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On Fibonacci (p, r)-cubes

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ABSTRACT

In this paper, it is pointed out that the definition of 'Fibonacci (p, r)-cube' in many papers (denoted by $I\Gamma_n^{(p,r)}$) is incorrect. The graph $I\Gamma_n^{(p,r)}$ is not the same as the original one (denoted by $O\Gamma_n^{(p,r)}$) introduced by Egiazarian and Astola. First, it is shown that $I\Gamma_n^{(p,r)}$ and $O\Gamma_n^{(p,r)}$ have different recursive structure. Then, it is proven that all the graphs $O\Gamma_n^{(p,r)}$ are partial cubes. However, only a small part of graphs $I\Gamma_n^{(p,r)}$ are partial cube. It is also shown that $I\Gamma_n^{(p,r)}$ and $O\Gamma_n^{(p,r)}$ have different medianicity. Finally, several questions are listed for further investigation.

Keywords: Fibonacci cube, Fibonacci (p, r)-cube, Partial cube, Median graph

2020 Mathematics Subject Classification: 05C75, 68R10.

1. Introduction

Let $B = \{0, 1\}$ and for $n \geq 1$ set

$$\mathcal{B}_n = \{b_1 b_2 \ldots b_n \mid b_i \in B, i \in 1 : n\}.$$

An element of $\mathcal{B}_n$ is called a *binary word* of length n (or simply a *word*). All words considered of this paper are binary.

The n-dimensional *hypercube* Q_n is the graph whose vertex set is $\mathcal{B}_n$, and two vertices are adjacent if and only if they differ in precisely one coordinate. The cube Q_3 is shown in Figure 1(a). Hypercubes play an important role in many areas of discrete mathematics and computer science. An excellent survey on hypercubes can be found in [15].

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The *Fibonacci cube* Γ_n [7] can be obtained from Q_n by removing all vertices that contain two consecutive 1s. It is a graph family that have been studied as alternatives for the classical hypercube topology for interconnection networks. The graph Γ_5 is shown in Figure 1 (b). For more results on application and structure of Γ_n, see the survey [12] and the recent book [4].

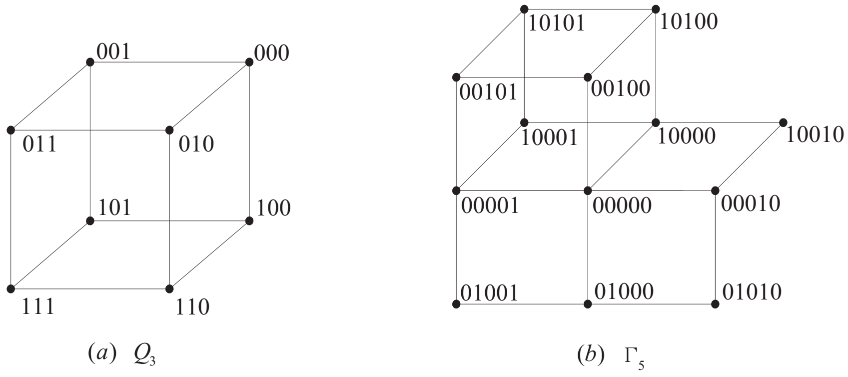


Fig. 1. The hypercube $Q_3(a)$, and the Fibonacci cube $\Gamma_5(b)$

When Fibonacci cubes were introduced, they soon became increasingly popular. Numerous variants and generalizations of Fibonacci cubes, the so called *Fibonacci-like cube*, are proposed and investigated such as in papers [1, 5, 8, 17, 20, 26]. Recently, many other Fibonacci-like cubes have also been introduced and studied, such as generalized Fibonacci cubes [9], generalized Lucas cubes [10], daisy cubes [13], Pell graphs [16], Fibonacci-run graph [3], Fibonacci p-graph [23], Metallic cubes [2] and Lucas-run graph [22].

In the present paper, a special attention is given to the graphs called 'Fibonacci (p, r)-cubes'. It was first introduced by Egiazarian and Astola [5]. In many papers, such as [12, 14, 18, 19, 25] and others, although it is pointed out that the graphs studied comes from [5], we find that it is not the same as given in [5]. For convenience, the graphs studied in [5] are called O-Fibonacci (p, r)-cubes, and the graphs studied in [12, 14, 18, 19, 25] are called I-Fibonacci (p, r)-cubes.

Let $p \geq 1$ and $r \geq 1$. Then for $n \geq 1$, $\alpha = a_1a_2 \ldots a_n$ is called a *O-Fibonacci (p, r)-word* ([5], where it is called Fibonacci (p, r)-code) if the following hold:

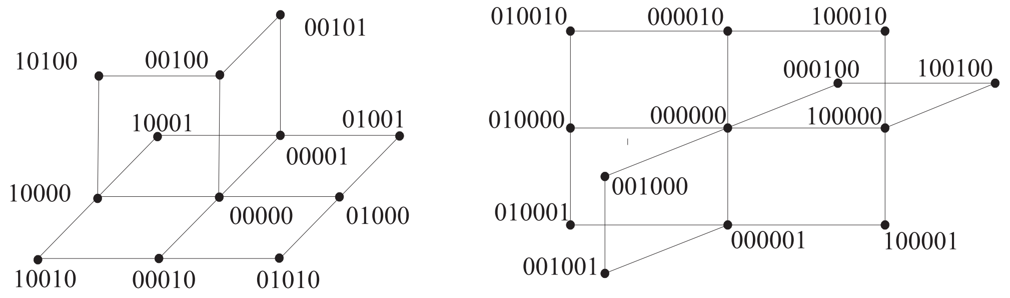
(1) if $a_i = 1$ then $a_{i+1} = \ldots = a_{i+(p-1)} = 0$, i.e. there is at least $p-1$ 0s between two 1s (which is called 'consecutive' 1s); and

(2) there are no more than r 'consecutive' 1s in α, i.e. ones, between which there are exactly $p-1$ zeroes.

For examples, $(100)^4 0^3 (100)^3 0 (100)^2 10$ is a O-Fibonacci $(3, 4)$-word of length 33, but $(100)^4 0^3 (100)^5 010$ is not a O-Fibonacci $(3, 4)$-word.

Definition 1.1. [5] Let $O\mathcal{F}_n^{(p,r)}$ be the set of all the O-Fibonacci (p, r)-words of length n. Then the O-Fibonacci (p, r)-cube $O\Gamma_n^{(p,r)}$ is the graph defined on the vertex set $O\mathcal{F}_n^{(p,r)}$, and two vertices being adjacent if they differ exactly in one coordinate.

It is easily seen that if $(p, r) = (1, 1)$, then a O-Fibonacci (p, r)-word is a word that contain no two consecutive 1s. Therefore, the O-Fibonacci $(1, 1)$-cube $O\Gamma_n^{(1,1)}$ is just the classical Fibonacci cube Γ_n. The graphs $O\Gamma_5^{(2,2)}$ and $O\Gamma_6^{(2,1)}$ are shown in Figure 2 (a) and (b), respectively.



