ARS COMBINATORIA 161 (2024) 29–47

Ars Combinatoria
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Combinatorial Press

$(n-2)$-fault-tolerant edge-pancyclicity of Möbius cubes MQ_n

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ABSTRACT

The n-dimensional Möbius cube MQ_n is an important variant of the hypercube Q_n, which possesses some properties superior to the hypercube. This paper investigates the fault-tolerant edge-pancyclicity of MQ_n, and shows that if $MQ_n (n \geq 5)$ contains at most $n-2$ faulty vertices and/or edges then, for any fault-free edge uv in $MQ_n^i (i = 0, 1)$ and any integer ℓ with $7 - i \leqslant \ell \leqslant 2^n - f_v$, there is a fault-free cycle of length ℓ containing the edge uv, where f_v is the number of faulty vertices. The result is optimal in some senses.

Keywords: Combinatorics, Möbius cube, Edge-pancyclicity, Fault-tolerant, Interconnection network

2020 Mathematics Subject Classification: 05C85, 68R10

1. Introduction

It is well known that interconnection networks are of interest in parallel and distributed computing systems because they determine the performance of the systems on a large scale. As topological structures, interconnection networks can be represented by a graph $G = (V, E)$, where V is the vertex-set of G and E is the edge-set of G. $|V(G)|$ and

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Received 10 January 2020; accepted 15 May 2020; published 31 December 2024.

DOI: 10.61091/ars161-03

$|E(G)|$ denote the numbers of vertices and edges of G, respectively. A path denoted by $(v_1, v_2, \ldots, v_k)$ is a sequence of vertices where two consecutive vertices are adjacent in G. A cycle is a path $(v_1, v_2, \ldots, v_k)$ where $v_1 = v_k$. A graph G is *pancyclic* if, for every $girth \leqslant \ell \leqslant |V(G)|$, G has a cycle of length ℓ. A graph G is *edge-pancyclic* if, for any edge e of G and every $girth \leqslant \ell \leqslant |V(G)|$, G has a cycle of length ℓ containing e. A graph G is *vertex-pancyclic* if, for any vertex u of G and every $girth \leqslant \ell \leqslant |V(G)|$, G has a cycle of length ℓ containing u. Obviously, if a graph is edge-pancyclic, it is also vertex-pancyclic. Edge-pancyclic and vertex-pancyclic on various interconnection networks were studied, including hypercubes, crossed cubes, twisted cubes, locally twisted cubes, augmented cubes, star graphs, and others.

Edge and/or vertex failures are inevitable when a large parallel and distributed computer system is running. Thus, the fault-tolerant capacity of network is an important issue for parallel and distributed computing. A graph G is *k-fault-tolerant pancyclic(resp., vertex-pancyclic, edge-pancyclic)* if $G - F$ is still pancyclic(resp., vertex-pancyclic, edge-pancyclic) for any $F \subset E(G) \cup V(G)$ with $|F| \leqslant k$. Pancyclicity and fault-tolerant pancyclicity have been widely studied for many well-known networks, see Xu and Ma [14] for a detail survey on these topics.

The Möbius cube has many properties superior to the hypercube. Though both the Möbius cubes and the ordinary hypercube have the same number of vertices and the same vertex degree, the diameter of the Möbius cube is approximately half that of the ordinary hypercube. Due to nearly half the diameter and better graph embedding capability as compared with its hypercube counterpart of the same size, the Möbius cubes have been proposed as promising candidates for interconnection topology, and have received considerable attention [1, 2, 3, 6, 7, 8, 11, 13, 5, 4].

With regard to the fault-tolerant Hamiltonicity of Möbius cubes, Huang et al. [7] showed that an n-dimensional Möbius cube is Hamiltonian in the presence of up to $n-2$ node and edge faults. As concerns the pancyclicity and fault-tolerant pancyclicity of Möbius cubes, Fan [3] proved that Möbius cubes are four-pancyclic. Hsieh and Chen [6] found that an n-dimensional Möbius cube with up to $n-2$ edge faults is four-pancyclic.

This paper investigates the fault-tolerant edge-pancyclicity of MQ_n, and shows that if $MQ_n (n \geq 5)$ contains at most $n-2$ faulty vertices and/or edges then, for any fault-free edge uv in $MQ_n^i (i = 0, 1)$ and any integer ℓ with $7 - i \leqslant \ell \leqslant 2^n - f_v$, there is a fault-free cycle of length ℓ containing the edge uv, where f_v is the number of faulty vertices.

The remainder of this paper is organized as follows. In Section 2, we recall the definition of MQ_n, and introduce some properties of MQ_n to be used in our proofs. In Section 3, we give the proof of our result. Finally, we give some concluding remarks in Section 4.

2. Definitions and properties

The n-dimensional Möbius cube, denoted by MQ_n, is such an undirected graph, its vertex set is the same as the vertex set of Q_n, the vertex $X = x_1 x_2 \cdots x_n$ connects to n other vertices Y_i, $(1 \le i \le n)$, where each Y_i satisfies one of the following equations:

$$Y_i = \begin{cases} x_1x_2\cdots x_{i-1}\bar{x}_i x_{i+1}\cdots x_n & \text{when } x_{i-1}=0, \\ x_1x_2\cdots x_{i-1}\bar{x}_i\bar{x}_{i+1}\cdots \bar{x}_n & \text{when } x_{i-1}=1. \end{cases}$$

From the above definition, X connects to Y_i by complementing the bit x_i if $x_{i-1}=0$ or by complementing all bits of $x_i, \cdots, x_n$ if $x_{i-1}=1$. The connection between X and Y_1 is undefined, so we can assume that x_0 is either equal to 0 or equal to 1, which gives us slightly different network topologies. If we assume $x_0=0$, we call the network a ¡®0-Möbius cube¡¯; and if we assume $x_0=1$, we call the network a ¡®1-Möbius cube¡¯, denoted by MQ_n^0 and MQ_n^1, respectively. The graphs shown in Fig.1 are MQ_4^0 and MQ_4^1, respectivily.

According to the above definition, it is not difficult to see that MQ_n^0 (respectively, MQ_n^1) can be recursively constructed from MQ_{n-1}^0 and MQ_{n-1}^1 by adding 2^{n-1} edges. MQ_n^0 is constructed by connecting all pairs of vertices that differ only in the 1-th bit, and MQ_n^1 is constructed by connecting all pairs of vertices that differ in the 1-th through the n-th bits. The superscript i of notation MQ_n^i, $i=0,1$, can be omitted if there is no ambiguity arise.

For convenience, we say that MQ_{n-1}^0 and MQ_{n-1}^1 are two sub-Möbius cubes of MQ_n, where MQ_{n-1}^0 (respectively, MQ_{n-1}^1) is an $(n-1)$-dimensional 0-Möbius cube (respectively, 1-Möbius cube) which includes all vertices $0x_2\cdots x_n$ (respectively, $1x_2\cdots x_n$), $x_i \in \{0,1\}$. More simply, let $L = MQ_{n-1}^0$ and $R = MQ_{n-1}^1$. In addition, we define the set of crossing edges of MQ_n to be $E_C = \{xy \in E(MQ_n) | x \in V(MQ_{n-1}^0) \wedge y \in V(MQ_{n-1}^1)\}$. For any edge $xy \in E_C$, vertices x and y are crossing vertices of each other. Indeed, there are 2^{n-1} crossing edges and 2^{n-1} pairs of crossing vertices in MQ_n.

The Möbius cube MQ_n was first proposed by Cull and Larson [1]. Like Q_n, MQ_n is an n-regular n-connected graph with 2^n vertices and $n2^{n-1}$ edges. Moreover, MQ_n has a diameter of $\lceil \frac{(n+2)}{2} \rceil$ for MQ_{n-1}^0 and $\lceil \frac{(n+1)}{2} \rceil$ for MQ_{n-1}^1. However, for $n \ge 4$, MQ_n is neither vertex-transitive nor edge-transitive.

Cull and Larson[1] first proved the existence of hamiltonian cycles in MQ_n by proving that in MQ_n^0 or MQ_n^1, there are 2^{n-k} disjoint cycles of length 2^k for any $k \ge 2$.

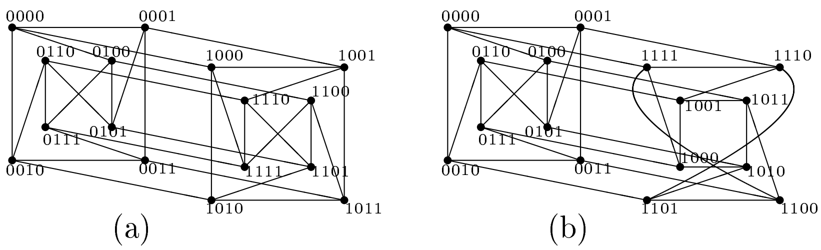

Fig. 1. (a) MQ_4^0, (b) MQ_4^1

We need the following two definitions.

