2 is on a chord of $C_{c(H_0)}$. Then $N_{H_0}(v_2) \in \{v_5, v_6, v_7, v_8\}$. If $v_2v_6 \in E(H_0)$, then H contains a subgraph $S_{1,5,11}(v_6, v_{6,7}, v_{5,6}v_5 \dots v_{2,3}, v_{2,6}v_2v_{3,4}v_4 \dots v_7v_{4,7})$, a contradiction. Hence $v_2v_6 \notin E(H_0)$, by symmetry, $v_2v_7 \notin E(H_0)$. Hence $v_4v_7 \in E(H_0)$ and $v_6v_9 \in E(H_0)$, we have that H contains a subgraph

$$S_{1,5,11}(v_9, v_{6,9}, v_{1,9}v_1 \dots v_{3,4}, v_{8,9}v_8 \dots v_4v_{4,7}),$$

a contradiction. Therefore $d_{H_0}(v_6) = d_{H_0}(v_7) = 2$, contradicting (1). This proves Case 2.

Case 3. $G \in \mathcal{B}_{6,8}$.

Subcase 3.1. $c(H_0) \geq 10$ and $|V(H_0)| \geq 10$.

By (2), $|V(C)| = 2c(H_0) - 2 \geq 18$. Hence $d_{H_0}(v_r) \geq 3$, where $v_r \in V(C_{c(H_0)})$. Then H contains all subgraphs $S_{1,2i+1,2j+1} \subseteq H[V(C) \cup \{u\}]$ with its center vertex v_r, positive integers $i + j = 7$, and $u \in N_H(v_r) \backslash \{v_{r-1,r}, v_{r,r+1}\}$, a contradiction.

Subcase 3.2. $c(H_0) = 9$(say $C_{c(H_0)} = v_1 v_2 \dots v_9 v_1$) and $|V(H_0)| \geq 10$.

By (2), $|V(C)| = 2c(H_0) - 2 \geq 16$. Firstly, suppose that $P^{2,s} \in \mathcal{P}'$ for any possibility $s \in \{4, 5, 6\}$. Then H contains a subgraph

$$S_{1,7,9} \subseteq S(v_2, v_3, P_H^{2,4} v_4 v_{4,5} v_5 v_{5,6}, v_1 v_{1,9} \dots v_6),$$
$$S_{1,7,9} \subseteq S(v_5, x_{2,5}^5, v_{4,5} v_4 v_{3,4} v_3 v_1 v_2 x_{2,5}^2, v_{5,6} v_6 \dots v_{1,9}),$$
$$S_{1,7,9} \subseteq S(v_6, x_{2,6}^6, v_{6,7} v_7 \dots v_{1,9}, v_{5,6} v_5 \dots v_3 v_1 v_2 x_{2,6}^2).$$

a contradiction. Hence $s \notin \{4, 5, 6\}$, by symmetry, $s \notin \{9, 8, 7\}$. Then, supposed that $P^{4,s} \in \mathcal{P}'$. Then H contains a subgraph $S_{1,7,9} \subseteq S(v_4, v_{4,5}, P_H^{4,6} v_6 v_{5,6} v_5 u, v_{3,4} v_3 \dots v_{7,8})$, where $u \in N_H(v_5) \backslash \{v_{4,5},\ v_{5,6}\}$, $S_{1,7,9} \subseteq S(v_4, v_{4,5}, P_H^{4,7} v_7 v_{6,7} v_6 v_{5,6}, v_{3,4} v_3 \dots v_{7,8})$, $S_{1,7,9} \subseteq S(v_8, x_{4,8}^8, v_{7,8} v_7 \dots v_{4,5}, v_{8,9} v_9 \dots v_4\, x_{4,8}^4)$ or $S_{1,7,9} \subseteq S(v_9, x_{4,9}^9, v_{1,9} v_1 \dots v_4 x_{4,9}^4, v_{8,9} v_8 \dots v_{4,5})$, a contradiction. Hence $P^{4,s} \notin \mathcal{P}'$, by symmetry, $P^{r,9} \notin \mathcal{P}'$. Finally, suppose that $P^{5,s} \in \mathcal{P}'$. Then H contains a subgraph $S_{1,7,9} \subseteq S(v_5, v_{5,6}, P_H^{5,7} v_7 v_{6,7} v_6 u,\ v_{4,5} v_4 \dots v_{8,9})$, where $u \in N_H(v_6) \backslash \{v_{6,7}, v_{5,6}\}$ or $S_{1,7,9} \subseteq S(v_5, v_{5,6}, P_H^{5,8} v_8 v_{7,8}\ v_7 v_{6,7}, v_{4,5} v_4 \dots v_{8,9})$, a contradiction. Hence $P^{5,s} \notin \mathcal{P}'$, by symmetry, $P^{r,8} \notin \mathcal{P}'$. Hence $P^{6,s} \notin \mathcal{P}'$ and $P^{7,s} \notin \mathcal{P}'$. Therefore $|V(H_0)| = 9$, a contradiction. This proves Claim 3.7. □

By Claims 3.6 and 3.7, $m^{H_0} = 4$. If $m^{H_0} = 4$, then $C_{c(H_0)}$ contains exactly six vertices of two triangles, where the vertices of each triangle are pairwise adjacent in $C_{c(H_0)}$. In the following of Theorem 1.6, we denote the another triangle by $v_{q-1} v_q v_{q+1} v_{q-1}$, by symmetry, $q \in \{5, 6, \dots, \lceil \frac{c(H_0)+3}{2} \rceil\}$. By Claim 3.6 (1), we have that $d_{H_0}(v_{q-1}) = d_{H_0}(v_q) = d_{H_0}(v_{q-1}) = 3$.

Claim 3.8. Suppose that $m^{H_0} = 4$ and $|V(H_0)| \geq 10$. Then $c(H_0) \neq 9$.