(a) $O\Gamma_5^{(2,2)}$ (b) $O\Gamma_6^{(2,1)}$

Fig. 2. O-Fibonacci (p, r)-cubes $O\Gamma_5^{(2,2)}(a)$ and $O\Gamma_6^{(2,1)}(b)$

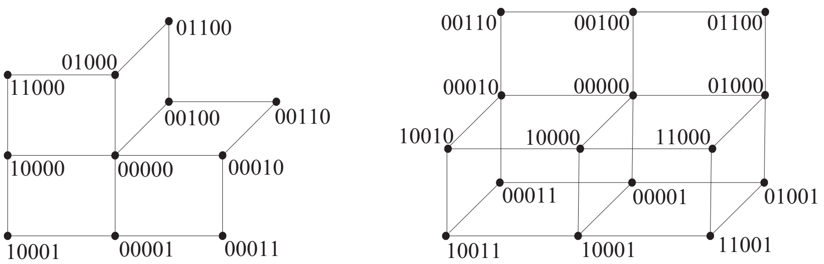
As mentioned above, the 'Fibonacci (p, r)-cubes' studied in [12, 14, 18, 19, 25] will be called I-Fibonacci (p, r)-cubes. They are defined as follows.

Let p, r and n be any positive integers. Then an *I-Fibonacci (p, r)-word* of length n is a word of length n in which there are at most r consecutive 1s and at least p element 0s between two sub-words composed of (at most r) consecutive 1s.

Definition 1.2. [19] Let $I\mathcal{F}_n^{(p,r)}$ denote the set of all I-Fibonacci (p, r)-words of length n. Then the I-Fibonacci (p, r)-cube $I\Gamma_n^{(p,r)}$ is the graph defined on the vertex set $I\mathcal{F}_n^{(p,r)}$ and two vertices are adjacent if they differ in exactly one coordinate.

Note that the cubes $I\Gamma_n^{(p,r)}$ is considered for $n \geq p$ and $n \geq r$ in the above papers. As $I\Gamma_n^{(p,r)}$ is not always trivial for the case $n < r$ or $n < p$, we consider the graph $I\Gamma_n^{(p,r)}$ for all $p \geq 1, r \geq 1$ and $n \geq 1$ in this paper.

For examples, the graphs $I\Gamma_5^{(3,2)}$ and $I\Gamma_5^{(2,2)}$ are shown in Figure 3 (a) and (b), respectively. Obviously, I-Fibonacci $(1, 1)$-cube $I\Gamma_n^{(1,1)}$ is just the classical Fibonacci cube Γ_n.



(a) $I\Gamma_5^{(3,2)}$ (b) $I\Gamma_5^{(2,2)}$

Fig. 3. I-Fibonacci (p, r)-cubes $I\Gamma_5^{(3,2)}(a)$ and $I\Gamma_5^{(2,2)}(b)$

We think that the main difference between the definitions of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ is the meaning of 'consecutive' 1s: the r 'consecutive 1s' in a vertex of $O\Gamma_n^{(p,r)}$ means the sub-word $(10^{p-1})^r$, but the r 'consecutive 1s' in a vertex of $I\Gamma_n^{(p,r)}$ means the sub-word 1^r.

For a binary word χ, we set $\chi^0 = \lambda$, where λ is the empty word. For convenience, if $n = 0$, then let $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ be the graphs with only one vertex λ.

Many Fibonacci like-cubes and some sub-cubes of hypercubes can be obtained from hypercubes by some word forbidden to appear in the words of hypercubes. From the point of view, the following note holds:

Remark 1.3. The cube $O\Gamma_n^{(p,r)}$ can be obtained from Q_n by removing all vertices that contain the words $(10^{p-1})^r1$ or 10^s1 for $s \leq p-2$ (if $p \geq 2$); and $I\Gamma_n^{(p,r)}$ can be obtained from Q_n by removing all vertices that contain the words 1^{r+1} or 10^s1 for $s \leq p-1$.

From Remark 1.3 and Definitions 1.1 and 1.2, $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are not isomorphic in general. For example, $O\Gamma_5^{(2,2)}$ (Figure 2 (a)) is not isomorphic to $I\Gamma_5^{(2,2)}$ (Figure 3 (b)). This fact can be further illustrated by the results of Sections 3 and 4 in the paper.

The rest of the paper is organized as follows. In Sect. 2, some necessary definitions and known results are introduced. In Sect. 3, the recursive structures of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are given. In Sect. 4, the graphs $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ which are partial cube and median graphs are determined. In the last section, some questions on $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are listed for further investigation.

2. Preliminaries

In this section, some definitions, notion and results needed in the paper are given. Let $\mathcal{A}$ be a set of some words. Then $\alpha\mathcal{A}$ is the set of the words obtained from $\mathcal{A}$ by appending a fixed word α in front of each of the elements of $\mathcal{A}$. Recall that *Fibonacci numbers* are defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let $\mathcal{F}_n$ be the vertex set of Fibonacci cube Γ_n. sThen for $n \geq 2$ the well known decomposition of Fibonacci cube can be obtained as follows [7], where $\mathcal{F}_0 = \{\lambda\}$ and $\mathcal{F}_1 = \{0, 1\}$:

$$\mathcal{F}_n = 0\mathcal{F}_{n-1} \cup 10\mathcal{F}_{n-2}. \tag{1}$$

The name of the cubes Γ_n is justified with the fact that for any $n \geq 0$, $|\mathcal{F}_n| = F_{n+2}$ [7]. By Eq. (1), the size of Γ_n can be shown in Eq. (2) for $n \geq 2$, and the recursive structure can be illustrated in Figure 4:

$$|E(\Gamma_n)| = |E(\Gamma_{n-1})| + |E(\Gamma_{n-2})| + F_n. \tag{2}$$

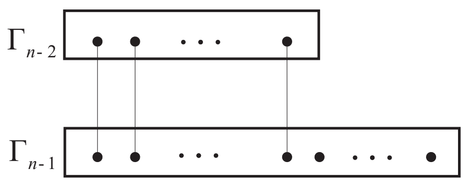


Fig. 4. The recursive structure of Γ_n

The *distance* $d_G(\alpha, \beta)$ between vertices α and β of a graph G is the length of a shortest α, β-path. Given two words α and β of the same length, their Hamming distance $H(\alpha, \beta)$

is the number of coordinates in which they differ. Let H and G be arbitrary (connected) graphs. Then a mapping $f : V(H) \to V(G)$ is an *isometric embedding* if $d_H(u, v) = d_G(f(u), f(v))$ holds for any $u, v \in V(H)$.

A *partial cube* is a connected graph that admits an isometric embedding into a hypercube [6]. It is well known that if α and β are vertices of Q_n, then $d_{Q_n}(\alpha, \beta) = H(\alpha, \beta)$. So we know that if G is a partial cube, then $d_G(\alpha, \beta) = H(\alpha, \beta)$ for any vertices α and β of G. There are more studies on determining which graphs are partial cubes. For example, some generalized Fibonacci and Lucas cubes [9, 10] as partial cubes are shown in [21, 24].