Definition 2.1. For any edge $e = (x_1x_2\cdots x_n, y_1y_2\cdots y_n) \in E(MQ_n)$, let $\lambda(e)$ be the smallest positive integer $i \in \{1,2,\ldots,n\}$ such that $x_i \ne y_i$, then e is called a λ-dimensional edge.

According to Definition 2.1, if we use u_L to denote a vertex in L, then u_R always denotes its unique neighbor in R, that is, $u_L u_R$ is an 1-dimensional edge. Let $e = uv$ be a i_1-dimensional edge, then denote $v = u^{i_1}$. Let $u^{i_1,\ldots,i_{j-1},i_j} = (u^{i_1,\ldots,i_{j-1}})^{i_j}$ for $j \geq 2$. Let $P^{t-1}(u_0) = (u_0, u_1, \ldots u_j \ldots u_{t-1})$ be a path of length $\ell = t-1$, then $u_j = u_{j-1}^{i_j}$ for $1 \leq j \leq t-1$. We use $(i_1, i_2, \ldots, i_{t-1})$ to denote $P^{t-1}(u_0)$ for short. If $u_0 u_{t-1} \in E$, then we use $(i_1, i_2, \ldots, i_{t-1})$ to denote cycle $C^t(u_0)$ of length $\ell = t$ containing edge $u_0 u_{t-1}$ for short.

For example, for $C^7(000001) = (000001, 001001, 011001, 010110, 110110, 101001, 100001)$ in MQ_n^0, we use $(3, 2, 3, 1, 2, 3)$ to denote the cycle $C^7(000001)$.

Definition 2.2. If $|F|= n-2$ and there exists a vertex w with $N_{MQ_n - F}(w) = \{w_1, w_2\}$, then w is called a weak 2-degree vertex and $\{w_1, w_2\}$ is called a w-weak vertex pair(or a weak vertex pair, for short).

Since there is no triangle in MQ_n, we can obtain the Proposition 2.3 as follows.

Proposition 2.3. *If $xy \in E(MQ_n)$, then (x, y) is not a weak vertex-pair in $MQ_n - F$ with $|F|\leq n-2$.*

Lemma 2.4. *(Xu et al. [9]) If for any vertex $u_L \in L(u_R \in R)$, $v_L(v_R)$ is a neighbor of $u_L(u_R)$ in $L(R)$ then, the distance between $u_R(u_L)$ and $v_R(v_L)$ is 1 or 2.*

Lemma 2.5. *(Xu et al. [11]) MQ_n is edge-pancyclic for $n \geq 2$.*

Lemma 2.6. *(Xu et al. [10]) For any two different vertices x and y with distance d in $MQ_n (n \geq 3)$, there exists an xy-path of every length ℓ from d to $2^n - 1$ except for $d+1$.*

Let F be a set of faulty elements in MQ_n. We need the following lemmas:

Lemma 2.7. *If any edge $u_L u_R \in E_C$ in $MQ_n^0 - F$ for $|F|\leq n-2$, there exists a fault-free 4-cycle or 5-cycle containing the edge $u_L u_R$.*

Proof. Let $u_L = 0x_2 \ldots x_n$, we show the lemma according to the following two cases.

Case 1. $x_2 = 0$.

We can find $(n-2)$ disjoint 4-cycles and a 5-cycle except the common edge $u_L u_R$ as follows.

$$\begin{cases} (i+2, \quad 1, \quad i+2), & 1 \leq i \leq n-2, \\ (2, \quad 1, \quad 3, \quad 2), & i = n-1. \end{cases}$$

Since $|F|\leq n-2$, there exists a fault-free 4-cycle or 5-cycle containing the edge $u_L u_R$. The lemma holds.

Case 2. $x_2 = 1$.

There exist at most $n-2$ disjoint 4-cycles as $C_i^4(u_L) = (i, 1, i) = (u_L, u_L^i, u_L^{i,1}, u_R)$ for $3 \leq i \leq n$ except the common edge $u_L u_R$.

If one of 4-cycle $C_i^4(u_L) (3 \leq i \leq n)$ is fault-free, the lemma holds.

If each 4-cycles $C_i^4(u_L)(3 \leq i \leq n)$ contains a faulty vertices. Consider 4-cycles $(3,1,3) = (u_L, u_L^3, u_L^{3,1}, u_R)$.

(1) If $u_L^{3,1} \in F$, then $u_L^3 \notin F$. We can find a fault free 5-cycle $(3,2,1,2) = (u_L, u_L^3, u_L^{3,2}, u_L^{3,2,1}, u_R)$ containing the edge $u_L u_R$.

(2) If $u_L^3 \in F$, then $u_L^{3,1} \notin F$. We can find a fault-free 5-cycle $(2,1,2,3) = (u_L, u_L^2, u_L^{2,1}, u_L^{3,1}, u_R)$ containing the edge $u_L u_R$.

The proof is completed. □

Lemma 2.8. *If any edge $u_L u_R \in E_C$ in $MQ_n^1 - F$ for $|F| \leq n-2$, there exists a fault-free 5-cycle containing the edge $u_L u_R$.*

Proof. Let $u_L = x_1 x_2 \dots x_i x_{i+1} \dots x_n$. There exist $(n-2)$ disjoint 5-cycles and a 4-cycle except the common edge $u_L u_R$ as follows.

$$\begin{cases} (i+1, & i+2, & 1, & i+1), & 1 \leq i \leq n-2 \wedge x_{i+1} = 0, \\ (i+1, & 1, & i+2, & i+1), & 1 \leq i \leq n-2 \wedge x_{i+1} = 1, \\ (n, & 1, & n), & & i = n-1. \end{cases}$$

Since $|F| \leq n-2$, there exists a fault-free 4-cycle or 5-cycle containing the edge $u_L u_R$. □

By Lemma 2.7 and 2.8, we can obtain the following result.

Corollary 2.9. *If any edge $u_L u_R \in E_C$ in $MQ_n - F$ for $|F| \leq n-2$, there exists a fault-free 4-cycle or 5-cycle containing the edge $u_L u_R$.*

Lemma 2.10. *(Xu et al. [9]) If $|F| \leq n-3$ and $n \geq 3$ then, for any fault-free edge e in MQ_n and any integer ℓ with $6 \leq \ell \leq 2^n - f_v$, there is a fault-free cycle of length ℓ containing the edge e in MQ_n.*

Lemma 2.11. *(Xu et al. [12]) If $F \subset V(MQ_n) \cup E(MQ_n)$ and $|F| \leq n-2$, then for any two distinct fault-free vertices u and v, there exists a fault-free path P_{uv} of every length ℓ with $2^{n-1}-1 \leq \ell \leq 2^n - f_v - 1 - \alpha$, where $\alpha = 0$ if vertices u and v is not a weak vertex-pair in $MQ_n - F$ and $\alpha = 1$ if vertices u and v form a weak vertex-pair in $MQ_n - F (n \geq 5)$.*

Lemma 2.12. *If $F \subset V(MQ_n^1)$ with $|F| \leq n-2 (n \geq 6)$, then for any edge $u_L v_L \in L$, there exists a fault-free 6-cycle containing the edge $u_L v_L$ in $MQ_n^1 - F$.*

Proof. By Lemma 2.10, the lemma holds for $|F| \leq n-3$. We only need to consider the case of $|F| = n-2$.

Let $u_L = x_1 x_2 \dots x_j \dots x_n$. We can assume that $|F_R| \leq |F_L|$. Otherwise, if $|F_L| \leq |F_R|$, then $|F_L| \leq \lfloor \frac{n-2}{2} \rfloor \leq n-4 (n \geq 6)$. By Lemma 2.10, there exists a fault-free 6-cycle containing the edge $u_L v_L$ in $L - F_L$, and so in $MQ_n^1 - F$.

If $|F_L|\leq n-4$, by Lemma 2.10, there exists a fault-free 6-cycle containing the edge u_Lv_L.

If $|F_L|\geq n-3$, then $|F_R|\leq 1$. We can prove the result according to the relationship between the v_L and u_L as follows.

Case 1. $v_L = u_L^j(j=2)$.

