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On Hermitian Cones in $\mathrm{PG}(3, q^2)$

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ABSTRACT

In this paper, we present a new combinatorial characterization of Hermitian cones in $\mathrm{PG}(3, q^2)$.

Keywords: projective space, Baer subplane, Hermitian arc

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1. Introduction

Algebraic varieties in a finite projective space $\mathrm{PG}(r, q)$ have few finite intersection sizes with all the members of one (or more than one) prescribed family of subspaces. Thus, it is natural to ask if it is possible to reconstruct their structure starting from these intersection numbers and possibly other additional arithmetic and/or geoemetric conditions. A first example for such a characterization problem is the famous theorem of B. Segre (1954) [9] which characterizes the non–degenerate conics of $\mathrm{PG}(2, q)$, q odd, as sets of $(q+1)$–points intersected by any line in at most two points. There is a wide literature devoted to this problem, mostly when there are only two possibilities for the intersection sizes (cf e.g. [2, 5, 7, 8, 10] and the references cited therein). Whereas less is known, as regards sets with more than two intersection sizes, (cf e.g. [3, 4, 10]).

Recently, some papers have been published concerning characterizations of cones with a Baer subplane or an Hermitian arc as base (director) curve in 3–dimensional finite projective spaces, as sets of points of $\mathrm{PG}(3, q^2)$ with three intersection numbers with respect to the planes [1, 6, 4, 3]. In [1], using a combination of combinatorial methods

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and algebraic geometry over finite fields, the author obtains a characterization of the Hermitian cone in $\mathrm{PG}(3,q^2)$ as a surface of degree $q+1$ and with intersection sizes with the planes as those of an Hermitian cone and using this result on cones she gets also a characterization of (nonsingular) Hermitian surfaces in $\mathrm{PG}(4,q^2)$.

In this paper, we present a combinatorial characterization of *Hermitian cones* of $\mathrm{PG}(3,q^2)$ starting from slightly weaker conditions than those assumed in [1].

Before to state our result, we recall some definitions to which we refer throughout the paper.

Let $\mathbb{P} = \mathrm{PG}(n,q)$ be the n–dimensional (desarguesian) projective pace of order q, and $m_1,\dots,m_s$ be s integers such that $0 \le m_1 < \cdots < m_s$ and for any integer h, $1 \le h \le n-1$, let $\mathcal{P}_h$ denote the family of all the h–dimensional subspaces of $\mathbb{P}$. A subset $\mathcal{K}$ of k points of $\mathbb{P}$ is a k–set of *class* $[m_1,\dots,m_s]_h$ if $|\mathcal{K}\cap\pi| \in \{m_1,\dots,m_s\}$ for every $\pi \in \mathcal{P}_h$. Furthermore, if for every $m_j \in \{m_1,\dots,m_s\}$ there is at least one subspace $\pi \in \mathcal{P}_h$ such that $|\mathcal{K}\cap\pi| = m_j$ $\mathcal{K}$ is of *type* $(m_1,\dots,m_s)_h$. The integers $m_1,\dots,m_s$ are the *intersection numbers* of $\mathcal{K}$ (with respect to the dimension h). If $h = 1,2$ one says that $\mathcal{K}$ is of *line* (*plane*)–*class* (*type*) $[m_1,\dots,m_s]_h$ (resp. $(m_1,\dots,m_s)_h$).

Let $\mathcal{K}$ be a subset of points of $\mathbb{P}$, a line (plane) intersecting $\mathcal{K}$ in exactly i points is called i–line (plane). If $i = 1$ a 1-line (plane) is called *tangent*. An *external line* is a 0–line.

An *Hermitian arc* (or *unital* of $PG(2,q^2)$ is a set of points of $\mathrm{PG}(2,q^2)$ of size q^3+1 and of line–type $(1,q+1)_1$.

Let π be a plane of $\mathrm{PG}(2,q^2)$ and $\mathcal{H}$ be a Hermitian arc in π, and V a point not in π. A *Hermitian cone* is the set of points of $\mathrm{PG}(3,q^2)$ which is the union of the lines through V and any point of $\mathcal{H}$.

We are going to prove the following result.