Proof. Assume, to the contrary, that $c(H_0) = 9$(say $C_{c(H_0)} = v_1 v_2 \dots v_9 v_1$). By (2), $|V(C)| = 2c(H_0) - 2 \geq 14$. In this case, $v_q \in \{v_5, v_6\}$, and $|V(\mathcal{D}_{H_0})| \geq 1$ and $E_{H_0}^1 \cap E_0 = \emptyset$.

Case 1. $G \in \mathcal{B}_{2,12}$.

Subcase 1.1. $v_q = v_5$.

Suppose that $N_{H_0}(v_8) \backslash \{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph $S_{1,3,13} \subseteq S(v_8, v_{8,9}, P^{r,8}\ x_{r,8}^r, v_{7,8} v_7 \dots v_9 x)$ with $x \in N_H(v_9)$, a contradiction. Hence v_8 is on a chord of $C_{c(H_0)}$, i.e.,

$$v_8 v_2 \in E(H_0) \text{ or } v_8 v_5 \in E(H_0). \tag{3}$$

Suppose that $N_{H_0}(v_9) \backslash \{v_8, v_1\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{r,9}$ for any possibility $r \in \{2, 5, 7, 9\}$, and H contains a subgraph $S_{1,3,13} \subseteq S(v_9, v_{1,9}, P_H^{r,9} x_{r,9}^r, v_{7,8} v_7 \dots v_3 v_1 v_2 x_1)$ with $x_1 \in\ N_H(v_2) \backslash\ \{v_3, v_1\}$ and $r \in \{5, 7\}$, $S_{1,3,13} \subseteq (v_4, v_5, v_{3,4} v_3 v_2, v_6 v_{6,7} \dots v_9 P_H^{9,9} v_9 \backslash v_9)$

or $S_{1,3,13} \subseteq (v_1, v_{1,9}, v_3v_{3,4}v_4, v_2P_H^{2,9}v_9v_{8,9}\dots v_5x_2)$ with $x_2 \in N_H(v_5)\backslash\{v_{4,5}, v_{5,6}\}$, a contradiction. Hence v_9 is on a chord of $C_{c(H_0)}$ $(v_5v_9 \in E(H_0))$. Similarly, $v_2v_7 \in E(H_0)$. Hence by Claim 3.6 (1), $v_8v_2 \notin E(H_0)$ and $v_8v_5 \notin E(H_0)$, contradicting (3).

Subcase 1.2. $v_q = v_6$.

Suppose that $N_{H_0}(v_8)\backslash\{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{r,8}$ for any possibility $r \in \{2, 4, 6, 8\}$. Then H contains a subgraph

$$S_{1,3,13} \subseteq (v_1, v_2, v_{1,9}v_9v_{8,9}, v_3v_{3,4,7}\dots v_8P_H^{8,8}v_8 \setminus v_8)$$

or

$$S_{1,3,13} \subseteq S(v_8, v_{8,9}, P_H^{r,8}x_{r,8}^r, v_{7,8}v_7\dots v_9x_1)$$

with $x_1 \in N_H(v_9)\backslash\ \{v_{1,9}, v_{8,9}\}$, a contradiction. Hence v_8 is on a chord of $C_{c(H_0)}$. Similarly, v_9 is on a chord of $C_{c(H_0)}$. Suppose that $N_{H_0}(v_2)\backslash\{v_3, v_1\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{2,r}$ for any possibility $r \in \{4, 6\}$, and H contains a subgraph $S_{1,3,13} \subseteq (v_5, v_6, v_{4,5}v_4v_{3,4}, v_7v_{7,8}\ \dots v_2P_H^{2,2}v_2\backslash v_2)$, $S_{1,3,13} \subseteq (v_9, x_2, v_{8,9}v_8v_{7,8}, v_{1,9}v_1\dots v_5v_7v_6P_H^{2,6}x_{2,6}^2)$ with $x_2 \in N_H(v_9)\backslash\ \{v_{8,9}, v_{1,9}\}$ or $S_{1,3,13} \subseteq S(v_4, v_{4,5}, P_H^{2,4}x_{2,4}^2, v_{3,4}v_3\dots v_7v_5v_6x_3)$ with $x_3 \in N_H(v_6)\backslash\ \{v_5, v_6\}$, a contradiction. Hence v_2 is on a chord of $C_{c(H_0)}$. Similarly, v_6 is on a chord of $C_{c(H_0)}$. Then we have that v_4 is on a chord of $C_{c(H_0)}$. Therefore, $|V(H_0)|= 9$, a contradiction, this proved Case 1.

In the proof of the following cases, for any path $P^{r,r} \in \mathcal{P}'$. Then H contains a subgraph $S_{1,2i+1,2j+1} \subseteq S(v_r, v_{r+1}, P_H^{r,r}v_r\backslash v_r, v_{r-1,r}v_{r-1}\dots v_{r+1,r+2})$, a contradiction. Therefore integer $1 \le r < s \le c(H_0)$ in $P^{r,s}$.

Case 2. $G \in \mathcal{B}_{4,10}$.

Suppose that $|P^{r,s}|\ge 4$, $S_{1,5,11} \subseteq H[V(C) \cup V(P_H^{r,s})]$ with its center vertex v_r, a contradiction. Then suppose that $|P^{r,s}|= 3$, say $P_H^{r,s} = v_rx_{r,s}^rx_{r,s}x_{r,s}^sv_s$.

Subcase 2.1. $v_q = v_5$.