There exist $n-1$ disjoint 6-cycles as $C_i^6(u_L)(1\leq i\leq n-1)$ except the common edge u_Lv_L as follows.

$$C_i^6(u_L) = \begin{cases} (1, & 3, & 1, & 2, & 3), & i=1, \\ (i+1, & 2, & 1, & 3, & 1), & i=2, \\ (i+1, & 5, & 2, & 5, & 4), & i=3, \\ (i+1, & n, & 2, & n, & i+1), & 4\leq i\leq n-2\wedge x_4=0, \\ (i+1, & 4, & 2, & 4, & i+1), & 4\leq i\leq n-2\wedge x_4=1, \\ (i+1, & 4, & 2, & 4, & n), & i=n-1. \end{cases}$$

Since $|F|\leq n-2$, there exists a fault-free 6-cycle containing the edge u_Lv_L.

Case 2. $v_L = u_L^j(3\leq j\leq n)$.

Case 2.1. $u_R, v_R\notin F$.

For $3\leq j\leq n-2$, there exist two disjoint u_Rv_R-paths as $P_i^3(u_R)$ of length $\ell=3$ in $R-F_R$ as follows.

$$P_i^3(u_R) = \begin{cases} (i+j-2, & j, & j-1), & i=1, \\ (i+j-2, & j+1, & j+2), & i=2\wedge x_{j-1}=x_{j+1}, \\ (i+j-2, & j+2, & j+1), & i=2\wedge x_{j-1}\neq x_{j+1}. \end{cases}$$

For $j\geq n-1$, there exist two disjoint u_Rv_R-paths as $P_i^3(u_R)$ of length $\ell=3$ in $R-F_R$ as follows.

$$P_i^3(u_R) = \begin{cases} (2, & j, & 2), & i=1, \\ (j-1, & j, & j-1), & i=2. \end{cases}$$

Since $|F_R|\leq 1$, there exists a fault-free u_Rv_R-path $P^3(u_R)$ of length $\ell=3$ in $R-F_R$. Then $C=u_Lu_R+P^3(u_R)+v_Rv_L+u_Lv_L$ is a fault-free 6-cycle containing the edge u_Lv_L.

Case 2.2. $|\{u_R,v_R\}\cap F|=1$. Without loss of generality, assume that $u_R\in F$. Let $F^{'}=F-\{u_R\}$, then $|F^{'}|=n-3$.

For $j=3$, there exist $n-2$ 6-cycles as $C_i^6(u_L)(1\leq i\leq n-2)$ containing the common edge u_Lv_L in $MQ_n^1-F^{'}$ as follows.

$$C_i^6(u_L) = \begin{cases} (i+1, & 1, & 3, & 1, & 2), & i=1, \\ (i+2, & 3, & 1, & 4, & 1), & i=2, \\ (i+2, & n, & 3, & n, & i+2), & 3\leq i\leq n-3\wedge x_2=0, \\ (i+2, & 3, & 1, & i+2, & 1), & 3\leq i\leq n-3\wedge x_2\neq 0, \\ (i+2, & 3, & 1, & n, & 1), & i=n-2. \end{cases}$$

For $4\leq j\leq n-2$, there exist $n-2$ 6-cycles as $C_i^6(u_L)(1\leq i\leq n-2)$ containing the common edge u_Lv_L in $MQ_n^1 - F'$ as follows:

$$C_i^6(u_L) = \begin{cases} (i+1, j, j+1, 2, j+1), i = 1, \\ (i+1, n, j, n, i+1), 2 \le i \le j-3 \wedge x_i = 0, \\ (i+1, j-1, j, j-1, i+1), 2 \le i \le j-3 \wedge x_i \ne 0, \\ (i+1, j+1, j+2, j, j-1), i = j-2 \wedge x_{j-2} = x_j \wedge x_{j-2} = x_{j+1}, \\ (i+1, j+2, j+1, j, j-1), i = j-2 \wedge x_{j-2} = x_j \wedge x_{j-2} \ne x_{j+1}, \\ (i+1, j, j+2, j+1, j-1), \\ i = j-2 \wedge x_{j-2} \ne x_j \wedge ((x_{j-1} = x_{j+1} \wedge x_{j-2} = 0) \vee (x_{j-1} \ne x_{j+1} \wedge x_{j-2} = 1)), \\ (i+1, j, j+1, j+2, j-1), \\ i = j-2 \wedge x_{j-2} \ne x_j \wedge ((x_{j-1} \ne x_{j+1} \vee x_{j-2} \ne 0) \wedge (x_{j-1} = x_{j+1} \vee x_{j-2} \ne 1)), \\ (i+2, j, 1, j+1, 1), i = j-1, \\ (i+2, 2, j, 2, i+2), j \le i \le n-2 \wedge x_{j-1} = 0, \\ (i+2, j, 1, i+2, 1), j \le i \le n-2 \wedge x_{j-1} \ne 0. \end{cases}$$

For $j = n-1$, there exist $n-2$ 6-cycles as $C_i^6(u_L)(1 \le i \le n-2)$ containing the common edge u_Lv_L in $MQ_n^1 - F'$ as follows.

$$C_i^6(u_L) = \begin{cases} (i+1, & n, & j, & n, & i+1), & i \le n-4 \wedge x_i = 0, \\ (i+1, & j-1, & j, & j-1, & i+1), & i \le n-4 \wedge x_i \ne 0, \\ (i+1, & n, & 1, & n-2, & 1), & i = n-3 \wedge x_{n-2} = 0, \\ (i+1, & 1, & n-1, & 1, & n-2), & i = n-3 \wedge x_{n-2} \ne 0, \\ (i+2, & n-1, & 1, & n, & 1), & i = n-2. \end{cases}$$

For $j = n$, there exist $n-2$ 6-cycles as $C_i^6(u_L) = (i+1, i, n, i, i+1)(1 \le i \le n-2)$ containing the common edge u_Lv_L in $MQ_n^1 - F'$.

Note that $n-2$ 6-cycles are disjoint in L and contain no u_R in $MQ_n^1 - F'$. Since $|F'| = n-3$, we have that there exists a fault-free 6-cycles in $MQ_n^1 - F'$, and so in $MQ_n^1 - F$.

Take edge $u_Lv_L = (001011, 001100)$ in MQ_6^1 for an example. We have $u_R = 110100$, $v_R = 110011$. Assume $110100 \in F$, then $110011 \notin F$. There exist four disjoint 6-cycles in L except a common edge $u_Lv_L = (001011, 001100)$ as follows(see Figure 2).

$$\begin{cases} (001011, & 011011, & 011100, & 011111, & 001111, & 001100), \\ (001011, & 000011, & 000010, & 000000, & 000100, & 001100), \\ (001011, & 001001, & 001110, & 110001, & 110011, & 001100), \\ (001011, & 001010, & 001101, & 110010, & 110011, & 001100). \end{cases}$$

□

Lemma 2.13. *If $F \subset R$ with $|F| = n-2 (n \ge 6)$, then for any edge $u_Rv_R \in R$, there exists a fault-free 6-cycle containing the edge u_Rv_R in $MQ_n^1 - F$.*

Proof. Let u_L, v_L be the adjacent vertices of u_R, v_R respectively in L and $u_R = x_1x_2 \ldots x_i \ldots x_n$.

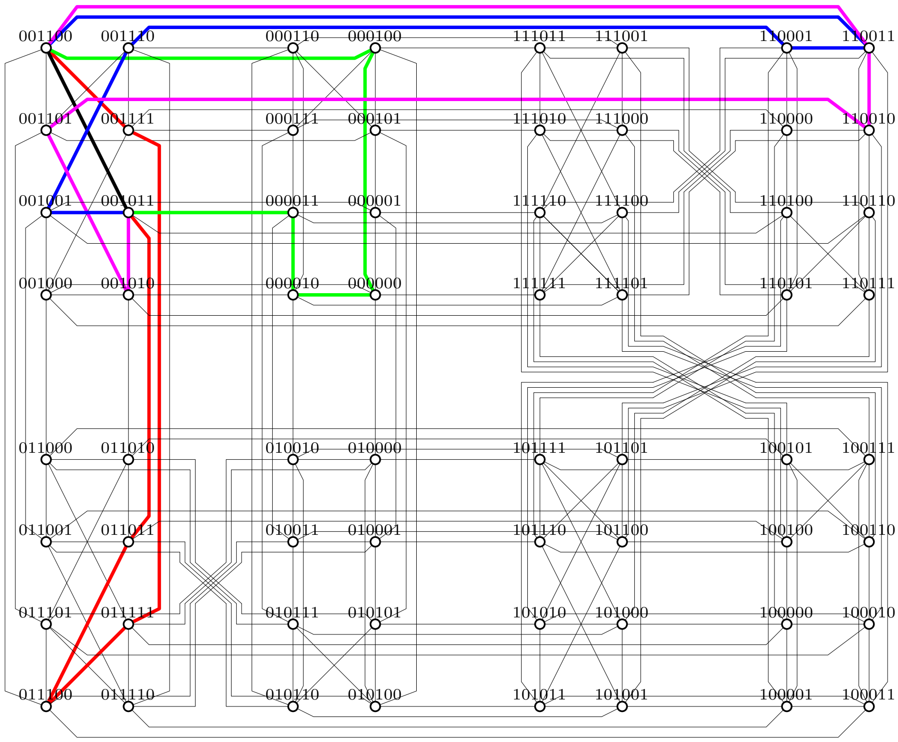

Fig. 2. Four disjoint 6-cycles in L except the common edge $(001011, 001100)$

Since $F \subset R$, $F_L = 0$. There exists a fault-free $u_L v_L$-path P of length $\ell = 3$ in L as follows.