Theorem 1.1. *Let q be a prime power and $s \ge q^2+1$ be an integer. A set $\mathcal{K}$ of points of $\mathrm{PG}(3,q^2)$ of plane-type $(s,(s-1)q+1,(s-1)q+s)_2$, such that*

(i) any line intersecting $\mathcal{K}$ in at least $q+2$ points is contained in $\mathcal{K}$,

(ii) if π is an $((s-1)q+1)$–plane with no line contained in $\mathcal{K}$, then at each point of $\pi\cap\mathcal{K}$ there is at least one tangent line,

is a Hermitian cone of $\mathrm{PG}(3,q^2)$.

Let us end this section by recalling the statement of the result in [1] referred to above.

Theorem 1.2 (Aguglia 2019 [1])**.** Let $\mathcal{S}$ be a surface of degree $q+1$ of $\mathrm{PG}(3,q^2)$. If $\mathcal{S}$ is of plane–type $(q^2+1,q^3+1,q^3+q^2+1)_2$ then $\mathcal{S}$ is a cone projecting a Hermitian curve in a plane π from a point V not in π.

In this theorem, (q^3+1)–planes not containing (q^2+1)–lines, for $q \neq 2$ are nonsingular Hermitian curves and so Hermitian arcs and therefore Theorem 1.1 shows that it is not necessary to assume that all the planar slices of the set are algebraic curves.

2. The proof

Throughout this section, the number of i–lines, $(1 \le i \le q^2+1)$, in a plane is denoted with b_i. For $i \in \{q^2+1, q^3+1, q^3+q^2+1\}$ the number of all i–planes of $\mathrm{PG}(3,q)$ and the number of all i–planes passing through a line ℓ of the space are denoted with c_i and $c_i(\ell)$, respectively.

The following Lemmata give the proof of Theorem 1.1.

Lemma 2.1. *$\mathcal{K}$ is of plane–type $(q^2+1, q^3+1, q^3+q^2+1)_2$.*

Proof. If $(s-1)q+s = q^4+q^2+1$, then $q+1|2$, which is not possible. Thus, there is a point of point of π not in $\mathcal{K}$. Then, by Remark 3.2[1] in [7] either $s \le q^3+q^2+1$ or $q=2$ and $15 = (s-1)2+s = 3s-2$. The latter possibilty gives a contradiction and so it follows that $(s-1)q+s \le q^3+q^2+1$. From $s \ge q^2+1$ it follows that $(s-1)q+s = q^3+q^2+1$, $s = q^2+1$ and so $\mathcal{K}$ is of plane–type $(q^2+1, q^3+1, q^3+q^2+1)_2$. □

Lemma 2.2. *There is at least one (q^2+1)–line.*

Proof. Let π be a (q^3+q^2+1)–plane, if there is a point of $\mathcal{K}$, say p, in π not on a (q^2+1)–line, then counting the number of points of $\pi \cap \mathcal{K}$ via the lines of π passing through p gives $q^3+q^2+1 = v_\pi \le 1+(q^2+1)q$, a contradiction. Hence, every point of $\pi \cap \mathcal{K}$ is on a (q^2+1)–line. □

In particular, (q^2+1)–lines do exist and a (q^3+q^2+1)–plane π contains no external line.

Lemma 2.3. *A (q^3+1)–plane which contains no (q^2+1)–line intersects $\mathcal{K}$ in a Hermitian arc, and so contains no external line.*

Proof. Let π be a (q^3+1)–plane with no line contained in $\mathcal{K}$, then every point of $\pi \cap \mathcal{K}$ belongs to exactly one tangent line and q^2 $(q+1)$–lines. Double counting gives $b_1 = q^3+1$ and $(q+1)b_{q+1} = (q^3+1)q^2$, thus $b_{q+1} = (q^2-q+1)q^2 = q^4-q^3+q^2$ and so the slice $\mathcal{K} \cap \pi$ is a set of points of π of size q^3+1 and of line type $(1, q+1)_1$ that is an hermitian arc. □

Lemma 2.4. *There is no external line to $\mathcal{K}$.*

[1] Remark 3.2 in [7]: *Let π_n be a projective plane of square order n and let X be a subset of points of π_n such that*

- *there is at least a point of π_n outside X,*
- *every line ℓ of π_n with $|\ell \cap X| \ge \sqrt{n}+2$ is contained in X,*

then either $|X| \le n\sqrt{n}+n+1$ or $n=4$ (and so π_n is desarguesian) $|X| = 15$ and X is the complement of a set of line–type $(0,2)_1$ in $\mathrm{PG}(2,4)$, that is of a 6–arc.