Suppose that $N_{H_0}(v_8)\backslash\{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{r,8}$ for any possibility $r \in \{2, 5\}$, and H contains a subgraph $S_{1,5,11}(v_9, x_1, v_{8,9}v_8P_H^{r,8}x_{r,8}^r, v_{1,9}v_1\dots v_{7,8})$ with $x_1 \in N_H(v_9)$ and $r \in \{2, 5\}$ a contradiction. Hence v_8 is in a chord of $C_{c(H_0)}$, i.e.,

$$v_8v_2 \in E(H_0) \text{ or } v_8v_5 \in E(H_0). \tag{4}$$

Suppose that $N_{H_0}(v_9)\backslash\{v_8, v_1\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{r,9}$ for any possibility $r \in \{2, 5, 7\}$, and H contains a subgraph $S_{1,5,11}(v_9, v_{1,9}, v_{8,9}v_8\dots v_{6,7}, P_H^{2,9}v_2v_1v_3v_{3,4}v_4v_6v_5x_2)$ with $x_2 \in N_H(v_5)$, $S_{1,5,11}(v_9,\ v_{1,9}, v_{8,9}v_8\dots v_{6,7}, P_H^{5,9}v_5v_6v_4v_{3,4}v_3v_1v_2x_3)$ with $x_3 \in N_H(v_2)$ or $S_{1,5,11}(v_8, x_2, v_{7,8}v_7\ P_H^{7,9}x_{7,9}^9, v_{8,9}v_9\dots\ v_{6,7})$ with $x_2 \in N_H(v_5)$, a contradiction. Hence v_9 is on a chord of $C_{c(H_0)}$ $(v_5v_9 \in E(H_0))$. Similarly, $v_2v_7 \in E(H_0)$. Hence by Claim 3.6 (1), $v_8v_2 \notin E(H_0)$ and $v_8v_5 \notin E(H_0)$, contradicting (4).

Subcase 2.2. $v_q = v_6$.

Suppose that $N_{H_0}(v_8)\backslash\{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{r,8}$ for any possibility $r \in \{2, 4, 6\}$, and H contains a subgraph $S_{1,5,11}(v_9, x_1, v_{8,9}v_8P_H^{r,8}x_{r,8}^r, v_{1,9}v_1\dots v_{7,8})$ for any possibility $r \in \{2, 4, 6\}$ and $x_1 \in N_H(v_9)$, a contradiction. Hence v_8 is on a chord of $C_{c(H_0)}$. Similarly, v_9 is on a chord of $C_{c(H_0)}$. Suppose that $N_{H_0}(v_2)\backslash\{v_3, v_1\} \subseteq V(\mathcal{D}_{H_0})$. Then there exists a path $P^{2,s}$ for any possibility $s \in \{4, 6\}$, and H contains a subgraph $S_{1,5,11}(v_4, v_{3,4}, v_{4,5}v_5v_7v_6x_2, P_H^{2,4}v_2v_3v_1v_{1,9}\dots v_{7,8})$ with $x_2 \in N_H(v_6)$ or $S_{1,5,11}(v_4, v_{3,4}, v_{4,5}v_5\dots v_{7,8}, v_{4,8}v_8v_{8,9}v_9v_{1,9}v_1v_3v_2P_H^{2,6}x_{2,6}^6)$, a contradiction. Thus v_2 is on a chord of $C_{c(H_0)}$. Similarly, v_6 is on a chord of $C_{c(H_0)}$. Hence v_4 is on a chord of $C_{c(H_0)}$. Therefore $|V(H_0)|= 9$, a contradiction. This proved Case 2.

Case 3. $G \in \mathcal{B}_{6,8}$.

Suppose that $|P^{r,s}|\geq 5$, $S_{1,7,9} \subseteq H[V(C) \cup V(P_H^{r,s})]$ with its center vertex v_r, a contradiction. Then suppose that $|P^{r,s}|= 3$ or $|P^{r,s}|= 4$, say $P_H^{r,s} = v_rx_{r,s}^rx_{r,s}x_{r,s}^sv_s$ or $P_H^{r,s} = v_rx_{r,s}^rx_{r,s}^1x_{r,s}^{12}x_{r,s}^2x_{r,s}^sv_s$.

Subcase 3.1. $v_q = v_5$.

Let $N_{H_0}(v_8)\backslash\{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$, there exists a path $P^{r,8}$ for any possibility $r \in \{2, 5\}$. Then H contains a subgraph $S_{1,7,9} \subseteq S(v_8, v_{7,8}, v_{8,9}v_9\dots v_{3,4}, P_H^{5,8}v_5v_4v_6v_{6,7}v_7x_1)$ with $x_1 \in N_H(v_7)$ or $S_{1,7,9} \subseteq S(v_8, v_{8,9}, v_{7,8}v_7\dots v_{3,4}, P_H^{2,8}v_2v_3v_1v_{1,9}v_9x_2)$ with $x_2 \in N_H(v_9)$, a contradiction. Hence v_8 is on a chord of $C_{c(H_0)}$, i.e.,

$$v_8v_2 \in E(H_0) \text{ or } v_8v_5 \in E(H_0). \tag{5}$$

Let $N_{H_0}(v_9)\backslash\{v_8, v_1\} \subseteq V(\mathcal{D}_{H_0})$, there exists a path $P^{r,9}$ for any possibility $r \in \{2, 5, 7\}$. If $|P^{r,s}|= 4$, then H contains a subgraph $S_{1,7,9}(v_3, v_2, v_{3,4}v_4v_6v_{6,7}\dots v_8, v_1v_{1,9}v_9P_H^{5,9}v_5)$, a contradiction. Hence $|P^{r,s}|= 3$ ($P_H^{r,s} = v_9x_{r,9}^9x_{r,9}x_{r,9}^rv_r$). Then H contains a subgraph

$$S_{1,7,9}(v_9, v_{1,9}, P_H^{2,9}v_2v_1v_3v_{3,4}, v_{8,9}v_8\dots v_6v_4v_5x_3)$$

with $x_3 \in N_H(v_5)$,

$$S_{1,7,9}(v_8, x_4, v_{8,9}v_9\dots v_{3,4}, v_{7,8}v_7v_{6,7}v_6v_4v_5P_H^{5,9}x_{5,9}^9)$$

with $x_4 \in N_H(v_8)$ or $S_{1,7,9}(v_7, v_{7,8}, P_H^{7,9}v_9v_{8,9}v_8x_5, v_{6,7}v_6\dots v_{1,9})$ with $x_5 \in N_H(v_8)$, a contradiction. Hence v_9 is on a chord of $C_{c(H_0)}$ ($v_5v_9 \in E(H_0)$). Similarly, $v_2v_7 \in E(H_0)$. Hence by Claim 3.6 (1), $v_8v_2 \notin E(H_0)$ and $v_8v_5 \notin E(H_0)$, contradicting (5).