$$P = \begin{cases} (2, & 4, & 3), & v_R = u_R^2 \wedge x_3 = 0, \\ (2, & 3, & 4), & v_R = u_R^2 \wedge x_3 \neq 0, \\ (j-1, & j, & j-1), & v_R = u_R^j (3 \leq j \leq n-1), \\ (2, & n, & 2), & v_R = u_R^n. \end{cases}$$

Then $C = v_R v_L + P + u_L u_R + u_R v_R$ is a fault-free 6-cycle containing the edge $u_R v_R$ in $MQ_n^1 - F$. □

Lemma 2.14. *For any edge $u_L u_R \in E_C$ in $MQ_n^i - F(i = 0, 1)(n \geq 6)$ with $|F| \leq n-2$, there exists a fault-free cycle of length $\ell = 7 - i, 7, 8$ containing the edge $u_L u_R$ in $MQ_n^i - F(i = 0, 1)$.*

Proof. Let $u_L = x_1 x_2 x_3 \ldots x_i \ldots x_n$. We prove the lemma according to edge $u_L u_R$ in MQ_n^0 or MQ_n^1 as follows.

Case 1. $u_L u_R \in E(MQ_n^0)$.

Case 1.1. $\ell = 7$.

For $1 \leq i \leq n-1$, we can find $n-1$ disjoint 7-cycles as $C_i^7(u_L)$ except the common edge u_Lu_R as follows.

$$C_1^7(u_L)=\begin{cases}(2,\ 3,\ n-1,\ 1,\ n-1,\ 2), & x_3=0\wedge x_2=0,\\ (2,\ 3,\ 2,\ 1,\ 4,\ 3), & x_3=0\wedge x_2\neq 0,\\ (2,\ 1,\ 2,\ 5,\ 4,\ 3), & x_3\neq 0\wedge x_2=0\wedge x_4=0,\\ (2,\ 1,\ 5,\ 2,\ 4,\ 3), & x_3\neq 0\wedge x_2=0\wedge x_4\neq 0,\\ (2,\ n,\ 1,\ n,\ 2,\ 3), & x_3\neq 0\wedge x_2\neq 0,\end{cases}$$

$$C_2^7(u_L)=\begin{cases}(3,\ 2,\ 3,\ 1,\ 2,\ 3), & x_3=0\wedge x_2=0,\\ (3,\ 2,\ 3,\ 1,\ 3,\ 2), & x_3=0\wedge x_2\neq 0,\\ (3,\ 4,\ 2,\ 1,\ 5,\ 2), & x_3\neq 0\wedge x_2=0\wedge x_4=0,\\ (3,\ 5,\ 2,\ 1,\ 4,\ 2), & x_3\neq 0\wedge x_2=0\wedge x_4\neq 0,\\ (3,\ 4,\ 2,\ 1,\ 4,\ 2), & x_3\neq 0\wedge x_2\neq 0\wedge x_4=0,\\ (3,\ 5,\ 2,\ 1,\ 5,\ 2), & x_3\neq 0\wedge x_2\neq 0\wedge x_4\neq 0,\end{cases}$$

$$C_i^7(u_L)=\begin{cases}(i+1,\ 2,\ 3,\ 1,\ 2,\ i+1), & 3\le i\le n-1\wedge x_2=0,\\ (i+1,\ 3,\ 2,\ 1,\ 2,\ i+1), & 3\le i\le n-1\wedge x_2\neq 0.\end{cases}$$

Since $|F|\le n-2$, there exists a fault-free 7-cycle containing the edge u_Lu_R.

Case 1.2. $\ell=8$.

For $1\le i\le n-1$, we can find $n-1$ disjoint 8-cycles as $C_i^8(u_L)$ except the common edge u_Lu_R as follows.

$$C_1^8(u_L)=\begin{cases}(2,\ 1,\ n-1,\ 3,\ n-1,\ 2), & x_2=0,\\ (2,\ 3,\ 2,\ 1,\ 2,\ 3), & x_2\neq 0,\end{cases}$$

$$C_i^8(u_L)=(i+1,\ 2,\ n,\ 2,\ 1,\ n),\ 2\le i\le n-2,$$

$$C_{n-1}^8(u_L)=\begin{cases}(n,\ 2,\ 1,\ n-1,\ 3,\ n-1), & x_2=0,\\ (n,\ 2,\ 1,\ 3,\ 4,\ 2), & x_2\neq 0\wedge x_3=0,\\ (n,\ 2,\ 1,\ 4,\ 3,\ 2), & x_2\neq 0\wedge x_3\neq 0.\end{cases}$$

Since $|F|\le n-2$, there exists a fault-free 8-cycle containing the edge u_Lu_R.

Take edge $(010010, 110010)$ in MQ_6^0 for an example, there exist 5 disjoint 8-cycles except the common edge $(010010, 110010)$ as follows(see Figure 3).

$$\begin{cases}(010010, 000010, 001010, 011010, 111010, 100101, 101101, 110010),\\ (010010, 011101, 001101, 001100, 011100, 111100, 111101, 110010),\\ (010010, 010110, 000110, 000111, 010111, 110111, 110110, 110010),\\ (010010, 010000, 000000, 000001, 010001, 110001, 110000, 110010),\\ (010010, 010011, 000011, 100011, 101011, 101100, 110011, 110010).\end{cases}$$

Case 2. $u_Lu_R\in E(MQ_n^1)$.

Case 2.1. $\ell=6$.

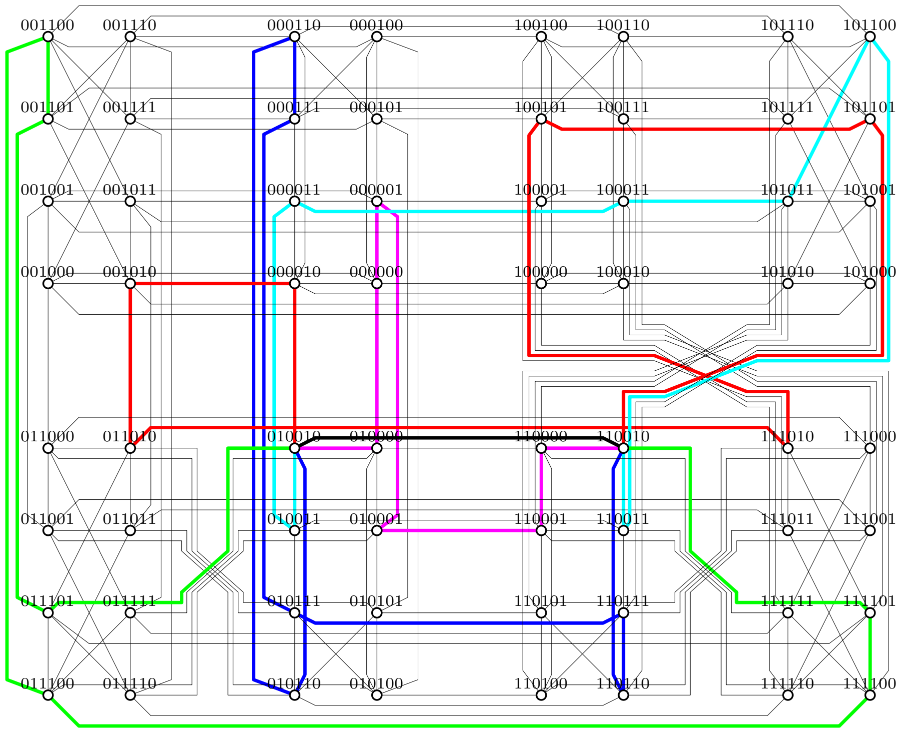

Fig. 3. 5 disjoint 8-cycles except the common edge $(010010, 110010)$ in MQ_6^0

For $1 \le i \le n-1$, we can find $n-1$ disjoint 6-cycles as $C_i^6(u_L)$ except the common edge $u_L u_R$ as follows.