Let $r \geq p+2$ and $n \geq r$. Then for some t with $p \leq t \leq r-2$, there exist vertices α and β of $I\Gamma_n^{(p,r)}$ such that 10^t1 and 11^t1 appear in the same coordinates of α and β, respectively. For convenience, we call there is a *distance-barrier* between the above vertices α and β. It can be shown that $d_{I\Gamma_n^{(p,r)}}(\alpha, \beta) \neq H(\alpha, \beta)$ by Remark 1.3. By the following result we know that not all $I\Gamma_n^{(p,r)}$ are partial cubes.

Lemma 2.1. *Let $p \geq 2$, α and β be any vertices of $I\Gamma_n^{(p,r)}$. Then $d_{I\Gamma_n^{(p,r)}}(\alpha, \beta) = H(\alpha, \beta)$ if and only if there does not exist distance-barrier between α and β.*

A *median* of vertices $u, v, w \in V(G)$ is a vertex of G that simultaneously lies on a shortest u, v-path, a shortest u, w-path, and a shortest v, w-path. The graph G is called a *median graph* if every triple of its vertices has a unique median. It is well known that a median graph must be a partial cube ([6], Proposition 12.4), and hypercube Q_n is a median graph for every $n \geq 1$ ([6], Proposition 3.7).

A subgraph H of a graph G is *median-closed* if, with any triple of vertices of H, their median is also in H. The following result gives a useful tool to prove that a graph is a median graph ([6], Corollary 14.9).

Theorem 2.2. [6] *A graph is a median graph if and only if it is a median-closed induced subgraph of a hypercube.*

It was shown that all Fibonacci cubes Γ_n are median graphs (of course are partial cubes) [11]. In this paper, the question for determining which $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are partial cubes and median graphs is solved completely.

Now we turn to consider some basic properties of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ in the rest of this section. By Definitions 1.1 and 1.2, the following results hold obviously.

Proposition 2.3. *Let r, r', p, p', n, n' be positive integers, $s = \min\{r, r'\}$ and $t = \min\{p, p'\}$. Then*

(a) $O\Gamma_n^{(1,r)} \cong I\Gamma_n^{(1,r)} \cong Q_n$ *for* $n \leq r$;

(b) $O\Gamma_n^{(1,1)} \cong I\Gamma_n^{(1,1)} \cong \Gamma_n$;

(c) $O\Gamma_n^{(p,r)} \cong O\Gamma_n^{(p,r')}$ for $n \leq sp$, and $O\Gamma_n^{(p,r)} \cong O\Gamma_n^{(p',r)}$ for $n \leq t$; *and*

(d) $I\Gamma_n^{(p,r)} \cong I\Gamma_n^{(p,r')}$ for $n \leq s$, and $I\Gamma_n^{(p,r)} \cong I\Gamma_n^{(p',r)}$ for $n \leq t+1$.

By Proposition 2.3 (1) and (2), $O\Gamma_n^{(p,r)} \cong I\Gamma_n^{(p,r)}$ for some special p and r. For examples,

$O\Gamma_3^{(1,3)} \cong I\Gamma_3^{(1,3)} \cong Q_3$ (as shown in Figure 1 (a)) and $O\Gamma_5^{(1,1)} \cong I\Gamma_5^{(1,1)} \cong \Gamma_5$ (as shown in Figure 1 (b)). It is obvious that all those graphs are connected. In general, we have the following result.

Proposition 2.4. *Let p, r and n be positive integers. Then both the graphs $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are connected.*

Proof. First we show that $I\Gamma_n^{(p,r)}$ is connected. It is obvious that 0^n is a vertex of $I\Gamma_n^{(p,r)}$ for any p, r and n. We claim that every vertex α of $I\Gamma_n^{(p,r)}$ is connected with 0^n by a $\alpha, 0^n$-path. In fact, let $\alpha = a_1a_2\dots a_n$ be any vertex of $I\Gamma_n^{(p,r)}$ differing from 0^n, and $a_{i_1} = \dots = a_{i_t} = 1$, where $t \geq 1$ and $i_1 \leq \dots \leq i_t$. Then the word α_j obtained from α by changing $a_{i_1}, \dots, a_{i_j}$ from 1 to 0 is also a vertex of $I\Gamma_n^{(p,r)}$, where $j = 1, \dots, t$. Obviously, $\alpha_t = 0^n$. If $j = 1$, then α and 0^n are adjacent vertices. Now suppose that $j \geq 2$. Then $\alpha \to \alpha_1 \to \dots \to \alpha_{j-1} \to 0^n$ is a path in $I\Gamma_n^{(p,r)}$, and so $I\Gamma_n^{(p,r)}$ is connected.

Similarly, we can show that $O\Gamma_n^{(p,r)}$ is connected by the facts that 0^n is a vertex of $O\Gamma_n^{(p,r)}$, and for any vertex α of $O\Gamma_n^{(p,r)}$ differing from 0^n, there exist a $\alpha, 0^n$-path. This completes the proof. □

3. Recursive Structure of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$

Although some of the structure of $O\Gamma_n^{(p,r)}$ was studied [5], we list them here to show they are different from that of $I\Gamma_n^{(p,r)}$.

3.1. Vertex sets of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$

Recall that $O\mathcal{F}_n^{(p,r)}$ and $I\mathcal{F}_n^{(p,r)}$ are the vertex sets of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$, respectively.

3.1.1. Vertex set of $O\Gamma_n^{(p,r)}$. In paper [5], it is shown that for $n \geq pr+1$, the set $O\mathcal{F}_n^{(p,r)}$ can be defined recursively by

$$O\mathcal{F}_n^{(p,r)} = \bigcup_{i=0}^{r} (10^{p-1})^i 0 O\mathcal{F}_{n-pi-1}^{(p,r)}, \tag{3}$$

with $O\mathcal{F}_0^{(p,r)} = \{\lambda\}$. For example, the first five (from $n = 1$) sets $O\mathcal{F}_n^{(2,2)}$ are thus:

$\{0, 1\}$,
$\{00, 01, 10\}$,
$\{000, 001, 010, 100, 101\}$,
$\{0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010\}$,
$\{00000, 00001, 00010, 00100, 00101, 01000, 01001, 01010, 10000, 10001, 10010, 10100\}$.

If $p = 1$ and $r = 1$, then we have $O\mathcal{F}_n^{(1,1)} = 0O\mathcal{F}_{n-1}^{(1,1)} \cup 10O\mathcal{F}_{n-2}^{(1,1)}$ by Eq. (3). This means that Eq. (1) can be obtained from Eq. (3) by Proposition 2.3(2).

