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On-line and Non-line Weighted Generalized Fibonomial Sums

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ABSTRACT

In this paper we consider some new weighted and alternating weighted generalized Fibonomial sums and the corresponding $q-$forms. A generalized form of weight sequences which contains squares in subscripts is discussed for the first time in the literature. The main key to get success in sums is an ability to change one sum into another that is simpler in some way. Thus, in order to prove these sums by doing some manipulations and tricks, our approach is to use classical $q-$analysis, in particular a formula of *Rothe*, a version of *Cauchy binomial theorem* and *Gauss identity*.

Keywords: Gaussian binomial coefficient, Fibonomial coefficient, Sums, q-binomial theorem

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1. Introduction

Binomial coefficients and their generalizations occur frequently in combinatorics, number theory, and discrete mathematics. There are many generalizations of the binomial coefficients in the literature. One of them is the sequential generalization, i.e. replacing the natural numbers by the terms of an arbitrary sequence (a_n) of real or complex numbers. A generalization which is obtained by choosing n^{th} Fibonacci number F_n instead of a_n is known as *Fibonomial coefficients*. For another well-known generalization of binomial coefficients, let q be a variable, and let $a_n = 1 + q + q^2 + \cdots + q^n$. Then we get the

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$q-$*binomial coefficient*, which is known to be a polynomial in q with nonnegative integer coefficients (a Gaussian polynomial). $q-$binomial coefficients have very rich properties and many of the properties of binomial coefficients can be proved more easily by using these coefficients. Both Fibonomial coefficients and $q-$binomial coefficients are interested in by several authors and so their various properties have been found. During this study, we will frequently use the relationships between Fibonomial coefficients and $q-$binomial coefficients.

For $n \geq m \geq 1$ the Fibonomial coefficient is defined by

$$nk_F := \frac{F_1F_2\ldots F_n}{(F_1F_2\ldots F_k)(F_1F_2\ldots F_{n-k})},$$

with $n0_F = nn_F = 1$ where F_n is the n^{th} Fibonacci number. For $n \geq 2$, Falcon and Plaza [4, 5] define two second order linear recurrences

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \qquad U_0 = 0,\ U_1 = 1, \\ V_n &= pU_{n-1} + V_{n-2}, \qquad V_0 = 2,\ V_1 = p, \end{aligned}$$

and named these sequences as $k-$Fibonacci numbers and $k-$Lucas numbers by taking k instead of p, respectively. The Binet forms of these numbers and their $q-$forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n\left(1+q^n\right) \quad ,$$

where α and β are the roots of the characteristic polynomial of the recurrences and $q = \frac{\beta}{\alpha} = -\alpha^{-2}$ so that $\alpha = \frac{\mathbf{i}}{\sqrt{q}}$. Using the sequence $\{U_n\}$, for $n \geq k \geq 1$, the generalized Fibonomial coefficient is defined by

$$nk_U := \frac{U_1U_2\ldots U_n}{(U_1U_2\ldots U_k)(U_1U_2\ldots U_{n-k})},$$

with $n0_U = nn_U = 1$. If we take the indices in a linear arithmetic progression, we obtain the generalized Fibonomial coefficients

$$nk_{U,m} := \frac{U_mU_{2m}\ldots U_{nm}}{(U_mU_{2m}\ldots U_{km})(U_mU_{2m}\ldots U_{(n-k)m})},$$

for a nonnegative integer m. The usual Fibonomial coefficients nk_F can be obtained by taking $m = p = 1$, and when $m = 1$ the coefficient $nk_{U,m}$ turns into the generalized Fibonomial coefficients nk_U. As in binomial coefficients, it is surprising that these quantities will always take integer values. The Fibonomial coefficients appear in several places in the literature (for more details, we refer to [2, 6, 17, 7]).

Throughout the paper, the set of natural numbers is denoted as usual $\mathbb{N}$. The $q-$Pochhammer symbol reads as $(x;y)_0 = 1$ and $(x;y)_n = (1-x)(1-xy)\cdots(1-xy^{n-1})$ for two indeterminates x and y and $n \in \mathbb{N}$. Then for $n, k \in \mathbb{N}$ the generalized Gaussian binomial coefficients are given by

$$nk_{x,y} := \frac{(x;y)_n}{(x;y)_k(x;y)_{n-k}},$$

with $nk_{x,y} = 0$ for $k < 0$ or $k > n$ which become the usual $q-$binomial coefficients nk_q for $x = y$.