$$C_i^6(u_L) = \begin{cases} (i+1, 1, i+3, i+2, i+1), 1 \le i \le n-4 \wedge x_i = x_{i+2} \wedge x_{i+1} = 0, \\ (i+1, i+2, i+3, 1, i+1), 1 \le i \le n-4 \wedge x_i = x_{i+2} \wedge x_{i+1} \ne 0, \\ (i+1, i+3, 1, i+2, i+1), 1 \le i \le n-4 \wedge x_i \ne x_{i+2} \wedge x_{i+1} = 0, \\ (i+1, i+2, 1, i+3, i+1), 1 \le i \le n-4 \wedge x_i \ne x_{i+2} \wedge x_{i+1} \ne 0, \end{cases}$$

$$C_{n-3}^6(u_L) = \begin{cases} (n-2, \ n-1, \ n-2, \ 1, \ n-1), & x_{n-2} = 0, \\ (n-2, \ n-1, \ 1, \ n-2, \ n), & x_{n-2} \ne 0, \end{cases}$$

$$C_{n-2}^6(u_L) = \begin{cases} (n-1, \ n-2, \ 1, \ n-2, \ n), & x_{n-2} = 0, \\ (n-1, \ 1, \ n-2, \ n-1, \ n-2), & x_{n-2} \ne 0, \end{cases}$$

$$C_{n-1}^6(u_L) = \begin{cases} (n, \ n-2, \ 1, \ n-1, \ n-2), & x_{n-2} = 0, \\ (n, \ n-2, \ 1, \ n-2, \ n-1), & x_{n-2} \ne 0. \end{cases}$$

Since $|F| \le n-2$, there exists a fault-free 6-cycle containing the edge $u_L u_R$.

Case 2.2. $\ell = 7$.

For $1 \le i \le n-1$, we can find $n-1$ disjoint 7-cycles as $C_i^7(u_L)$ except the common edge $u_L u_R$ as follows.

$$C_i^7(u_L) = \begin{cases} (i+1, i+3, i+2, 1, i+3, i+1), & 1 \le i \le n-4 \wedge x_{i+1} = 0, \\ (i+1, i+3, 1, i+2, i+3, i+1), & 1 \le i \le n-4 \wedge x_{i+1} \ne 0, \end{cases}$$

$$C_{n-3}^7(u_L) = \begin{cases} (n-2, \ 1, \ n, \ n-2, \ n, \ n-1), & x_{n-2} = 0, \\ (n-2, \ 1, \ n, \ n-1, \ n-2, \ n), & x_{n-2} \ne 0, \end{cases}$$

$$C^7_{n-2}(u_L) = \begin{cases} (n-1, n-2, n-3, 1, n-3, n), x_{n-2} = 0 \wedge x_{n-3} = 0, \\ (n-1, n, n-3, 1, n-3, n-2), x_{n-2} = 0 \wedge x_{n-3} \neq 0 \wedge x_{n-4} = 0, \\ (n-1, n-3, n-2, 1, n-3, n), x_{n-2} = 0 \wedge x_{n-3} \neq 0 \wedge x_{n-4} \neq 0, \\ (n-1, 2, n-2, 2, 1, n-2), x_{n-2} \neq 0, \end{cases}$$

$$C^7_{n-1}(u_L) = \begin{cases} (n, n-3, 1, n-3, n, n-2), x_{n-2} = 0 \wedge x_{n-3} = 0, \\ (n, n-2, n-3, 1, n-3, n), x_{n-2} = 0 \wedge x_{n-3} \neq 0 \wedge x_{n-4} = 0, \\ (n, n-2, n-1, 1, n, n-2), x_{n-2} = 0 \wedge x_{n-3} \neq 0 \wedge x_{n-4} \neq 0, \\ (n, 2, n-1, 2, 1, n-1), x_{n-2} \neq 0. \end{cases}$$

Since $|F| \leq n-2$, there exists a fault-free 7-cycle containing the edge $u_L u_R$.

Case 2.3. $\ell = 8$.

For $1 \leq i \leq n-1$, we can find $n-1$ disjoint 8-cycles as $C^8_i(u_L)$ except the common edge $u_L u_R$ as follows.

$$C^8_i(u_L) = \begin{cases} (i+1, n, 1, n, i+3, i+2), 1 \leq i \leq n-4 \wedge x_i = x_{i+2} \wedge x_{i+1} = 0, \\ (i+1, i+2, i+3, n, 1, n), 1 \leq i \leq n-4 \wedge x_i = x_{i+2} \wedge x_{i+1} \neq 0, \\ (i+1, i+3, n, 1, n, i+2), 1 \leq i \leq n-4 \wedge x_i \neq x_{i+2} \wedge x_{i+1} = 0, \\ (i+1, i+2, n, 1, n, i+3), 1 \leq i \leq n-4 \wedge x_i \neq x_{i+2} \wedge x_{i+1} \neq 0, \end{cases}$$

$$C^8_{n-3}(u_L) = \begin{cases} (n-2, \ 1, \ n-1, \ 1, \ n, \ 1), \ x_{n-2} = 0, \\ (n-2, \ 1, \ n, \ 1, \ n-1, \ 1), \ x_{n-2} \neq 0, \end{cases}$$

$$C^8_{n-2}(u_L) = \begin{cases} (n-1, \ 2, \ n, \ 3, \ 1, \ 2), \ x_2 = 0, \\ (n-1, \ 2, \ n, \ 1, \ 3, \ 2), \ x_2 \neq 0, \end{cases}$$

$$C^8_{n-1}(u_L) = \begin{cases} (n, n-2, 1, 2, n-2, 2), x_2 = 0 \wedge x_{n-2} = 0, \\ (n, n-1, n-2, n-3, 1, n-3), x_2 = 0 \wedge x_{n-2} \neq 0 \wedge x_{n-3} = 0, \\ (n, n-1, n-3, 1, n-3, n-2), x_2 = 0 \wedge x_{n-2} \neq 0 \wedge x_{n-3} \neq 0, \\ (n, 2, 3, 2, 1, 3), x_2 \neq 0 \wedge x_3 = 0, \\ (n, 2, 4, 3, 1, 2), x_2 \neq 0 \wedge x_3 \neq 0. \end{cases}$$

Since $|F| \leq n-2$, there exists a fault-free 8-cycle containing the edge $u_L u_R$.

Take edge $(010010, 101101)$ in MQ^1_6 for an example, there exist 5 disjoint 8-cycles except the common edge $(010010, 101101)$ as follows (see Figure 4):

$$\begin{cases} (010010, 000010, 001010, 001101, 001100, 110011, 110010, 101101), \\ (010010, 011101, 011110, 011111, 100000, 100001, 100101, 101101), \\ (010010, 010110, 101001, 101011, 010100, 010101, 101010, 101101), \\ (010010, 010000, 000000, 000001, 111110, 110001, 101110, 101101), \\ (010010, 010011, 000011, 001011, 011011, 100100, 101100, 101101). \end{cases}$$

□

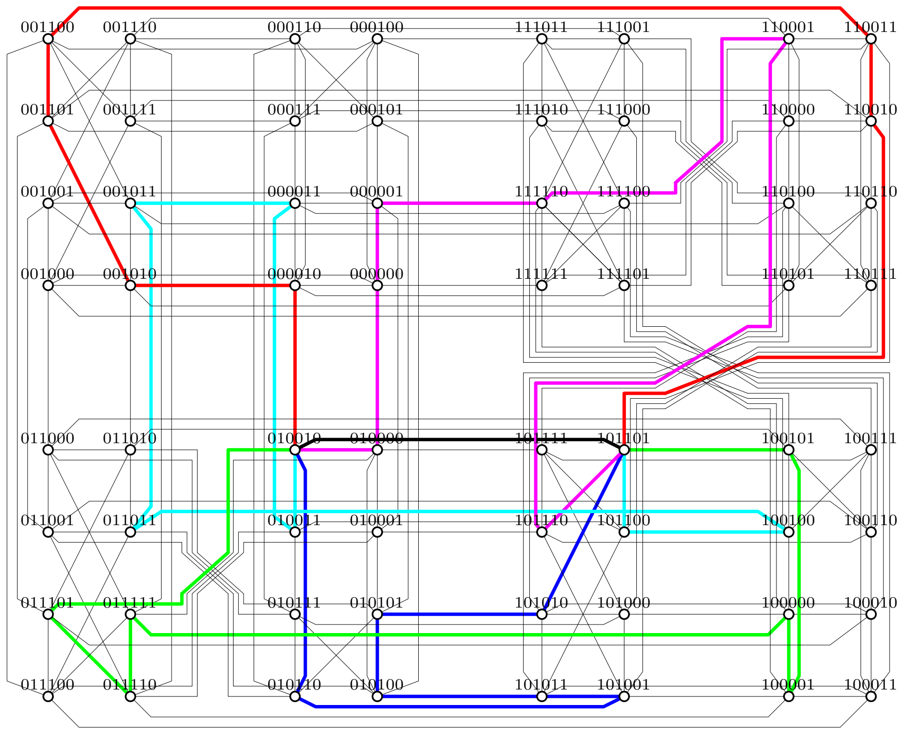

Fig. 4. 5 disjoint 8-cycles except the common edge $(010010, 101101)$ in MQ_6^1

3. Fault-tolerant edge-pancyclicity of MQ_n

In this section, we investigate the fault-tolerant edge-Pancyclicity of MQ_n and show that MQ_n is $(n-2)$-fault-tolerant edge-Pancyclic.