Subcase 3.2. $v_q = v_6$.

Let $N_{H_0}(v_8)\backslash\{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$, there exists a path $P^{r,8}$ for any possibility $r \in \{2, 4, 6\}$. If $|P^{r,s}|= 4$, then H contains a subgraph

$$S_{1,7,9}(v_8, v_{7,8}, P_H^{4,8}v_4v_{4,5}v_5v_6, v_{8,9}v_9\dots v_{3,4})$$

or

$$S_{1,7,9}(v_2, v_1, v_3v_{3,4}\dots v_7, P_H^{2,8}v_8v_{8,9}v_9v_{1,9}),$$

a contradiction. Hence $|P^{r,s}|= 3$ ($P_H^{r,s} = v_8x_{r,8}^8x_{r,8}x_{r,8}^rv_r$). Then H contains a subgraph

$$S_{1,7,9}(v_4, x_1, v_{4,5}v_5\dots v_{8,9}, v_{3,4}v_3\dots v_8P_H^{2,8}x_{2,8}^2) \quad \text{with } x_1 \in N_H(v_4),$$
$$S_{1,7,9}(v_8, v_{7,8}, v_{8,9}v_9\dots v_{3,4}, P_H^{4,8}v_4v_{4,5}v_5v_7v_6x_2) \quad \text{with } x_2 \in N_H(v_6),$$
$$S_{1,7,9}(v_4, x_3, v_{3,4}v_3\dots v_{8,9}, v_{4,5}v_5v_7v_6P_H^{6,8}v_8v_{7,8}) \quad \text{with } x_3 \in N_H(v_4),$$

a contradiction. Hence v_8 is on a chord of $C_{c(H_0)}$. Similarly, v_9 is on a chord of $C_{c(H_0)}$. Let $N_{H_0}(v_2)\backslash\{v_3,\ v_1\} \subseteq V(\mathcal{D}_{H_0})$, there exists a path $P^{2,r}$ for any possibility $r \in \{4,6\}$. Then H contains a subgraph $S_{1,7,9} \subseteq S(v_9, v_{4,9}, v_{1,9}v_1v_2P_H^{2,6}v_6, v_{8,9}v_8v_{7,8}v_7v_5v_{4,5}v_4v_{3,4}v_3)$ or $S_{1,7,9} \subseteq S(v_8, v_{4,8}, v_{8,9}v_9 \dots v_{3,4}, v_{7,8}v_7 \dots v_4P_H^{2,4}x_{2,4}^2)$, a contradiction. Hence v_2 is on a chord of $C_{c(H_0)}$. Similarly, v_6 is on a chord of $C_{c(H_0)}$. Hence v_4 is on a chord of $C_{c(H_0)}$. Therefore, $|V(H_0)|= 9$, a contradiction. This proves Claim 3.8. □

Thus we can get that $c(H_0) \geq 10$ and $|V(H_0)|\geq 10$ and $m^{H_0} = 4$, where $E^1_{H_0} \cap E_0 = \emptyset$. By (2), $|V(C)|= 2c(H_0) - 2 \geq 16$. For any path $P^{r,r} \in \mathcal{P}'$. Then H contains a subgraph $S_{1,2i+1,2j+1} \subseteq S(v_r, v_{r+1}, P_H^{r,r}v_r\backslash v_r, v_{r-1,r}v_{r-1} \dots v_{r+1,r+2})$, a contradiction. Therefore integer $1 \leq r < s \leq c(H_0)$ in $P^{r,s}$.

Claim 3.9. Suppose that $m^{H_0} = 4$. Then $|V(H_0)|= c(H_0) \geq 10$.

Proof. Assume, to the contrary, that $|V(H_0)|> c(H_0) \geq 10$. If $c(H_0) \geq 11$, then $|V(C)|= 2c(H_0) - 4 \geq 18$. Hence $d_{H_0}(v_r) \geq 3$ with $v_r \in V(C_{c(H_0)})$, Then H contains all subgraphs $S_{1,2i+1,2j+1} \subseteq H[V(C) \cup \{u\}]$ with its center vertex v_r, and $u \in N_H(v_r)\backslash\{v_{r-1,r}, v_{r,r+1}\}$, a contradiction. Therefore $c(H_0) = 10$(say $C_{c(H_0)} = v_1v_2 \dots v_{10}v_1$).

Case 1. $G \in \mathcal{B}_{2,12}$.

We can get that $S_{1,3,13} \subseteq H[V(C) \cup V(P_H^{r,s})]$ with its center vertex $v_r \in V(C_{c(H_0)})$, a contradiction.

Case 2. $G \in \mathcal{B}_{4,10}$.

If $|P^{r,s}|\geq 4$, then $S_{1,5,11} \subseteq H[V(C) \cup V(P_H^{r,s})]$ with its center vertex $v_r \in V(C_{c(H_0)})$, a contradiction. Hence $|P^{r,s}|= 3$, i.e., $P_H^{r,s} = v_rx^r_{r,s}x_{r,s}x^s_{r,s}v_s$.

Subcase 2.1. $v_q = v_5$.