For convenience, if $n \geq 1$ and $-p \leq n - pi - 1 < 0$ for some i $(1 \leq i \leq r)$, then let $(10^{p-1})^i 0 O\mathcal{F}_{n-pi-1}^{(p,r)}$ be the set containing only one word, and this word is the prefix of length n of $(10^{p-1})^i0$; if $n-pi-1<-p$, then let $(10^{p-1})^i0O\mathcal{F}^{(p,r)}_{n-pi-1}=\emptyset$. This means that Eq. (3) also holds for $1\le n\le pr$, and so we have

$$|O\mathcal{F}^{(p,r)}_n|=\sum_{i=0}^{r}|O\mathcal{F}^{(p,r)}_{n-pi-1}|, \tag{4}$$

where $|O\mathcal{F}^{(p,r)}_{n-pi-1}|=1$ if $-p\le n-pi-1<0$, and $|O\mathcal{F}^{(p,r)}_{n-pi-1}|=0$ if $n-pi-1<-p$.

In paper [5], *Fibonacci (p,r)-number* $OF^{(p,r)}_n$ is defined as follows with $OF^{(p,r)}_n=0$ if $n\le 0$, and $OF^{(p,r)}_n=1$ if $1\le n\le p+1$:

$$OF^{(p,r)}_n=\sum_{i=0}^{r}OF^{(p,r)}_{n-pi-1}. \tag{5}$$

It is easily seen that if $p=r=1$, then $OF^{(p,r)}_n=F_n$. By Eqs. (4) and (5), it is known that $|V(O\Gamma^{(p,r)}_n)|=|O\mathcal{F}^{(p,r)}_n|=OF^{(p,r)}_{n+p+1}$. By this result and Proposition 2.3(2), $|V(\Gamma_n)|=|\mathcal{F}_n|=|O\mathcal{F}^{(1,1)}_n|=OF^{(1,1)}_{n+1+1}=F_{n+2}$ holds for the classical Fibonacci cubes [7].

3.1.2. Vertex set of $I\Gamma^{(p,r)}_n$. On the vertex set of $I\Gamma^{(p,r)}_n$, we have the following result.

Theorem 3.1. *Let $p\ge 1, r\ge 1, n\ge p+r$ and $I\mathcal{F}^{(p,r)}_0=\{\lambda\}$. Then $I\mathcal{F}^{(p,r)}_n$ satisfies:*

$$I\mathcal{F}^{(p,r)}_n=0I\mathcal{F}^{(p,r)}_{n-1}\cup 10^pI\mathcal{F}^{(p,r)}_{n-p-1}\cup\ldots\cup 1^r0^pI\mathcal{F}^{(p,r)}_{n-p-r}. \tag{6}$$

Proof. It is easy to see that $I\mathcal{F}^{(p,r)}_n\supseteq 0I\mathcal{F}^{(p,r)}_{n-1}\cup 10^pI\mathcal{F}^{(p,r)}_{n-p-1}\cup\ldots\cup 1^r0^pI\mathcal{F}^{(p,r)}_{n-p-r}$. Let $\alpha\in I\mathcal{F}^{(p,r)}_n$ and suppose that the coordinate of the first 0 of α is i. Then $1\le i\le r+1$ by the definition of I-Fibonacci (p,r)-word and then the following holds. If $i=1$, then $\alpha=0\beta$ for some $\beta\in I\mathcal{F}^{(p,r)}_{n-1}$. If $2\le i\le r+1$, then α has the form of $1^{i-1}0^p\gamma$, where $\gamma\in I\mathcal{F}^{(p,r)}_{n-p-(i-1)}$. It implies that $I\mathcal{F}^{(p,r)}_n\subseteq 0I\mathcal{F}^{(p,r)}_{n-1}\cup 10^pI\mathcal{F}^{(p,r)}_{n-p-1}\cup\ldots\cup 1^r0^pI\mathcal{F}^{(p,r)}_{n-p-r}$. This completes the proof. □

It is easy to see that if $p=1$ and $r=1$, then Eq. (1) can be obtained from Eq. (6) by Proposition 2.3 (2).

For convenience, if $1\le n<p+i$ for some $i\in[r]$, then let $1^i0^pI\mathcal{F}^{(p,r)}_{n-p-i}$ be the set consisting of only the word which is the prefix of length n of 1^i0^p. It can be seen that if $i<j$ and $n<p+i$, then $1^i0^pI\mathcal{F}^{(p,r)}_{n-p-i}=1^j0^pI\mathcal{F}^{(p,r)}_{n-p-j}$. So for $n<i$, let $1^i0^pI\mathcal{F}^{(p,r)}_{n-p-i}=\emptyset$. Then for $1\le n<p+r$, the set $I\mathcal{F}^{(p,r)}_n$ also can be determined by Eq. (6).

For example, the first few $I\mathcal{F}^{(2,2)}_n$ are thus:
$I\mathcal{F}^{(2,2)}_1=\{0,1\}$,
$I\mathcal{F}^{(2,2)}_2=\{00,01,10,11\}$,
$I\mathcal{F}^{(2,2)}_3=\{000,001,010,011,100,110\}$,
$I\mathcal{F}^{(2,2)}_4=\{0000,0001,0010,0011,0100,0110,1000,1001,1100\}$,
$I\mathcal{F}^{(2,2)}_5=\{00000,00001,00010,00011,00100,00110,01000,01001,01100,10000,10001,$ $10010,10011,11000,11001\}$.

By Theorem 3.1 and the above analysis, the following result holds.

Corollary 3.2. Setting $|I\mathcal{F}_n^{(p,r)}|=0$ for $n<-p$ and $|I\mathcal{F}_n^{(p,r)}|=1$ for $-p\le n\le 0$, we have

$$|I\mathcal{F}_n^{(p,r)}|=|I\mathcal{F}_{n-1}^{(p,r)}|+|I\mathcal{F}_{n-p-1}^{(p,r)}|+\ldots+|I\mathcal{F}_{n-p-r}^{(p,r)}|. \quad (7)$$

By Eqs. (3) and (6), it is easy to see that if $p=1$ or $r=1$, then $O\mathcal{F}_n^{(p,r)}=I\mathcal{F}_n^{(p,r)}$ and so $O\Gamma_n^{(p,r)}\cong I\Gamma_n^{(p,r)}$. For $p>1$, $r>1$ and $n=0$ or 1, $O\mathcal{F}_n^{(p,r)}=I\mathcal{F}_n^{(p,r)}$ and $O\Gamma_n^{(p,r)}\cong I\Gamma_n^{(p,r)}$. But for $n>1$, $|I\mathcal{F}_n^{(p,r)}|>|O\mathcal{F}_n^{(p,r)}|$ by Eqs. (4) and (7). So the following result holds.

Corollary 3.3. Let $p\ge 1, r\ge 1$ and $n\ge 0$. Then $O\Gamma_n^{(p,r)}\ncong I\Gamma_n^{(p,r)}$ if and only if $p>1, r>1$ and $n>1$.

The above result implies that $O\Gamma_n^{(p,r)}\ncong I\Gamma_n^{(p,r)}$ from the general sense. However, there are exist some $p>1$ and $p'>1$, $r>1$ and $r'>1$, and $n>1$ and $n'>1$ such that $O\Gamma_n^{(p,r)}\cong I\Gamma_{n'}^{(p',r')}$. For example, it can be shown that $O\Gamma_4^{(2,2)}\cong I\Gamma_4^{(3,2)}$, as illustrated in Figure 5.