Let F be a set of faulty elements in MQ_n, $F_L = F \cap L$, $F_R = F \cap R$, $F_C = F \cap E_C$, $F^v = F \cap V(MQ_n)$, $F^e = F \cap E(MQ_n)$ $F_L^v = F_L \cap V(L)$ and $F_R^v = F_R \cap V(R)$, $f_v = |F^v|$, $f_e = |F^e|$.

Theorem 3.1. *If $f_v + f_e \le n-2$ and $n \ge 5$ then, for any fault-free edge e in $MQ_n^i (i = 0, 1)$ and any integer ℓ with $7 - i \le \ell \le 2^n - f_v$, there is a fault-free cycle of length ℓ containing the edge e.*

In this section, we will give the proof of Theorem 3.1. The theorem follows from Lemma 2.10 if $|F| \leqslant n-3$. We only need to consider $|F| = n-2$. Start with the following lemma.

Lemma 3.2. *If Theorem 3.1 holds for any subset $F \subset V(MQ_n)$ with $|F| = n-2$, then Theorem 3.1 also holds for any subset $F' \subset V(MQ_n) \cup E(MQ_n)$ with $|F'| = n-2$.*

Proof. The lemma holds for $f_e = 0$ by hypothesis of Lemma 3.2. Assume that the lemma holds for any t with $0 < t \le n-3$ and $f_e = t$. We prove the lemma holds for $f_e = t+1$. Let xy be an edge in F.

Case 1. $x \in F$ or $y \in F$.

Without loss of generality, assume $x \in F$. Let $F' = F - xy$, then $|F'| = n-3$. By Lemma 2.10, this result has been proved.

Case 2. $x, y \notin F$.

Let $F' = F + \{x\} - \{xy\}$, then $|F'| = n-2$, and F' contains at most t edges and $f_v + 1$ vertices. By induction hypothesis, for every integer ℓ with $7 - i \le \ell \le 2^n - f_v - 1$, there is a fault-free ℓ-cycle containing the edge e in $MQ_n^i - F'(i = 0, 1)$. By Proposition 2.1, for any $e = uv \in E(MQ_n - F)$, (u, v) is not a weak vertex-pair in $MQ_n - F$. For $\ell = 2^n - f_v$, by Lemma 2.11, there is a fault-free uv-hamiltonian path in $MQ_n - F$, *i.e.*, there is a fault-free ℓ-cycle containing the edge e in $MQ_n - F$.

The proof of lemma is completed. □

Proof of Theorem 3.1. By Lemma 3.2, we only need to prove the theorem with $F \subset V(MQ_n)$. The proof proceeds by induction on $n \geqslant 5$. The result holds for $n = 5$ by developing computer program using depth first searching technique combining with backtracking and branch and bound algorithm.

Assume that the theorem holds for any k with $6 \le k < n$. Then we must show the theorem holds for n.

Let $e = uv$ be a fault-free edge in MQ_n. By Proposition 2.3, (u, v) is not a weak vertex-pair in $MQ_n - F$. Let $2^{n-1} \le \ell \le 2^n - f_v$ and $\ell = \ell' + 1$, where $2^{n-1} - 1 \le \ell' \le 2^n - f_v - 1$ then, by Lemma 2.11, there exists a fault-free uv-path P of length ℓ' in MQ_n. Then $P + uv$ is a fault-free cycle of length ℓ containing the edge e in MQ_n. Thus, we only need to consider ℓ with $7 - i \le \ell \le 2^{n-1} - 1$ in $MQ_n^i (i = 0, 1)(n \ge 6)$.

Case 1. $|F_R| \le |F_L|$.

Case 1.1. $e \in E(L - F_L)$. Let $e = u_L v_L$.

By Lemma 2.12, there exists a fault-free 6-cycle containing the edge e in $MQ_n^1 - F$. Thus, we only need to consider the length of $7 \le \ell \le 2^{n-1} - 1$.

Case 1.1.1. $|F_L| \le n - 3$. Then $|F_R| \le n - 4$.

Case 1.1.1.1. $7 \le \ell \le 2^{n-1} - |F_L^v|$.

By induction hypothesis, there is a fault-free cycle of length ℓ containing the edge e in L, so in MQ_n.

Case 1.1.1.2. $2^{n-1} - |F_L^v| + 1 \le \ell \le 2^{n-1} - 1$.

Write $\ell = \ell_1 + 1 + \ell_2$, where $2^{n-2} - |F_L^v| + 1 \le \ell_1 \le 2^{n-2} - 1$ and $\ell_2 = 2^{n-2} - 1$. Since $2^{n-2} - |F_L^v| + 1 \ge 2^{n-2} - |F_L| + 1 \ge 2^{n-2} - n + 4 > 7$ for $n \ge 6$ and $|F_L| \le n - 3$, by induction hypothesis, there is a fault-free cycle C of length ℓ_1 containing the edge $u_L v_L$ in L. Note that a cycle of length ℓ_1 contains a matching M with $|M| = \lfloor \frac{\ell_1}{2} \rfloor$. Consider the following inequality.

$$\begin{aligned} \lfloor \tfrac{\ell_1}{2} \rfloor - |F_C| - |F_R| - |\{e\}| \ &\ge \left\lfloor \frac{2^{n-2} - |F_L^v| + 1}{2} \right\rfloor - |F_C| - |F_R| - 1 \\ &\ge 2^{n-3} - |F| - 1 \ \ge 2^{n-3} - n + 1. \end{aligned}$$

Let $f(x) = 2^{x-3} - x + 1$. Since $f'(x) = 2^{x-3} \ln 2 - 1 \ge 0$ for $x \ge 6$, $f(x)$ is an increasing function, which implies that $\lfloor \frac{\ell_1}{2} \rfloor - |F_C| - |F_R| \ge f(6) = 2^{6-3} - 6 + 1 = 3$. It follows that, there is such an edge, say $x_L y_L$, in M that $x_L y_L \ne u_L v_L$ and $|\{x_R, y_R, x_L x_R, y_L y_R\} \cap F| = 0$. By Lemma 2.11, there is a fault-free $x_R y_R$-path P of length ℓ_2 in R. Then $C - x_L y_L + y_L y_R + P + x_R x_L$ is a fault-free cycle of length $\ell \, (= \ell_1 + 1 + \ell_2)$ containing e (see Figure 5 (a)).

Case 1.1.2. $|F_L|= n-2$. In this case, $|F_R|= 0$.

Let u_R and v_R be neighbors of u_L and v_L in R, respectively.

Let $\ell = \ell' + 3$, where $4 \leq \ell' \leq 2^{n-1} - 4$. By Lemma 2.4, $d_{u_R v_R} \leq 2$. Since $|F_R|= |F_C|= 0$, by Lemma 2.6, there is a fault-free $u_R v_R$-path P of length ℓ' in R, and so $P + v_R v_L + v_L u_L + u_L u_R$ is a fault-free cycle of length ℓ containing the edge $u_L v_L$ (see Figure 5 (b)).

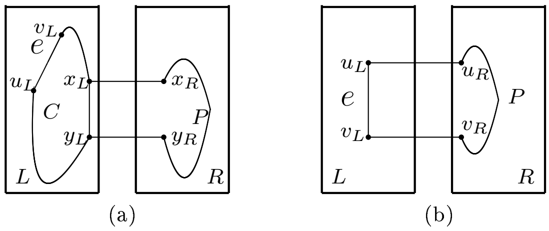

Fig. 5. The illustrations of Case 1.1

Case 1.2. $e \in E(R - F_R)$. Let $e = u_R v_R$.

Case 1.2.1. $|F_L|\leq n-3$. Then $|F_R|\leq n-4$.

Case 1.2.1.1. $7 - i \leq \ell \leq 2^{n-1} - |F_R^v|$ in $MQ_n^i (i = 0, 1)$.

By induction hypothesis, there is a fault-free cycle of length ℓ containing the edge e in R, so in $MQ_n^i (i = 0, 1)$.