Firstly, suppose that $N_{H_0}(v_{10})\backslash\{v_1, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,5,11}(v_{10}, v_{1,10}, v_{9,10}v_9 \dots v_{7,8}, P_H^{2,10}v_2v_1v_3 \dots v_7),$$
$$S_{1,5,11}(v_{10}, v_{9,10}, v_{1,10}v_1v_2v_3v_{3,4}, P_H^{5,10}v_5v_4v_6 \dots v_9),$$
$$S_{1,5,11}(v_{10}, v_{1,10}, v_{9,10}v_9v_{8,9}v_8v_{7,8}, P_H^{7,10}v_7v_{6,7} \dots v_1),$$
$$S_{1,5,11}(v_8, v_{8,9}, P_H^{8,10}v_{10}v_{9,10}, v_{7,8}v_7v_{6,7} \dots v_{1,10}),$$

a contradiction. Hence $N_{H_0}(v_{10}) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_7) \subseteq V(C_{c(H_0)})$. Then, suppose that $N_{H_0}(v_9)\backslash\{v_8,\ v_{10}\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,5,11}(v_6, v_5, v_{6,7}v_7 \dots v_{8,9}, v_4v_{3,4}v_3v_2P_H^{2,9}v_9v_{9,10}v_{10}v_{1,10})$$

or

$$S_{1,5,11}(v_9, v_{9,10}, v_{8,9}v_8 \dots v_{6,7}, P_H^{5,9}v_5v_4 \dots v_{10}),$$

a contradiction. Hence $N_{H_0}(v_9) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_8) \subseteq V(C_{c(H_0)})$. Finally, suppose that $N_{H_0}(v_2)\backslash\{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,5,11}(v_4, v_{3,4}, v_5P_H^{2,5}v_2, v_6v_{6,7} \dots v_1),$$

a contradiction. Hence $N_{H_0}(v_2) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_5) \subseteq V(C_{c(H_0)})$. Therefore, $|V(H_0)|= c(H_0) = 10$, a contradiction.

Subcase 2.2. $v_q = v_6$.

Firstly, suppose that $N_{H_0}(v_{10})\backslash\{v_1, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph $S_{1,5,11}(v_7, v_6, v_{7,8}\ v_8v_{8,9}v_9v_{9,10}, v_5v_{4,5}v_4 \ldots v_{10}P_H^{r,10}x_{r,10})$ for any possibility $r \in \{2, 4, 6, 8\}$, a contradiction. Hence $N_{H_0}(v_{10}) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_8) \subseteq V(C_{c(H_0)})$. Secondly, suppose that $N_{H_0}(v_9)\backslash\{v_8, v_{10}\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,5,11}(v_7, v_6, v_5v_{4,5}v_4v_{3,4}v_3, v_{7,8}v_8 \ldots v_2x_{2,9}^2x_{2,9}),$$
$$S_{1,5,11}(v_1, v_2, v_3v_{3,4}v_4x_{4,9}^4x_{4,9}, v_{1,10}v_{10} \ldots v_{4,5}),$$
$$S_{1,5,11}(v_1, v_2, v_3v_{3,4} \ldots v_5, v_{1,10}v_{10}v_{9,10}v_9 \ldots v_6x_{6,9}^6x_{6,9}),$$

a contradiction. Hence $N_{H_0}(v_9) \subseteq V(C_{c(H_0)})$. Finally, suppose that $N_{H_0}(v_2)\backslash\{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph $S_{1,5,11}(v_4, v_{3,4}, P_H^{2,4}v_2v_3, v_{4,5}v_5 \ldots v_{1,10})$ or

$$S_{1,5,11}(v_1, v_2, v_3v_{3,4}v_4v_{4,5}v_5, v_{1,10}v_{10} \ldots v_6x_{2,6}^6x_{2,6}),$$

a contradiction. Hence $N_{H_0}(v_2) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_6) \subseteq V(C_{c(H_0)})$. Therefore, $N_{H_0}(v_4) \subseteq V(C_{c(H_0)})$ and $|V(H_0)|= c(H_0) = 10$, a contradiction.

Subcase 2.3. $v_q = v_7$.

Suppose that $N_{H_0}(v_9)\backslash\{v_8, v_{10}\} \subseteq V(\mathcal{D}_{H_0})$, then H contains a subgraph

$$S_{1,5,11}(v_8, v_7, v_{8,9}v_9P_H^{r,9}, x_{r,9}^r, v_6v_{5,6} \ldots v_{10}) \quad \text{for any possibility } r \in \{2, 4, 5, 7\}.$$

a contradiction. Hence $N_{H_0}(v_9) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_{10}) \subseteq V(C_{c(H_0)})$, $N_{H_0}(v_5) \subseteq V(C_{c(H_0)})$ and $N_{H_0}(v_4) \subseteq V(C_{c(H_0)})$. Next, suppose that $N_{H_0}(v_2)\backslash\{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph $S_{1,5,11}(v_1, v_2,\ v_{1,10}v_{10} \ldots v_{8,9}, v_3v_{3,4}\ \ldots v_7P_H^{2,7}x_{2,7}^2)$, a contradiction. Hence $N_{H_0}(v_2) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_7) \subseteq V(C_{c(H_0)})$. Therefore, $|V(H_0) = c(H_0)|= 10$, a contradiction. This proves Case 2.

Case 3. $G \in \mathcal{B}_{6,8}$.

If $|P^{r,s}|\geq 5$, then $S_{1,7,9} \subseteq H[V(C) \cup V(P_H^{r,s})]$ with its center vertex $v_r \in V(C_{c(H_0)})$, a contradiction. Hence $|P^{r,s}|= 3$ or $|P^{r,s}|= 4$, i.e., $P_H^{r,s} = v_rx_{r,s}^rx_{r,s}x_{r,s}^sv_s$ or $P_H^{r,s} = v_rx_{r,s}^rx_{r,s}^1x_{r,s}^{12}x_{r,s}^2x_{r,s}^sv_s$.

Subcase 3.1. $v_q = v_5$.