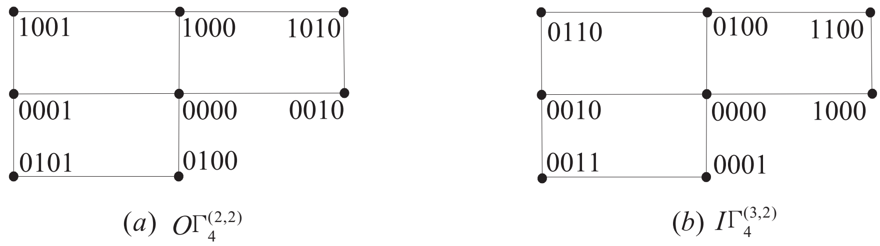


Fig. 5. Graphs $O\Gamma_4^{(2,2)}(a)$ and $I\Gamma_4^{(3,2)}(b)$

3.2. Edge sets of $O\Gamma_n^{(p,r)}$ *and* $I\Gamma_n^{(p,r)}$

The recursive structure on the edge sets of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are studied in this subsection.

3.2.1. Edge set of $O\Gamma_n^{(p,r)}$. We show that the iterative formula of the size of $O\Gamma_n^{(p,r)}$ previously given ([5], Property 2) was erroneous and determine its correct expression. First we take $O\Gamma_n^{(2,3)}$ as an example to understand easily the structure of the edge set of $O\Gamma_n^{(p,r)}$. By Eq. (3), for $n\ge 7$, $O\mathcal{F}_n^{(2,3)}=0O\mathcal{F}_{n-1}^{(2,3)}\cup 100O\mathcal{F}_{n-3}^{(2,3)}\cup 10100O\mathcal{F}_{n-5}^{(2,3)}\cup 1010100O\mathcal{F}_{n-7}^{(2,3)}$.

Inside each subgraph of $O\Gamma_n^{(p,r)}$ induced by $(10)^t0O\mathcal{F}_{n-2t-1}^{(2,3)}$ the edges are inherited from $O\Gamma_{n-2t-1}^{(2,3)}$, $t=0,1,2$ and 3. We need to determine the edges between these four subgraphs. Let $0\le i<j\le 3$. Then by the fact $0(10)^{j-i-1}0O\mathcal{F}_{n-2j-1}^{(2,3)}\subseteq O\mathcal{F}_{n-2i-1}^{(2,3)}$, it is known that $(10)^i00(10)^{j-i-1}0O\mathcal{F}_{n-2j-1}^{(2,3)}$ is a subset of $(10)^i0O\mathcal{F}_{n-2i-1}^{(2,3)}$. It is easily seen that $(10)^j0O\mathcal{F}_{n-2j-1}^{(2,3)}=(10)^i10(10)^{j-i-1}0O\mathcal{F}_{n-2j-1}^{(2,3)}$. Let α be a vertex of $(10)^j0O\mathcal{F}_{n-2j-1}^{(2,3)}$. Then $\alpha=(10)^i10(10)^{j-i-1}0\beta$ for some $\beta\in O\mathcal{F}_{n-2j-1}^{(2,3)}$. Obviously, there exist a vertex $\alpha'=(10)^i00(10)^{j-i-1}0\beta\in O\mathcal{F}_{n-2i-1}^{(2,3)}$, and so α is adjacent to α'. Therefore, there are $|O\mathcal{F}_{n-2j-1}^{(2,3)}|$ edges between the subsets $(10)^j0O\mathcal{F}_{n-2j-1}^{(2,3)}$ and $(10)^i0O\mathcal{F}_{n-2i-1}^{(2,3)}$. So we know that the decomposition of $O\Gamma_n^{(2,3)}$ can be shown as in Figure 6, and

$$\begin{aligned}|E(O\Gamma_n^{(2,3)})| &= |E(O\Gamma_{n-1}^{(2,3)})|+|E(O\Gamma_{n-3}^{(2,3)})|\\ &\quad + |E(O\Gamma_{n-5}^{(2,3)})|+|E(O\Gamma_{n-7}^{(2,3)})|\\ &\quad + |O\mathcal{F}n-3^{(2,3)}|+2|O\mathcal{F}n-5^{(2,3)}|+3|O\mathcal{F}n-7^{(2,3)}|\\ &= \sum^{3} t=0 \left(|E(I\Gamma_{n-2t-1}^{(2,3)})|+t|V(I\Gamma_{n-2t-1}^{(2,3)})|\right).\end{aligned}$$

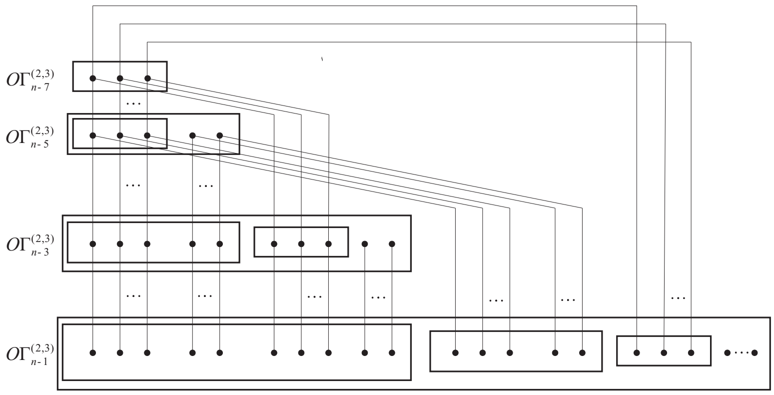


Fig. 6. The decomposition of $O\Gamma_n^{(2,3)}$

In general, we can get the structure of the edge set of $O\Gamma_n^{(p,r)}$ as follows. By Eq. (3) we know that the vertex set of $O\Gamma_n^{(p,r)}$ can be decomposed into $r+1$ disjoint subsets for $n \geq pr+1$: $O\mathcal{F}_n^{(p,r)} = \bigcup_{t=0}^{r}(10^{p-1})^t 0 O\mathcal{F}_{n-pt-1}^{(p,r)}$. So the graph $O\Gamma_n^{(p,r)}$ can be decomposed into $r+1$ disjoint subgraphs isomorphic to $O\Gamma_{n-tp-1}^{(p,r)}$ for $t=0,1,\ldots,r$, respectively. Further, for $0 \leq i < j \leq r$, it can be found that there are $|V(O\Gamma_{n-jp-1}^{(p,r)})|= |O\mathcal{F}_{n-jp-1}^{(p,r)}|$ edges connecting the subgraphs $O\Gamma_{n-ip-1}^{(p,r)}$ and $O\Gamma_{n-jp-1}^{(p,r)}$ (of $O\Gamma_n^{(p,r)}$). So there are $\sum_{t=0}^{r}(t|O\mathcal{F}_{n-pt-1}^{(p,r)}|)$ edges between these $r+1$ subgraphs. So we have the following result.