Proof. Assume on the contrary, that there is an external line, say ℓ. All the planes on ℓ are (q^2+1)–planes, and so $k = |\mathcal{K}| = (q^2+1)^2$. Let t be a tangent line, counting the number of points of $\mathcal{K}$ via the planes on t, yields

$$q^4 + 2q^2 + 1 = k = 1 + c_{q^2+1}(t)q^2 + c_{q^3+1}(t)q^3 + c_{q^3+q^2+1}(t)(q^3+q^2),$$

$$q^4 + 2q^2 = (q^2 + 1 - c_{q^3+1}(t) - c_{q^3+q^2+1}(t))q^2 + c_{q^3+1}(t)q^3 + c_{q^3+q^2+1}(t)(q^3+q^2),$$

$$q^2 = c_{q^3+1}(t)q^2(q-1) + c_{q^3+q^2+1}(t)q^3,$$

$$1 = c_{q^3+1}(t)(q-1) + c_{q^3+q^2+1}(t)q,$$

thus $c_{q^3+1}(t) = 1$, $q = 2$ and $c_{q^3+q^2+1} = 0$.

So, $q = 2$ and no 13–plane contains a tangent line. Let π be a 13–plane, on each point of $\pi \cap \mathcal{K}$ there are at most two 5–lines, and there exists a point, say p_0, of $\pi \cap \mathcal{K}$ on exactly two 5–lines, one 3–line and two 2–lines. Let p_1, p_2 be the other two points of the 3–line on p_0. Since on each point of π there is at least one 5–line, p_1 belongs to at least one 5–line, and since any 5–line not on p_0 has to intersect the two 2–lines on p_0, it follows that on p_2 there is no 5–line, a contradiction. □

Hence, $\mathcal{K}$ is a blocking set[2]. Thus, (q^2+1)–planes intersects $\mathcal{K}$ in exactly one (q^2+1)–line.

If there is no $(q+1)$–line, let π be a (q^3+q^2+1)–plane, then counting its number of points via the lines on a point of π not in $\mathcal{K}$ gives $q^3+q^2+1 \leq (q^2+1)q$, a contradiction. Thus, $(q+1)$–lines do exist.

Lemma 2.5. $|\mathcal{K}| = q^5 + q^2 + 1$.

Proof. Let $k = |\mathcal{K}|$. If π is a (q^2+1)–plane and ℓ is the (q^2+1)–line $\pi \cap \mathcal{K}$, then counting the number of points of $\mathcal{K}$ via the planes on ℓ gives $k \leq q^2 + 1 + q^2 \cdot q^3 = q^5 + q^2 + 1$.

Let ℓ be a $(q+1)$–line, counting the number of points of $\mathcal{K}$ via the planes on ℓ gives

$$\begin{aligned} q + 1 + c_{q^3+1}(\ell)(q^3 - q) + (q^2 + 1 - c_{q^3+1}(\ell))(q^3 + q^2 - q) &= q^5 + q^4 + q^2 + 1 - c_{q^3+1}(\ell)q^2 \\ &= k \leq q^5 + q^2 + 1 \end{aligned}$$

and so

$$c_{q^3+1}(\ell) \geq q^2.$$

It follows that either $c_{q^3+1}(\ell) = q^2+1$ and $k = q^5+1$ or $c_{q^3+1}(\ell) = q^2$ and $k = q^5+q^2+1$. If $k = q^5+1$, any (q^3+q^2+1)–plane contains no $(q+1)$–line, and so if π is a (q^3+q^2+1)–plane and p is a point of $\pi \setminus \mathcal{K}$, then $q^3+q^2+1 \leq (q^2+1)q = q^3+q$, which is not possible. Therefore, $k = q^5+q^2+1$ and on a $(q+1)$–line there passes exactly one (q^3+q^2+1)–plane. □

Lemma 2.6. *Any line intersects* $\mathcal{K}$ *in* $1, q, q+1$ *or* q^2+1 *points.*

[2] Note that now we are in the conditions of Theorem 1.2 in [3]. However, we prefer to give an independent proof of our result.