Firstly, suppose that $N_{H_0}(v_9)\backslash\{v_8, v_{10}\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,7,9}(v_9, x_{5,9}^9, v_{9,10}v_{10} \ldots v_{3,4}, v_{8,9}v_8 \ldots v_6v_4v_5x_{5,9}^5),$$
$$S_{1,7,9}(v_9, x_{2,9}^9, v_{9,10}v_{10}v_{1,10}v_1v_3v_2x_{2,9}^2, v_{8,9}v_8 \ldots v_{3,4}),$$
$$S_{1,7,9}(v_6, v_5, v_4v_{3,4} \ldots v_{10}, v_{6,7}v_7v_{7,8} \ldots v_9P_H^{7,9}x_{7,9}^7),$$

a contradiction. Hence $N_{H_0}(v_9) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_8) \subseteq V(C_{c(H_0)})$. Then, suppose that $N_{H_0}(v_7)\backslash\{v_1, v_9\} \subseteq V(\mathcal{D}_{H_0})$, we can easily get that H contains a subgraph

$$S_{1,7,9}(v_7, x_{2,7}^7, v_{7,8}v_8 \ldots v_{1,10}, v_{6,7}v_6 \ldots v_3v_1v_2x_{2,7}^2),$$
$$S_{1,7,9}(v_7, x_{7,10}^7, v_{7,8}v_8v_{8,9}v_9v_{9,10}v_{10}x_{7,10}^{10}, v_{6,7}v_6 \ldots v_{1,10}),$$
$$S_{1,7,9}(v_7, v_{6,7}, v_{7,8}v_8 \ldots v_{1,10}, P_H^{5,7}v_5v_4v_{3,4} \ldots v_1),$$

a contradiction. Hence $N_{H_0}(v_7) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_{10}) \subseteq V(C_{c(H_0)})$. Finally, suppose that $N_{H_0}(v_2)\backslash\{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph $S_{1,7,9} \subseteq S(v_4, v_5, v_{3,4}v_3v_1v_2P_H^{2,5}x_{2,5}^5, v_6v_{6,7}\dots v_{10})$, a contradiction. Hence $N_{H_0}(v_2) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_5) \subseteq V(C_{c(H_0)})$. Therefore, $|V(H_0)|= 10$, a contradiction.

Subcase 3.2. $v_q = v_6$.

Firstly, suppose that $N_{H_0}(v_8)\backslash\{v_7, v_9\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,7,9}(v_1, v_2, v_3v_{3,4}\dots v_7, v_{1,10}v_{10}\dots v_8P_H^{8,10}x_{8,10}^{10}),$$
$$S_{1,7,9}(v_1, v_2, v_3v_{3,4}v_4v_{4,5}v_5v_7v_{7,8}, v_{1,10}v_{10}v_{9,10}v_9v_{8,9}v_8P_H^{6,8}x_{6,8}^6),$$
$$S_{1,7,9}(v_8, x_{4,8}^8, v_{7,8}v_7\dots v_4x_{4,8}^4, v_{8,9}v_9\dots v_{3,4})$$

or

$$S_{1,7,9}(v_8, x_{2,8}^8, v_{7,8}v_7\dots v_{3,4}, v_{8,9}v_9\dots v_1v_3v_2x_{2,8}^2),$$

a contradiction. Hence $N_{H_0}(v_8) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_{10}) \subseteq V(C_{c(H_0)})$. Then, suppose that $N_{H_0}(v_9)\backslash \{v_{10}, v_8\} \subseteq V(\mathcal{D}_{H_0})$, we have that H contains a subgraph

$$S_{1,5,11}(v_8, v_7, v_{8,9}v_9P_H^{r,9}, x_{r,9}^r, v_6v_{5,6}\dots v_{10}) \quad \text{for any possibility } r \in \{2, 4, 5, 7\}.$$

a contradiction. Hence $N_{H_0}(v_9) \subseteq V(C_{c(H_0)})$. Finally, suppose that $N_{H_0}(v_2)\backslash\{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$, we can get that H contains a subgraph

$$S_{1,7,9} \subseteq S(v_3, v_{3,4}, v_2P_H^{2,4}v_4v_{4,5}v_5v_6, v_1v_{1,10}\dots v_7),$$

or

$$S_{1,7,9}(v_2, x_{2,6}^2, v_3v_{3,4}\dots v_6x_{2,6}^6, v_1v_{1,10}\dots v_7),$$

a contradiction. Hence $N_{H_0}(v_2) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_6) \subseteq V(C_{c(H_0)})$. Therefore, $N_{H_0}(v_4) \subseteq V(C_{c(H_0)})$ and $|V(H_0)|= 10$, a contradiction.

Subcase 3.3. $v_q = v_7$.

Suppose that $N_{H_0}(v_9)\backslash\{v_8, v_{10}\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph

$$S_{1,7,9}(v_9, x_{2,9}^9, v_{9,10}v_{10}\dots v_1v_3v_2x_{2,9}^2, v_{8,9}v_8\dots v_{3,4}),$$
$$S_{1,7,9}(v_9, x_{4,9}^9, v_{8,9}v_8\dots v_{4,5}, v_{9,10}v_{10}v_{1,10}\dots v_4x_{4,9}^4),$$
$$S_{1,7,9}(v_9, x_{5,9}^9, v_{8,9}v_8\dots v_5x_{5,9}^5, v_{9,10}v_{10}\dots v_{4,5}),$$
$$S_{1,7,9}(v_1, v_2, v_3v_{3,4}\dots v_6, v_{1,10}v_{10}v_{9,10}v_9P_H^{7,9}v_7v_8),$$

a contradiction. Hence $N_{H_0}(v_9) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_{10}) \subseteq V(C_{c(H_0)})$, $N_{H_0}(v_5) \subseteq V(C_{c(H_0)})$ and $N_{H_0}(v_4) \subseteq V(C_{c(H_0)})$. Next, we suppose that $N_{H_0}(v_2)\backslash\{v_1, v_3\} \subseteq V(\mathcal{D}_{H_0})$. Then H contains a subgraph $S_{1,7,9}(v_2, x_{2,7}^2, v_3v_{3,4}\dots v_6, v_1v_{1,10}\dots v_7x_{2,7}^7)$, a contradiction. Hence $N_{H_0}(v_2) \subseteq V(C_{c(H_0)})$, by symmetry, $N_{H_0}(v_7) \subseteq V(C_{c(H_0)})$. Therefore, $|V(H_0)|= 10$, a contradiction. This proves Claim 3.9. □

Claim 3.10. Suppose that $m^{H_0} = 4$ and $|V(H_0)|= c(H_0)$. Then $c(H_0) \neq 10$.