Theorem 3.4. *Let $n \geq pr+1$. Then*

$$|E(O\Gamma_n^{(p,r)})|= \sum_{t=0}^{r}(|E(O\Gamma_{n-pt-1}^{(p,r)})|+t|O\mathcal{F}_{n-pt-1}^{(p,r)}|). \tag{8}$$

3.2.2. Edge set of $I\Gamma_n^{(p,r)}$. First, we also take $I\Gamma_n^{(2,3)}$ as an example to better understand the structure of the edge set of $O\Gamma_n^{(p,r)}$. By Eq. (6), we know that $I\mathcal{F}_n^{(2,3)}$ can be decomposed into four disjoint subsets for $n \geq 5$: $0I\mathcal{F}_{n-1}^{(2,3)}$, $100I\mathcal{F}_{n-3}^{(2,3)}$, $1100I\mathcal{F}_{n-4}^{(2,3)}$ and $11100I\mathcal{F}_{n-5}^{(2,3)}$.

Inside each subgraph of $I\Gamma_n^{(p,r)}$ induced by $0I\mathcal{F}_{n-1}^{(2,3)}$ and $1^t00I\mathcal{F}_{n-2-t}^{(2,3)}$ ($t \in [3]$) the edges are inherited from $I\Gamma_{n-1}^{(2,3)}$ and $I\Gamma_{n-2-t}^{(2,3)}$, respectively. Now we consider the edges between the above four subsets. It is easily seen that $01^{t-1}00I\mathcal{F}_{n-2-t}^{(2,3)} \subset 0I\mathcal{F}_{n-1}^{(2,3)}$. So for every vertex $\alpha \in 1^t00I\mathcal{F}_{n-2-t}^{(2,3)}$, there exist a vertex $\alpha' \in 01^{t-1}00I\mathcal{F}_{n-2-t}^{(2,3)}$ such that there is an edge between α and α'. So there are $|I\mathcal{F}_{n-2-t}^{(2,3)}|$ edges between $1^t00I\mathcal{F}_{n-2-t}^{(2,3)}$ and $0I\mathcal{F}_{n-1}^{(2,3)}$ for $t \in [3]$. Suppose $1 \leq i < j \leq 3$, $\beta \in 1^j00I\mathcal{F}_{n-2-j}^{(2,3)}$ and $\beta' \in 1^i00I\mathcal{F}_{n-2-i}^{(2,3)}$. If

$j-i\geq 2$, then β and β' are not adjacent in $I\Gamma_n^{(p,r)}$. If $j=i+1$, then by the fact $1^{j-1}000I\mathcal{F}_{n-2-j}^{(2,3)}\subset 1^i00I\mathcal{F}_{n-2-i}^{(2,3)}$, we know that there exist a vertex $\beta''\in 1^i00I\mathcal{F}_{n-2-i}^{(2,3)}$ such that β' and β'' are adjacent in $I\Gamma_n^{(p,r)}$. This implies that for $1\leq i<j\leq 3$, there exist edges between $1^j00I\mathcal{F}_{n-2-j}^{(2,3)}$ and $1^i00I\mathcal{F}_{n-2-i}^{(2,3)}$ only if $j=i+1$, and there are $|I\mathcal{F}_{n-2-j}^{(2,3)}|$ edges between them. Hence, we know that the decomposition of $I\Gamma_n^{(2,3)}$ can be shown as in Figure 7, and $|E(I\Gamma_n^{(2,3)})|=|E(I\Gamma_{n-1}^{(2,3)})|+\sum_{t=1}^{3}(|E(I\Gamma_{n-2-t}^{(2,3)})|+2|I\mathcal{F}_{n-2-t}^{(2,3)}|)-|I\mathcal{F}_{n-3}^{(2,3)}|$.

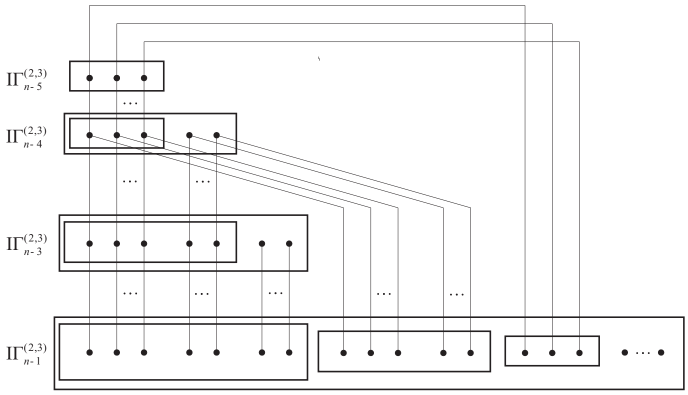


Fig. 7. The decomposition of $I\Gamma_n^{(2,3)}$

In general, we have the following result.

Theorem 3.5. *$n\geq p+r$. Then*

$$|E(I\Gamma_n^{(p,r)})|=|E(I\Gamma_{n-1}^{(p,r)})|+\sum_{t=1}^{r}(|E(I\Gamma_{n-p-t}^{(p,r)})|+2|I\mathcal{F}_{n-p-t}^{(p,r)}|)-|I\mathcal{F}_{n-p-1}^{(p,r)}|. \tag{9}$$

Proof. By Eq. (6), $I\mathcal{F}_n^{(p,r)}=0I\mathcal{F}_{n-1}^{(p,r)}\cup 10^pI\mathcal{F}_{n-p-1}^{(p,r)}\cup\ldots\cup 1^r0^pI\mathcal{F}_{n-p-r}^{(p,r)}$. So the graph $I\Gamma_n^{(p,r)}$ can be decomposed into $r+1$ disjoint subgraphs isomorphic to $I\Gamma_{n-1}^{(p,r)}$ (induced by the set $0I\mathcal{F}_{n-1}^{(p,r)}$) and $I\Gamma_{n-p-t}^{(p,r)}$ (induced by the set $1^t0^{p-1}I\mathcal{F}_{n-p-t}^{(p,r)}$) for $t\in[r]$, respectively. To achieve the desired result, we need to consider the edges between the above subgraphs. First, we consider $0I\mathcal{F}_{n-1}^{(p,r)}$ and $1^t0^pI\mathcal{F}_{n-p-t}^{(p,r)}$, $t\in[r]$. Let α be a vertex of $1^t0^pI\mathcal{F}_{n-p-t}^{(p,r)}$. Then $\alpha=1^t0^p\alpha'$ for some $\alpha'\in I\mathcal{F}_{n-p-t}^{(p,r)}$. It can be seen that the vertex $\beta=01^{t-1}0^p\alpha'\in 0I\mathcal{F}_{n-1}^{(p,r)}$, and so there are $|I\mathcal{F}_{n-p-t}^{(p,r)}|$ edges between $0I\mathcal{F}_{n-1}^{(p,r)}$ and $1^t0^pI\mathcal{F}_{n-p-t}^{(p,r)}$. Now we consider the edges between $1^i0^pI\mathcal{F}_{n-p-i}^{(p,r)}$ and $1^j0^pI\mathcal{F}_{n-p-j}^{(p,r)}$ for $1\leq i<j\leq r$. Obviously, if $j\geq i+2$, then there is not edges between them. Suppose $j=i+1$ and let $\alpha\in 1^j0^pI\mathcal{F}_{n-p-j}^{(p,r)}$. Then $\alpha=1^j0^p\alpha'=1^i10^p\alpha'$ for some $\alpha'\in I\mathcal{F}_{n-p-j}^{(p,r)}$. As $\beta=1^i00^p\alpha'\in 1^i0^pI\mathcal{F}_{n-p-i}^{(p,r)}$ and α and β are adjacent, we know that there are $|I\mathcal{F}_{n-p-j}^{(p,r)}|$ edges between $1^i0^pI\mathcal{F}_{n-p-i}^{(p,r)}$ and $1^j0^pI\mathcal{F}_{n-p-j}^{(p,r)}$ for $j=i+1$. Therefore, there are altogether $2\sum_{t=1}^{r}|I\mathcal{F}_{n-p-t}^{(p,r)}|-|I\mathcal{F}_{n-p-1}^{(p,r)}|$ edges connecting these $r+1$ subgraphs. This completes the proof. □