Case 1.2.1.2. $2^{n-1} - |F_R^v|+1 \leq \ell \leq 2^{n-1} - 1$.

The proof is similar to Case 1.1.1.2.

Case 1.2.2. $|F_L|= n-2$. In this case, $|F_R|= 0$.

Since $|F_R|= 0$, by Lemma 2.5, there exists a fault-free cycle of length ℓ with $7 - i \leq \ell \leq 2^{n-1} - 1$ containing the edge e in R, and so in $MQ_n^i - F (i = 0, 1)$.

Case 1.3. $e \in (E_C - F_C)$. Let $e = u_L u_R$.

By Lemma 2.14, there exists a fault-free cycle of length $\ell = 7 - i, 7, 8$ containing the edge $u_L u_R$ in $MQ_n^i - F (i = 0, 1)$. Thus, we only need to consider the length $9 \leq \ell \leq 2^{n-1} - 1$.

Case 1.3.1. $|F_L|\leq n-3$. Then $|F_R|\leq n-4$.

By Corollary 2.1., there exists a fault-free 4-cycle C (or 5-cycle) containing the edge $u_L u_R$.

Case 1.3.1.1. $C = C^4 = (u_L, s_L, s_R, u_R)$.

For $9 \leq \ell \leq 2^{n-1} - |F_R^v|-1$. Let $\ell = \ell' + 2$, where $7 \leq \ell' \leq 2^{n-1} - |F_R^v|-3$. By induction hypothesis, there is a fault-free cycle C' of length ℓ' containing the edge $u_R s_R$ in R. Then $C = C' - u_R s_R + s_R s_L + s_L u_L + u_L u_R$ is a fault-free cycle of length ℓ containing the edge $e = u_L u_R$.

For $2^{n-1} - |F_R^v|\leq \ell \leq 2^{n-1} - 1$. Let $\ell = \ell_1 + \ell_2$, where $\ell_1 = 2^{n-2}$ and $2^{n-2} - |F_R^v|\leq \ell_2 \leq 2^{n-2} - 1$. Since $2^{n-2} > 7$ and $2^{n-2} - |F_R^v|> 7$ for $n \geq 6$, by induction hypothesis, there exists a fault-free cycle C_1 of length ℓ_1 containing the edge $u_L s_L$ in L and a fault-free cycle C_2 of length ℓ_2 containing the edge $u_R s_R$ in R. Then $C = C_1 - u_L s_L + s_L s_R + C_2 - s_R u_R + u_R u_L$ is a fault-free cycle of length ℓ containing the edge $e = u_L u_R$ in MQ_n (see Figure 6(a)).

Case 1.3.1.2. $C = C^5 = (u_L, w_L, w_R, x, u_R)$.

Since $|F_R|\leq n-4$. Let $F_R^1 = F_R - \{w_R\}$, then $|F_R^1|= |F_R|-1 \leq n-3$.

For $9 \leq \ell \leq 2^{n-1} - |F_R^v|-1$. Let $\ell = \ell' + 3$, where $6 \leq \ell' \leq 2^{n-1} - |F_R^v|-4$. By induction hypothesis, there is a fault-free cycle C' of length ℓ' containing the edge u_Rx in $R-F^1_R$. Then $C = C' - u_Rx + xw_R + w_Rw_L + w_Lu_L + u_Lu_R$ is a fault-free cycle of length ℓ containing the edge $e = u_Lu_R$.

For $2^{n-1} - |F^v_R| \leq \ell \leq 2^{n-1} - 1$. Let $\ell = \ell_1 + \ell_2 + 2$, where $\ell_1 = 2^{n-2} - 1$ and $2^{n-2} - |F^v_R| - 1 \leq \ell_2 \leq 2^{n-2} - 2$. Since $2^{n-2} - |F^v_R| - 1 > 7$ for $n \geq 6$, by induction hypothesis, there exists a fault-free cycle C_1 of length ℓ_2 containing the edge xu_R in $R - F^1_R$ and, by Lemma 2.11, there exists a fault-free u_Lw_L-path P of length ℓ_1 in L. Then $C = C_1 - u_Rx + xw_R + w_Rw_L + P + u_Lu_R$ is a fault-free cycle of length ℓ containing the edge $e = u_Lu_R$ in MQ_n (see Figure 6(b)).

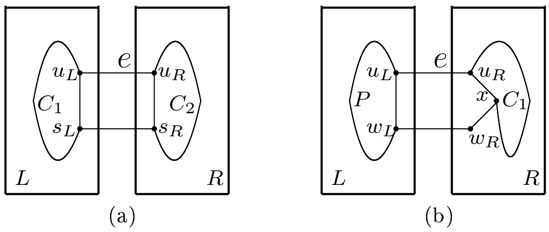

Fig. 6. The illustration of Case 1.3.1.1. and Case 1.3.1.2.

Case 1.3.1.3. $C = C^5 = (u_L, x, w_L, w_R, u_R)$.

For $9 \leq \ell \leq 2^{n-1} - |F^v_R| - 1$. Let $\ell = \ell' + 3$, where $6 \leq \ell' \leq 2^{n-1} - |F^v_R| - 4$. By induction hypothesis, there is a fault-free cycle C' of length ℓ' containing the edge u_Rw_R in R. Then $C = C' - u_Rw_R + w_Rw_L + w_Lx + xu_L + u_Lu_R$ is a fault-free cycle of length ℓ containing the edge $e = u_Lu_R$.

For $2^{n-1} - |F^v_R| \leq \ell \leq 2^{n-1} - 1$. Let $\ell = \ell_1 + \ell_2 + 1$, where $\ell_1 = 2^{n-2}$ and $2^{n-2} - |F^v_R| - 1 \leq \ell_2 \leq 2^{n-2} - 2$. Since $2^{n-2} - |F^v_R| - 1 > 7$ for $n \geq 6$, by induction hypothesis, there exists a fault-free cycle C_1 of length ℓ_2 containing the edge u_Rw_R in R and, by Lemma 2.11, there exists a fault-free u_Lw_L-path P of length ℓ_1 in L. Then $C = C_1 - u_Rw_R + w_Rw_L + P + u_Lu_R$ is a fault-free cycle of length ℓ containing the edge u_Lu_R in MQ_n (see Figure 7(a)).

Case 1.3.2. $|F_L| = n-2$. In this case, $|F_R| = 0$.

Since $|F_L| = n-2$, there exists a fault-free neighbor v_L of u_L in L. Let v_R be neighbors of v_L in R.

Let $\ell = \ell' + 3$, where $6 \leq \ell' \leq 2^{n-1} - 4$. By Lemma 2.4, $d_{u_Rv_R} \leq 2$. By Lemma 2.6, there is a fault-free u_Rv_R-path P of length ℓ' in R. So $P + u_Ru_L + u_Lv_L + v_Lv_R$ is a fault-free cycle of length ℓ containing the edge u_Lu_R.

Case 2. $|F_L| \leq |F_R|$.

Case 2.1. $e \in E(R - F_R)$. Let $e = u_Rv_R$.

Case 2.1.1. $|F_R| \leq n-3$. Then $|F_L| \leq n-4$.

For $7 - i \leq \ell \leq 2^{n-1} - |F^v_R|$ in $MQ^i_n (i = 0, 1)$. By induction hypothesis, there is a fault-free cycle of length ℓ containing the edge e in R, so in $MQ^i_n (i = 0, 1)$.

For $2^{n-1} - |F^v_R| + 1 \leq \ell \leq 2^{n-1} - 1$. The proof is similar to Case 1.1.1.2.

Case 2.1.2. $|F_R| = n-2$. In this case, $|F_L| = 0$.

By Lemma 2.13, there exists a fault-free 6-cycle containing the edge e in $MQ^1_n - F$. Thus, we only need to consider the length of $7 \leq \ell \leq 2^{n-1} - 1$.

Let u_L and v_L be neighbors of u_R and v_R in L, respectively.

Let $\ell = \ell' + 3$, where $4 \leq \ell' \leq 2^{n-1} - 4$. By Lemma 2.4, $d_{u_L v_L} \leq 2$. Since $|F_L|= |F_C|= 0$, by Lemma 2.6, there is a fault-free $u_L v_L$-path P of length ℓ' in L, and so $P + v_L v_R + v_R u_R + u_R u_L$ is a fault-free cycle of length ℓ containing the edge $u_R v_R$.

Case 2.2. $e \in E(L - F_L)$. Let $e = u_L v_L$.

Case 2.2.1. $|F_R|\leq n-3$. Then $|F_L|\leq n-4$.