Proof. Assume, to the contrary, that $|V(H_0)|= c(H_0) = 10$.

Case 1. $G \in \mathcal{B}_{2,12}$.

Firstly, suppose that $v_q = v_5$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_7)\backslash\{v_6, v_8\} \subseteq \{v_2, v_{10}\}$. Then H contains a subgraph

$$S_{1,3,13}(v_7, v_{2,7}, v_{7,8}v_8v_{5,8}, v_{6,7}v_6 \dots v_{8,9})$$

or

$$S_{1,3,13}(v_7, v_{7,10}, v_{7,8}v_8v_{5,8}(v_{2,8}), v_{6,7}v_6 \dots v_{8,9}),$$

a contradiction. Therefore $d_{H_0}(v_7) = 2$, a contradiction.

Then, suppose that $v_q = v_6$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_9)\backslash\{v_8, v_{10}\} \subseteq \{v_2, v_4, v_6\}$. We can get that H contains a subgraph $S_{1,3,13}(v_9, v_{2,9}, v_{9,10}v_{10}v_{6,10}(v_{4,10}), v_{8,9}v_8 \dots v_{1,10})$, $S_{1,3,13}(v_9, v_{6,9}, v_{9,10}v_{10}v_{4,10}, v_{8,9}v_8 \dots v_{1,10})$ or $S_{1,3,13}(v_9, v_{4,9}, v_{9,10}v_{10}v_{r,10}, v_{8,9}v_8 \dots v_{1,10})$ for any possibility $r \in \{2, 4, 6\}$, a contradiction. Therefore $d_{H_0}(v_9) = 2$, a contradiction.

Finally, suppose that $v_q = v_7$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_7)\backslash\{v_8, v_6\} \subseteq \{v_2, v_4, v_{10}\}$. Then H contains a subgraph

$$S_{1,3,13}(v_6, v_{5,6}, v_7v_{2,7}v_2, v_8v_{8,9} \dots v_1v_3v_{3,4}v_4v_{4,5}v_5v_{5,9}(v_{5,10})),$$
$$S_{1,3,13}(v_4, v_{4,7}, v_{4,5}v_5x_1, v_{3,4}v_3 \dots v_{5,6}) \text{ with } x_1 \in \{v_{2,5}, v_{2,9}, v_{2,10}\},$$
$$S_{1,3,13}(v_{10}, v_{7,10}, v_{9,10}v_9x_2, v_{1,10}v_1 \dots v_{8,9}) \text{ with } x_2 \in \{v_{2,9}, v_{4,9}, v_{5,9}\},$$

a contradiction. Therefore $d_{H_0}(v_7) = 2$, a contradiction. This proves Case 1.

Case 2. $G \in \mathcal{B}_{4,10}$.

Firstly, suppose that $v_q = v_5$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_7)\backslash\{v_6, v_8\} \subseteq \{v_2, v_{10}\}$.Then H contains a subgraph

$$S_{1,5,11}(v_7, v_{2,7}, v_{7,8}v_8 \dots v_9v_{5,9}, v_{6,7}v_6 \dots v_{9,10})$$

or

$$S_{1,5,11}(v_7, v_{7,10}, v_{7,8}v_8 \dots v_9v_{5,9}(v_{2,9}), v_{6,7}v_6 \dots v_{9,10}),$$

a contradiction. Therefore $d_{H_0}(v_7) = 2$, a contradiction.

Then, suppose that $v_q = v_6$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_{10})\backslash\{v_9, v_1\} \subseteq \{v_4, v_6\}$. We can get that H contains a subgraph $S_{1,5,11}(v_{10}, v_{4,10}, v_{9,10}v_9v_{8,9}v_8v_{2,8}(v_{4,8}), v_{1,10}v_1 \dots v_{7,8})$ or $S_{1,5,11}(v_{10}, v_{6,10}, v_{9,10}v_9\, v_{8,9}v_8v_{2,8}(v_{4,8}), v_{1,10}v_1 \dots v_{7,8})$, a contradiction. Therefore $d_{H_0}(v_{10}) = 2$, a contradiction.

Finally, suppose that $v_q = v_7$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_9)\backslash\{v_8, v_{10}\} \subseteq \{v_2, v_4, v_5\}$. Then H contains a subgraph

$$S_{1,5,11}(v_9, v_{2,9}, v_{8,9}v_8v_6v_7v_{4,7}(v_{7,10}), v_{9,10}v_{10} \dots v_{5,6}),$$
$$S_{1,5,11}(v_9, v_{4,9}, v_{8,9}v_8v_6v_7u, v_{9,10}v_{10} \dots v_{5,6}) \text{ for any possibility } u \in \{v_{7,10}, v_{2,7}, v_{4,7}\}$$

or

$$S_{1,5,11}(v_9, v_{5,9}, v_{8,9}v_8v_6v_7u, v_{9,10}v_{10} \dots v_{5,6}) \text{ for any possibility } u \in \{v_{7,10}, v_{2,7}, v_{4,7}\},$$

a contradiction. Therefore $d_{H_0}(v_9) = 2$, a contradiction. This proves Case 2.

Case 3. $G \in \mathcal{B}_{6,8}$.