If $p = 1$ and $r = 1$, then by Eqs. (8) and (9) we have

$$|E(O\Gamma_n^{(1,1)})| = |E(O\Gamma_{n-1}^{(1,1)})| + |E(O\Gamma_{n-2}^{(1,1)})| + |O\mathcal{F}_{n-2}^{(1,1)}|, \textit{and}$$
$$|E(I\Gamma_n^{(1,1)})| = |E(I\Gamma_{n-1}^{(1,1)})| + |E(I\Gamma_{n-2}^{(1,1)})| + |I\mathcal{F}_{n-2}^{(1,1)}|,$$

respectively. This means that Eq. (2) can be obtained from both Eqs. (8) and (9).

4. Relation to Hypercubes

Both partial cubes and median graphs are important and well-studied classes of graphs. The graphs $I\Gamma_n^{(p,r)}$ and $O\Gamma_n^{(p,r)}$ which are partial cubes and median graphs are determined.

4.1. $I\Gamma_n^{(p,r)}$ and $O\Gamma_n^{(p,r)}$ as partial cubes

Both graphs $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ are induced subgraphs of hypercubes. It is natural to ask whether they can be isometrically embedded into hypercubes. First we consider $O\Gamma_n^{(p,r)}$.

Theorem 4.1. *Let $p \geq 1$ and $r \geq 1$. Then for any $n \geq 1$, $O\Gamma_n^{(p,r)}$ is a partial cube.*

Proof. Let $\alpha = a_1 a_2 \dots a_n$ and $\beta = b_1 b_2 \dots b_n$ be any two vertices of $O\Gamma_n^{(p,r)}$. Suppose that the Hamming distance $H(\alpha, \beta)$ between α and β is s, and $a_{i_j} \neq b_{i_j}$ for all $j \in [s]$. The desired result can be obtained by showing $d_{O\Gamma_n^{(p,r)}}(\alpha, \beta) = H(\alpha, \beta)$ for all $s \geq 1$. This can be shown by using induction on s. Obviously if $s = 1$, then $d_{O\Gamma_n^{(p,r)}}(\alpha, \beta) = 1 = H(\alpha, \beta)$ by Definition 1.1. Suppose that $s \geq 2$ and $d_{O\Gamma_n^{(p,r)}}(\mu, \nu) = H(\mu, \nu)$ holds for any two vertices μ and ν of $O\Gamma_n^{(p,r)}$ with $H(\mu, \nu) = s - 1$. Without loss of generality, suppose that $a_{i_1} = 1$ and $b_{i_1} = 0$. Let α' be the word obtained from α by changing a_{i_1} from 1 to 0. Then $H(\alpha, \alpha') = 1$, $H(\alpha', \beta) = s - 1$ and α' is a O-Fibonacci (p, r)-word of length n, that is, $\alpha' \in O\mathcal{F}_n^{(p,r)}$. As $d_{O\Gamma_n^{(p,r)}}(\alpha', \beta) = H(\alpha', \beta) = s - 1$ by the induction hypothesis, we know $d_{O\Gamma_n^{(p,r)}}(\alpha, \beta) = H(\alpha, \alpha') + H(\alpha', \beta) = 1 + s - 1 = s$. This completes the proof. □

By Theorem 4.1, all $O\Gamma_n^{(p,r)}$ are partial cubes. However, this does not hold for $I\Gamma_n^{(p,r)}$. For $n \geq p$ and $n \geq r$, the cubes $I\Gamma_n^{(p,r)}$ which are partial cubes have been determined [25]. Now for all the cases $n \geq 1$, $p \geq 1$ and $r \geq 1$, the results are listed as follows.

Theorem 4.2. *Let $p \geq 1, r \geq 1$ and $n \geq 1$. Then $I\Gamma_n^{(p,r)}$ is a partial cube if and only if it is one of the following cases:*

(a) $p = 1$, $r \geq 1$, and $n \geq 1$;
(b) $p \geq 2$, $r \leq p + 1$ and $n \geq 1$; and
(c) $p \geq 2$, $r \geq p + 2$ and $n < r$.

Proof. First we consider the case $p = 1$ and $r \geq 1$. If $n \geq r$, then $I\Gamma_n^{(1,r)}$ is a partial cube ([25], Lemma 2.2). If $n < r$, then $I\Gamma_n^{(p,r)} \cong Q_n$ by Proposition 2.3, and so $I\Gamma_n^{(p,r)}$ is a partial cube. It means that if (a) holds, then $I\Gamma_n^{(1,r)}$ is a partial cube.

If $p \geq 2$ and $r \leq p + 1$, then it is obvious that there is not a distance-barrier between any two vertices of $I\Gamma_n^{(p,r)}$. So if (b) holds, then $I\Gamma_n^{(1,r)}$ is partial cube by Lemma 2.1.

Now we turn to consider the case $p \geq 2$ and $r \geq p+2$. If $n \geq r$, then it was shown that $I\Gamma_n^{(p,r)}$ is not a partial ([25], Lemma 2.5). If $n < r$, then there is not a distance-barrier between any two vertices of $I\Gamma_n^{(p,r)}$, and so $I\Gamma_n^{(p,r)}$ is a partial cube by Lemma 2.1.

According to the above analysis, $I\Gamma_n^{(p,r)}$ is a partial cube if and only if one of $(a), (b)$ and (c) holds. □

4.2. $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$ as median graphs

It is well known that a median graph must be a partial cube. In this subsection, we show that $O\Gamma_n^{(p,r)}$ (resp. $I\Gamma_n^{(p,r)}$) being median graphs is only a small part of the $O\Gamma_n^{(p,r)}$ (resp. $I\Gamma_n^{(p,r)}$) which are partial cubes.

Note that for $n \geq p$ and $n \geq r$, the graphs $I\Gamma_n^{(p,r)}$ which are median graphs has been determined [18]. For the cases $p \geq 1$, $r \geq 1$ and $n \geq 1$, graphs $I\Gamma_n^{(p,r)}$ as median graphs are list as follows.