In the literature there exists several sums involving Gaussian $q-$binomial coefficients with weight functions. Also some sums including Fibonomial coefficients are evaluated. By taking $q = \beta/\alpha$, we can see that there exists a correspondence between these two classes of sums and hence using an appropriate convenience gives us that we can evaluate one class of sums from another. Thus our approach is essentialy based on these connections, that is

$$nk_{U,m} = \alpha^{mk(n-k)} nk_{q^m}.$$

Fibonomial coefficients and $q-$binomial coefficients have very strong relationships because they can be easily converted to each other. In this way, while an identity associated with Fibonomial coefficients is proved, it is written in the form of $q-$binomial coefficient and proof is made accordingly. Also a proven $q-$binomial identity is true not only for a specific selection of q, but also for all real or complex values.

We recall some well known identities related to the $q-$identities. *Gauss identity* is given as

$$\sum_{k=0}^{n} (-1)^k 2nk_q = \prod_{k=1}^{n} (1 - q^{2k-1}).$$

Then for a nonnegative integer m, we have

$$\sum_{k=0}^{n} (-1)^k 2nk_{q^m} = \prod_{k=1}^{n} (1 - q^{m(2k-1)}). \tag{1}$$

A version of *Cauchy binomial theorem* is stated as

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} nk_q x^k = (x;q)_n = \prod_{k=1}^{n} (1 + xq^k),$$

and *Rothe's formula* is given by

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} nk_q x^k = (x;q)_n = \prod_{k=0}^{n-1} (1 - xq^k),$$

see [1].

Now we recall some well-known results about the sums involving Fibonomial coefficients from the current literature. These sums are computed explicitly by writing everything in terms of q and using the Cauchy binomial theorem and Rothe's formula.

• In [10, 16], the authors consider some Fibonomial sums with weights generalized Fibonacci and Lucas numbers.

• In [11], some variations of $q-$Dixon identity which have results in terms of Fibonomial sums are examined.

• In [12, 18, 14, 13], the authors are interested in the sums with terms finite products of generalized Fibonacci and Lucas numbers and squares of Fibonomial coefficients. An example can be given as

$$\sum_{k=0}^{2n+1}(-1)^k q^{k^2-2kn-3k}(1-q^{2k})^2 2n+1k_q^2 = 2(-1)^{n+1}q^{-n^2-2n-2}\frac{(1+q)\left(1-q^{2n+1}\right)^2}{1+q^{2n}}2n+1n_{q^2}.$$

They give a systematic approach to evaluate these kinds of sums. In [15], sums with a new kind of coefficients are examined. They consider the coefficients as products of two Gaussian $q-$binomial coefficients with a parametric rational weight function. Also some applications to Fibonomial sums are given. To compute these sums, the partial fraction decomposition technique is used.

• In [8], a class of sums of triple aerated Fibonomial coefficients with generalized Fibonacci number coefficients are studied.

• In [3], quadratic sums of Gaussian $q-$binomial coefficients with two additional parameters are evaluated. These results include various known results on square sums of the Gaussian $q-$binomial coefficients when the parameters are specialized.

• In [9, 21, 22, 23, 24], various weighted Fibinomial sums are calculated.

• In [19, 20], various divisibility properties of Fibonomial coefficients are considered.

In this paper, we will usually deal with the following types of sums:

$$\sum_k a_{m,n,k}an+bk_{U,m}, \quad \sum_k (-1)^k a_{m,n,k}an+bk_{U,m}, \quad \text{and} \quad \sum_k (-1)^k a_{m,n,n^2,k,k^2}an+bk_{U,m},$$

where a, b are integers. The first of the above sums will be called as *on-line weighted*, the second is called as *on-line alternate weighted* and the third is called as *non-line alternate weighted* sum. In particular, a generalized version of the sum of the third type will be given for the first time in the literature.

In the paper, inspired by some of the previous results and earlier partial $q-$binomial sums formulæ, we shall derive some interesting new kinds of generalized Fibonomial sums with generalized Fibonacci and Lucas numbers weighted. We compute these sums by using Cauchy binomial theorem or Rothe's formula after converting them into forms involving the Gaussian $q-$binomial coefficients. These steps can be seen by the following diagram:

$$\sum_k a_{m,n,k}an+bk_{U,m} \quad \xrightarrow{\text{Convert to } q-\text{form}} \quad \sum_k a_{m,n,k}(q)an+bk_{q^m}$$

$$\nwarrow \qquad \swarrow$$

$$f(m,n,q)$$

Obtained closed form

To summarize, we will present the following situations in this paper:

- Sums of the Gaussian $q-$binomial coefficients.
- Partial sums of the Gaussian $q-$binomial coefficients.
- New weighted sums containing square subscripts of generalized Fibonacci and Lucas numbers which will be given for the first time in this paper.
- New weighted sums of the generalized Fibonomial coefficients.
- New $q-$identities for readers' convenience.