Firstly, suppose that $v_q = v_5$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_7) \backslash \{v_6, v_8\} \subseteq \{v_2, v_{10}\}$. Then H contains a subgraph

$$S_{1,7,9}(v_7, v_{2,7}, v_{7,8}v_8 \dots v_{10}v_{5,10}(v_{7,10}), v_{6,7}v_6 \dots v_{1,10})$$

or

$$S_{1,7,9}(v_7, v_{7,8}, v_{7,10}v_{10}v_{9,10}v_9v_{8,9}v_8\, v_{5,8}(v_{2,8}), v_{6,7}v_6 \dots v_{1,10}),$$

a contradiction. Therefore $d_{H_0}(v_7) = 2$, a contradiction.

Then, suppose that $v_q = v_6$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_{10}) \backslash \{v_9, v_1\} \subseteq \{v_4, v_6\}$. We can get that H contains a subgraph $S_{1,7,9}(v_6, v_{6,10}, v_5v_{4,5} \dots v_2v_{2,8}(v_{2,9}), v_{6,7}v_7 \dots v_1)$ or $S_{1,7,9}(v_{10}, v_{4,10}, v_{1,10}v_1 \dots v_{4,5},\ v_{9,10}v_9 \dots v_7v_5v_6u)$ for any possibility $u \in \{v_{2,6}, v_{6,9}, v_{6,10}\}$, a contradiction. Therefore $d_{H_0}(v_{10}) = 2$, a contradiction.

Finally, suppose that $v_q = v_7$. Since $\kappa'(H_0) \geq 3$, $N_{H_0}(v_9) \backslash \{v_8, v_{10}\} \subseteq \{v_2, v_4, v_5\}$. If $v_4v_9 \in E(H_0)$ or $v_5v_9 \in E(H_0)$, then H contains a subgraphs

$$S_{1,7,9}(v_9, v_{4,9}(v_{5,9}), v_{9,10}v_{10}v_{1,10}v_1v_3v_2u, v_{8,9}v_8 \dots v_{3,4})$$

for any possibility $u \in \{v_{2,5}, v_{2,9}, v_{2,7}\}$, a contradiction. Therefore $v_4v_9 \notin E(H_0)$ and $v_5v_9 \notin E(H_0)$. By symmetry, $v_4v_{10} \notin E(H_0)$ and $v_5v_{10} \notin E(H_0)$. Therefore $v_2v_9 \in E(H_0)$ and $v_7v_{10} \in E(H_0)$, we can get that $d_{H_0}(v_4) = 2$, a contradiction. This proves Claim 3.10.

By Claims 3.1 and 3.4-3.10, we have that $|V(H_0)| = c(H_0) \geq 11$ and $m^{H_0} = 4$. By (2), $|V(C)| = 2c(H_0) - 2 \geq 18$. Since $\kappa'(H_0) \geq 3$, $v_r \in N_{H_0}(v_2) \backslash \{v_1, v_3\}$. We can get that H contains subgraphs $S_{1,3,13} \subseteq S(v_2, v_{2,r}, v_1v_{1,c(H_0)}v_{c(H_0)}, v_3v_{3,4} \dots v_{c(H_0)-1})$, $S_{1,5,11} \subseteq S(v_2, v_{2,r}, v_1v_{1,c(H_0)}v_{c(H_0)} \dots v_{c(H_0)-1},\ v_3v_{3,4}v_4v_{4,5} \dots v_{c(H_0)-2})$ and

$$S_{1,7,9} \subseteq S(v_2, v_{2,r}, v_1v_{1,c(H_0)} \dots v_{c(H_0)-2}, v_3v_{3,4} \dots v_{c(H_0)-3}),$$

a contradiction. This completes the proof of Theorem 1.6. □

4. Concluding remark

Remark 4.1. We construct a family of 3-connected non-Hamilton-connected graph in Figure 5 with integer $m_1 \geq 3$, and there is no Hamiltonian (a, b)-path in Figure 5. Then we can easily find that these graphs are $\{K_{1,3}, \Gamma_0\}$-free and $B_{2i+1,2j}$-free, these graphs are also $B_{2i,2j+1}$-free with positive integers $i+j = 7$. Thus this example shows that our results of Theorem 1.6 are sharp.

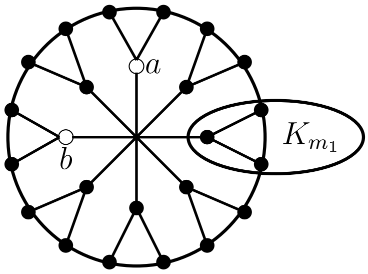

Fig. 5. A family of 3-connected non-Hamilton-connected graph

Remark 4.2. We can now update the discussion of potential triples $K_{1,3}$, Γ_0 and X of connected graphs that might imply Hamilton-connectedness of a 3-connected $\{K_{1,3}, \Gamma_0, X\}$-free graph, summarized in [21] and [22]. In this paper, we focus on inducing the hourglass on the result of forbidden subgraph pairs, there are the following possibilities for X (see Figure 1 for the graphs Z_i, $B_{i,j}$ and $N_{i,j,k}$). We summarize the current status of the problem in the following table, where integer $i, j, k \geq 1$, and we can get that

The graph X	Possible	Best Known	Reference	Open
P_i	$i \leq 16$	P_{16}	[19], [12]	–
Z_{2i}	$i \leq 7$	Z_{14}	[22]	–
$B_{2i,2j}$	$i + j \leq 7$	$i + j \leq 7$	This paper	–
$N_{2i,2j,2k}$	$i + j + k \leq 7$	$i + j + k \leq 7$	[21]	–

The proof of results in [19] depends on the pairs of forbidden subgraphs, while this method of the present paper does not depend on the results of a pair of forbidden subgraphs and we may prove the results directly.

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Conflict of interest

The authors declare no conflict of interest.

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