Proof. Let ℓ be an h–line, with $2 \leq h \leq q$, then on ℓ there is no (q^2+1)–plane. So,

$$q^5 + q^2 + 1 = h + c_{q^3+1}(\ell)(q^3 + 1 - h) + (q^2 + 1 - c_{q^3+1})(\ell))(q^3 + q^2 + 1 - h),$$

$$c_{q^3+1}(\ell) + h = q^2 + q + 1,$$

but

$$q^2 + 1 + h \geq c_{q^3+1}(\ell) + h = q^2 + q + 1,$$

and so $h = q$ and $c_{q^3+1}(\ell) = q^2 + 1$.

Hence every line of PG$(3, q^2)$ intersects $\mathcal{K}$ in $1, q, q+1$ or q^2+1 points.

□

By the above proof, it follows that all the planes through a q–line are (q^3+1)–planes.

The basic equations for k–sets of points of PG$(3, q^2)$ of plane–type $(q^2+1, q^3+1, q^3+q^2+1)_2$ are

$$\begin{aligned} c_{q^2+1} + c_{q^3+1} + c_{q^3+q^2+1} =&(q^2+1)(q^4+1),\\ (q^2+1)c_{q^2+1} + (q^3+1)c_{q^3+1} + (q^3+q^2+1)c_{q^3+q^2+1} =&k(q^4+q^2+1),\\ q^2(q^2+1)c_{q^2+1} + q^3(q^3+1)c_{q^3+1} + (q^3+q^2)(q^3+q^2+1)c_{q^3+q^2+1} =&k(k-1)(q^2+1). \end{aligned}$$

Being $k = q^5+q^2+1$ it follows that $c_{q^2+1} = q^3+1$, $c_{q^3+1} = q^6$ and $c_{q^3+q^2+1} = q^4-q^3+q^2$.

Lemma 2.7. *A (q^3+1)–plane contains at most one (q^2+1)–line.*

Proof. Let π be a (q^3+1)–plane, and assume on the contrary that it contains at least two (q^2+1)–line, say ℓ and ℓ_1. Let p be a point of $\ell_1 \setminus \{\ell_1 \cap \ell\}$. All lines on p contained in π intersect $\mathcal{K}$ in at least q points, so the usual counting on p gives $q^3+1 = |\pi \cap \mathcal{K}| \geq q^2+1+q^2(q-1) = q^3+1$, that is in π on p there is exactly one (q^2+1)–line and q^2 q–lines. Thus, in π on each point of $\ell_1 \setminus \{\ell_1 \cap \ell\}$ there are exactly q^2 q–lines and one (q^2+1)–line, namely ℓ_1. Hence, $b_q \geq q^4$. Since all the planes on a q–line are (q^3+1)–planes, it follows that $c_{q^3+1} \geq b_q \cdot q^2 + 1 \geq q^4 \cdot q^2 + 1$, a contradiction being $c_{q^3+1} = q^6$. □

Let ℓ be a (q^2+1)–line, and let x and y be the number of (q^2+1)–planes and of (q^3+1)–planes on ℓ, respectively. Counting the number of points of $\mathcal{K}$ via the lines on ℓ gives

$$\begin{aligned} q^5 + q^2 + 1 =&q^2 + 1 + y(q^3 - q^2) + (q^2 + 1 - x - y)q^3,\\ q^3 =&xq^3 + yq^2, \end{aligned}$$

that is $q = xq + y$. So, either $x = 0$ and $y = q$ or $x = 1$ and $y = 0$. It follows that on a (q^2+1)–line there is at most one (q^2+1)–plane. Let $\mathcal{L}$ be the set of all the (q^2+1)–lines contained into exactly one (q^2+1)–plane. Double countings give $|\mathcal{L}| = c_{q^2+1} = q^3+1$.

Lemma 2.8. *No (q^3+1)–plane contains a (q^2+1)–line.*

Proof. Let π be a (q^3+1)–plane, and assume on the contrary that it contains at least one (q^2+1)–line, say ℓ. By the previous Lemma, ℓ is the unique (q^2+1)–line contained in π. It follows that on every point of $\pi \cap \mathcal{K} \setminus \ell$ there are only q and $q+1$–lines. If ℓ_1 is a (q^2+1)–line different from ℓ and contained in a (q^2+1)–plane, it intersects π in a point of ℓ, otherwise there is no (q^2+1)–plane on ℓ_1.