For $7 - i \leq \ell \leq 2^{n-1} - |F_L^v|$ in $MQ_n^i - F(i = 0, 1)$. Since $|F_L|\leq n-4$, by Lemma 2.10, there exists a fault-free cycle of length ℓ containing the edge e in $MQ_n^i - F(i = 0, 1)$.

For $2^{n-1} - |F_L^v|+1 \leq \ell \leq 2^{n-1} - 1$. The proof is similar to Case 1.1.1.2.

Case 2.2.2. $|F_R|= n-2$. In this case, $|F_L|= 0$.

Since $|F_L|= 0$, by Lemma 2.5, there exists a fault-free cycle of length ℓ with $7 - i \leq \ell \leq 2^{n-1} - 1$ containing the edge e in L, and so in $MQ_n^i(i = 0, 1)$.

Case 2.3. $e \in (E_C - F_C)$. Let $e = u_L u_R$.

By Lemma 2.14, there exists a fault-free cycle of length $\ell = 7-i, 7, 8$ containing the edge $u_L u_R$ in $MQ_n^i - F(i = 0, 1)$. Thus, we only need to consider the length $9 \leq \ell \leq 2^{n-1} - 1$.

Case 2.3.1. $|F_R|\leq n-3$. Then $|F_L|\leq n-4$.

By Corollary 2.1., there exists a fault-free 4-cycle C (or 5-cycle) containing the edge $u_L u_R$.

Case 2.3.1.1. $C = C^4 = (u_L, s_L, s_R, u_R)$.

The proof is similar to Case 1.3.1.1.

Case 2.3.1.2. $C = C^5 = (u_L, w_L, w_R, x, u_R)$.

For $9 \leq \ell \leq 2^{n-1} - |F_L^v|-1$. Let $\ell = \ell' + 3$, where $6 \leq \ell' \leq 2^{n-1} - |F_L^v|-4$. Since $|F_L|\leq n-4$, by Lemma 2.10, there is a fault-free cycle C' of length ℓ' containing the edge $u_L w_L$ in $L - F_L$. Then $C = C' - u_L w_L + w_L w_R + w_R x + x u_R + u_R u_L$ is a fault-free cycle of length ℓ containing the edge $e = u_L u_R$.

For $2^{n-1}-|F_L^v|\leq \ell \leq 2^{n-1}-1$. Let $\ell = \ell_1+\ell_2+1$, where $2^{n-2}-|F_L^v|-1 \leq \ell_1 \leq 2^{n-2}-2$ and $\ell_2 = 2^{n-2}$. Since $2^{n-2}-|F_L^v|-1 > 7$ for $n \geq 6$, by induction hypothesis, there exists a fault-free cycle C_1 of length ℓ_1 containing the edge $u_L w_L$ in $L - F_L$ and, by Lemma 2.11, there exists a fault-free $u_R w_R$-path P of length ℓ_2 in R. Then $C = C_1 - u_L w_L + w_L w_R + P + u_R u_L$ is a fault-free cycle of length ℓ containing the edge $e = u_L u_R$ in MQ_n (see Figure 7(b)).

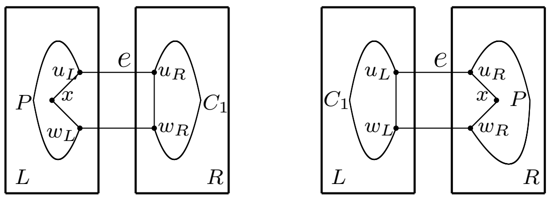

Fig. 7. The illustration of Case 1.3.1.3. and Case 2.3.1.2.

Case 2.3.1.3. $C = C^5 = (u_L, x, w_L, w_R, u_R)$.

The proof is similar to Case 1.3.1.3.

Case 2.3.2. $|F_R|= n-2$. In this case, $|F_L|= 0$.

Since $|F_R|= n-2$, there exists a fault-free neighbor v_R of u_R in R. Let v_L be the neighbor of v_R in L. Let $\ell = \ell' + 3$, where $6 \leq \ell' \leq 2^{n-1} - 4$. By Lemma 2.4, $d_{u_L v_L} \leq 2$. By Lemma 2.6, there is a fault-free $u_L v_L$-path P of length ℓ' in L. So $P + u_L u_R + u_R v_R + v_R v_L$ is a fault-free cycle of length ℓ containing the edge u_Lu_R.

The proof of the theorem is completed. □

4. Conclusion and remarks

As one of the most fundamental networks for parallel and distributed computation, cycles are suitable for developing simple algorithms with low communication cost. Edge and/or vertex failures are inevitable when a large parallel computer system is put in use. Therefore, the fault-tolerant capacity of a network is a critical issue in parallel computing. The fault-tolerant edge-pancyclicity of an interconnection network is a measure of its capability of implementing ring-structured parallel algorithms in a communication-efficient fashion in the presence of faults.

In view of the fact that the hypercube network Q_n contains only even cycles, MQ_n is superior to Q_n in fault-tolerant pancyclicity. This shows that, when the MQ_n is used to model the topological structure of a large-scale parallel processing system, our result implies that the system has larger capability of implementing ring-structured parallel algorithms in a communication-efficient fashion in the hybrid presence of edge and vertex failures than one of the hypercube network.

We make some remarks on the optimality of our result in the following sense.

(1) For $\ell = 4$, in MQ_5^0, taking $e = (11011, 11100)$, there are only two cycles: $(11011, 11010, 11101, 11100)$, $(11011, 01011, 01100, 11100)$ of length 4 containing the edge e (we calculate it by computer).

If $F = \{01011, 11101\}$, then there exists no fault-free cycle of length 4 containing the edge e in $MQ_5^0 - F$. In MQ_5^1, taking $e = (11101, 11110)$, there exists only one cycle $(11101, 11100, 11111, 11110)$ of length 4 containing the edge e. If $F = \{11100\}$ or $F = \{11111\}$, then there exists no fault-free cycle of length 4 containing the edge e in $MQ_5^1 - F$.

(2) For $\ell = 5$, in MQ_5, taking $e = (11110, 11111)$, the cycles of length 5 containing the edge e are as *Table* 1 and *Table* 2(we calculate it by computer):

Table 1. Cycles of length 5 containing the edge $e = (11110, 11111)$ in MQ_5^0

(11110 11101 11010 11000 11111)	(11110 11101 10010 10000 11111)
(11110 11001 11011 11100 11111)	(11110 10001 10011 11100 11111)

Table 2. Cycles of length 5 containing the edge $e = (11110, 11111)$ in MQ_5^1

(11110 11101 11010 11000 11111)	(11110 11101 10010 10000 11111)
(11110 11101 00010 00000 11111)	(11110 11001 11011 11100 11111)
(11110 10001 10011 11100 11111)	(11110 00001 00011 11100 11111)

If $F = \{11100, 11101\}$, then there exists no fault-free cycle of length 5 containing the edge e in $MQ_5 - F$.

(3) For $\ell = 6$, in MQ_n^0, taking $e = (01011, 01100)$, the cycles of length 6 containing the edge e are as $Table$ 3(we calculate it by computer):

Table 3. Cycles of length 6 containing the vertex $e = (01011, 01100)$ in MQ_5^0

(01011 01010 01000 00000 00100 01100)	(01011 01010 01101 01110 01111 01100)
(01011 01010 01101 00101 00100 01100)	(01011 01010 01101 11101 11100 01100)
(01011 01010 00010 00000 00100 01100)	(01011 01010 11010 11011 11100 01100)
(01011 01010 11010 11101 11100 01100)	(01011 01010 11010 11101 01101 01100)
(01011 01001 01000 01010 01101 01100)	(01011 01001 01000 00000 00100 01100)
(01011 01001 00001 00000 00100 01100)	(01011 01001 00001 00101 00100 01100)
(01011 01001 00001 00101 01101 01100)	(01011 01001 11001 11011 11100 01100)
(01011 00011 00010 00000 00100 01100)	(01011 00011 00010 01010 01101 01100)
(01011 00011 00001 00000 00100 01100)	(01011 00011 00001 00101 00100 01100)
(01011 00011 00001 00101 01101 01100)	(01011 11011 11010 11101 11100 01100)
(01011 11011 11010 11101 01101 01100)	(01011 11011 11010 01010 01101 01100)
(01011 11011 11100 11101 01101 01100)	(01011 11011 11100 11111 01111 01100)

If $F = \{00100, 01101, 11100\}$, then there exists no fault-free cycle of length 6 containing the edge e in $MQ_5^0 - F$.

Acknowledgements

The research is supported by National Natural Science Foundation of China under Grant No.U22A2005, No. U21A20491, No. U1936109, No.U1908214, No. KLSA201906 and Key Research and Development Program of Zhejiang (2024SSYS0001).

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