Theorem 4.3. *Let $p \geq 1, r \geq 1$ and $n \geq 1$. Then $I\Gamma_n^{(p,r)}$ is a median graph if and only if it is one of the following cases:*

(a) $p = 1$, $r \geq 2$ and $r \geq n \geq 1$;

(b) $p \geq 2$, $r \geq 3$ and $2 \geq n \geq 1$; and

(c) $r \leq p$, $r \leq 2$ and $n \geq 1$.

Proof. We distinguish three cases: (1) $p = 1$ and $r \geq 2$, (2) $p \geq 2$ and $r \geq 3$, and (3) $r \leq p$ and $r \leq 2$. It has been shown that if (1) or (3) holds for $n \geq p$ and $n \geq r$, or (2) hold for $n \geq 3$, then $I\Gamma_n^{(p,r)}$ is not a median graph ([25], Lemma 4.2 and Corollary 4.4).

If (1) holds and $n < r$, then $I\Gamma_n^{(p,r)} \cong Q_n$ by Proposition 2.3(1). It is obvious that if (2) happens and $2 \geq n \geq 1$, then $I\Gamma_n^{(p,r)} \cong Q_n$. It is well known that Q_n is a median graph. If $n < p$ and (3) holds, then $I\Gamma_n^{(p,r)} \cong I\Gamma_n^{(n,r)}$ by Proposition 2.3 (3). It has been known that $I\Gamma_n^{(n,r)}$ is a median graph if (3) happens ([25], Corollary 4.4). According to the above analysis, $I\Gamma_n^{(p,r)}$ is a median graph if and only if (a), (b), or (c) holds. □

The following result determines the graphs $O\Gamma_n^{(p,r)}$ which are median graphs.

Theorem 4.4. *Let $p \geq 1, r \geq 1$ and $n \geq 1$. Then $O\Gamma_n^{(p,r)}$ is a median graph if and only if one of the following cases holds:*

(a') $p \geq 1, r = 1$ and $n \geq 1$;

(b') $p = 1, r \geq 2$ and $r \geq n \geq 1$; and

(c') $p \geq 2$, $r \geq 2$ and $n \leq pr$.

Proof. We also distinguish three cases by p and r: $(1')$ $p \geq 1$ and $r = 1$, $(2')$ $p = 1$ and $r \geq 2$, and $(3')$ $p \geq 2$ and $r \geq 2$. By Corollary 3.3, we know that $O\Gamma_n^{(1,r)} \cong I\Gamma_n^{(1,r)}$ and $O\Gamma_n^{(p,1)} \cong I\Gamma_n^{(p,1)}$. So if (a') or (b') holds, then $O\Gamma_n^{(p,r)}$ is a median graph by Theorem 4.3 (a) and (c). Now we turn to consider case $(3')$. For the case $p \geq 2$, $r \geq 2$ and $n \leq pr$, let

$$\chi = x_1x_2\dots x_n,$$
$$\eta = y_1y_2\dots y_n,$$
$$\rho = p_1p_2\dots p_n,$$

and

$$\omega = w_1w_2\dots w_n,$$

where χ,η and ρ are vertices of $O\Gamma_n^{(p,r)}$, and ω is the median of χ,η and ρ. It is well known that the median of the triple in Q_n is obtained by the majority rule ([6], Proposition 3.7): the ith coordinate of the median is equal to the element that appears at least twice among the x_i, y_i, and p_i. Without loss of generality, suppose that among x_1, y_1 and p_1 there at least two 1s. Then $w_1 = 1$. Suppose the second 1 contained in ω is w_i. As χ,η are vertices of $O\Gamma_n^{(p,r)}$ and there are at least two 1 among x_i, y_i and p_i, we know $i \geq p+1$. By considering the coordinate of the next element 1 in ω, we can find that the number of 0s between two 1 is at least $p-1$ in ω. Since the length of ω is not more than pr, there are at most r continue '1' in ω. Therefore, ω is a vertex of $O\Gamma_n^{(p,r)}$, and so $O\Gamma_n^{(p,r)}$ is a median graph for this case.

For any $p \geq 2$, $r \geq 2$ and $n > pr$, let

$$\alpha = 10^{p-1}10^{p-1}0(0^{p-1}1)^{r-2}0^{n-pr-1},$$
$$\beta = 10^{p-1}00^{p-1}1(0^{p-1}1)^{r-2}0^{n-pr-1},$$

and

$$\gamma = 00^{p-1}10^{p-1}1(0^{p-1}1)^{r-2}0^{n-pr-1}.$$

Then α,β and γ are vertices of $O\Gamma_n^{(p,r)}$. Set

$$\mu = 10^{p-1}10^{p-1}1(0^{p-1}1)^{r-2}0^{n-pr-1}.$$

It is easy to see that α,β and γ are pairwise at distance 2 in $O\Gamma_n^{(p,r)}$. By the majority rule, the unique candidate for their median is μ. Since there are $r+1$ 'consecutive' 1s in μ, it does not belong to $O\Gamma_n^{(p,r)}$ and so $O\Gamma_n^{(p,r)}$ is not median-closed induced subgraph of hypercube. Hence, $O\Gamma_n^{(p,r)}$ is not a median graph by Theorem 2.2 for this case. This completes the proof. □

5. Concluding Remarks

In this section, two questions are listed for further study of $O\Gamma_n^{(p,r)}$ and $I\Gamma_n^{(p,r)}$.

Corollary 3.3 shows that $O\Gamma_n^{(p,r)} \ncong I\Gamma_n^{(p,r)}$ for almost all of p and r. However, there may be some p,r,n and p',r',n' such that $O\Gamma_n^{(p,r)} \cong I\Gamma_{n'}^{(p',r')}$. As an example, $O\Gamma_4^{(2,2)} \cong I\Gamma_4^{(3,2)}$ is shown in Figure 5. A natural question that arises is the following:

Question 5.1. *For which values of p,r,n and p',r',n', $O\Gamma_n^{(p,r)} \cong I\Gamma_{n'}^{(p',r')}$?*

The *eccentricity* $e(v)$ of a vertex v of a graph G is the maximum of its distances to other vertices in G, and the *diameter* $d(G)$ of G are the maximum of the vertex eccentricities. The diameter of $O\Gamma_n^{(p,r)}$ was determined ([5], Property 4). But the diameter of $I\Gamma_n^{(p,r)}$ has not been studied. So the following questions are listed.

Question 5.2. *What is the diameter of* $I\Gamma_n^{(p,r)}$*?*

As mentioned above the diameter of a graph G is the greatest distance between any two vertices in G. Theorem 4.1 shows that every graph $O\Gamma_n^{(p,r)}$ is a partial cube, and so the distance between any two vertices of $O\Gamma_n^{(p,r)}$ is the Hamming distance of them. However, Theorem 4.2 shows that only a small part of all graphs $I\Gamma_n^{(p,r)}$ are partial cube. Therefore, it seems that determining the diameter of $I\Gamma_n^{(p,r)}$ is a rather difficult task.

Conflict of interest

The author declares no conflict of interest.

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