Since $|\mathcal{L}|= q^3+1$ there are at least q^3 (q^2+1)–lines different from ℓ_1. Let ℓ_2 a line of $\mathcal{L} \setminus \ell_1$, it has to intersect both ℓ and ℓ_1 since there is one (q^2+1)–plane on ℓ_2. If ℓ_2 intersect ℓ in a point different from $\ell \cap \ell_1$, from Lemma 2.7 it follows that the three lines ℓ, ℓ_1 and ℓ_2 belong to a (q^3+q^2+1)–plane having no tangent lines. Thus, the usual counting argument on the point $\ell \cap \ell_1$ in this plane gives $q^3+q^2+1 \geq q^2+1+q^2+(q^2-1)q = q^3+2q^2-q+1$, a contradiction.

Hence all the (q^2+1)–lines of $\mathcal{L}$ are concurrent in a point of ℓ, say p. Thus, on p there are at least the q^3+1 (q^2+1)–lines of $\mathcal{L}$ and the line ℓ. Counting the number of points of $\mathcal{K}$ via the lines on p gives $|\mathcal{K}| \geq q^2+1+(q^3+1)q^2 = q^5+2q^2+1$, a contradiction, and the statement is proved. □

Since a (q^3+1)–plane with no (q^2+1)–line intersects $\mathcal{K}$ in an Hermitian arc, it follows that there are no q–lines.

Let π be a (q^3+q^2+1)–plane, and let ℓ be a (q^2+1)–line contained in π. Let p be a point of $\pi \cap \mathcal{K} \setminus \ell$ and let ℓ_1 a (q^2+1)–line of π on p. Let p_0 be the point $\ell \cap \ell_1$. Let $x_0 (\leq q^2-1)$ be the number of $(q+1)$–lines of π on p_0. Assume that there is no tangent line on p_0. Counting the number of points of $\pi \cap \mathcal{K}$ via the lines passing through p_0 gives

$$q^3+q^2+1 = 1 + x_0 q + (q^2+1-x_0)q^2,$$

that is

$$x_0(q-1) = q^2(q-1),$$

and so $x_0 = q^2$, a contradiction since $x_0 \leq q^2-1$.

Thus in π, and so in every (q^3+q^2+1)–plane there is at least one tangent line. So in every (q^3+q^2+1)–plane all the (q^2+1)–lines are concurrent in a point on which there are all the tangent lines to $\mathcal{K}$ of the plane, too. If ℓ is a $(q+1)$–line of a (q^3+q^2+1)–plane, since on each of its point there is a (q^2+1)–line and since these lines have to be concurrent we have that the number of all the (q^2+1)–lines of a (q^3+q^2+1) plane is $(q+1)$.

So, in every (q^3+q^2+1)–plane, the (q^2+1)–lines are part of a pencil containing no $(q+1)$–line.

Let ℓ be a (q^2+1)–line and π_0 be a (q^3+1)–plane. Let p be the point of $\mathcal{K}$ in which ℓ intersects π_0. Through ℓ there are exactly one (q^2+1)–plane, say π_1 and q^2 (q^3+q^2+1)–planes π_i, $i = 2, \ldots, q^2+1$. The plane π_1 intersects $\pi_0 \cap \mathcal{K}$ in the only tangent line at p, the planes π_i intersects $\pi_0 \cap \mathcal{K}$ in the remaining q^2 $(q+1)$–lines.

Any other (q^2+1)–line, say ℓ_1 intersects ℓ, otherwise there is no (q^2+1)–plane on ℓ_1 and so counting the number of points of $\mathcal{K}$ through the planes on ℓ_1 gives $q^5+q^2+1 = k \geq q^2+1+(q^2+1)q^3$.

All the (q^2+1)–lines different from ℓ are concurrent in a point V of ℓ, otherwise let ℓ_1 and ℓ_2 be two (q^2+1)–lines intersecting ℓ in two different points. Then ℓ_1 and ℓ_2 are disjoint and so skew. Hence on each of them there pass only $(q^3 + q^2 + 1)$ planes, and so counting the number of points of $\mathcal{K}$ via the planes on one of the ℓ_i, $i = 1, 2$, gives a number bigger than $q^5 + q^2 + 1$, a contradiction. Thus, there is a point V on ℓ such that on it there are $(q^3 + 1)$ $(q^2 + 1)$–lines and all the other lines are tangent ones.

Hence, $\mathcal{K}$ is the Hermitian cone with vertex V and base $\pi \cap \mathcal{K}$, and the proof is complete.

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