2. Sums: with the Exact Closed Forms

We give our main results in this section. We find identities of several sums. To prove the identities, the technique is to translate everything in terms of a variable q, and then to use Rothe's identity and Cauchy Binomial theorem from classical $q-$calculus.

2.1. Non-line weighted sums

We first consider the sums with non-line weighted. The following theorem gives some identities in this kind.

Theorem 2.1. *Let n and m be nonnegative integers. Then we have*

(i) For n is odd

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m}U_{m(k-n)^2} = 2(p^2+4)^{\frac{n-1}{2}}\prod_{k=1}^{n}U_{m(2k-1)},$$

and for n is even

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m}V_{m(k-n)^2} = 2(p^2+4)^{\frac{n}{2}}\prod_{k=1}^{n}U_{m(2k-1)}.$$

(ii) $\sum\limits_{k=0}^{4n}(-1)^k 4nk_U V_{(2n-k)^2} = 2(p^2+4)^n \frac{U_{8n-2}}{U_{4n-1}}\prod\limits_{k=1}^{2n-2}U_{2k+1}.$

(iii) $\sum\limits_{k=0}^{4n}(-1)^k 4nk_U U_{(2n-k)^2} = 0.$

(iv) $\sum\limits_{k=0}^{4n+2}(-1)^k 4n+2k_U V_{(2n+1-k)^2} = 0.$

(v) For n is odd,

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m}V_{m(k^2-2nk)} = 2(p^2+4)^{\frac{n+1}{2}}(-1)^{(m+1)}U_{mn^2}\prod_{k=1}^{n}U_{m(2k-1)},$$

and for n is even,

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m}V_{m(k^2-2nk)} = (p^2+4)^{\frac{n}{2}}V_{mn^2}\prod_{k=1}^{n}U_{m(2k-1)}.$$

(vi) Let m be an even integer. Then

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m}U_{m(k^2-2nk)} = (p^2+4)^{\frac{n-1}{2}}V_{mn^2}\prod_{k=1}^{n}U_{m(2k-1)},$$

for n is odd, and

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m}U_{m(k^2-2nk)} = -(p^2+4)^{\frac{n}{2}}U_{mn^2}\prod_{k=1}^{n}U_{m(2k-1)},$$

for n is even.

(vii) Let m be an odd integer. Then

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m} U_{m(k^2-2nk)} = -(p^2+4)^{\frac{n-1}{2}} V_{mn^2} \prod_{k=1}^{n} U_{m(2k-1)},$$

for n is odd, and

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m} U_{m(k^2-2nk)} = -(p^2+4)^{\frac{n}{2}} U_{mn^2} \prod_{k=1}^{n} U_{m(2k-1)},$$

for n is even.

Proof. We will give the proof of the identity (i). Other identities can be similarly shown. To prove this, we use Gauss identity given in (1). Replacing q by α/β, we find that (1) reduces to

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m} \beta^{m(k-n)^2} = (-1)^n(\alpha-\beta)^n \prod_{k=1}^{n} U_{m(2k-1)}. \tag{2}$$

Similarly if q is replaced by β/α, we obtain

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m} \alpha^{m(k-n)^2} = (-1)^n(\beta-\alpha)^n \prod_{k=1}^{n} U_{m(2k-1)}. \tag{3}$$

If n is odd, we subtract (2) from (3) to get

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m} U_{m(k-n)^2} = 2(p^2+4)^{\frac{n-1}{2}} \prod_{k=1}^{n} U_{m(2k-1)},$$

and if n is even, we add (2) to (3) to get

$$\sum_{k=0}^{2n}(-1)^k 2nk_{U,m} V_{m(k-n)^2} = 2(p^2+4)^{\frac{n}{2}} \prod_{k=1}^{n} U_{m(2k-1)}.$$

This completes the proof. □

We obtain the following results given by L. Carlitz in "The Fibonacci Quarterly, Advanced problems and solutions, 10(6)(1972), page 630, problem H-202".

Corollary 2.2. For k is odd,

$$\sum_{j=0}^{2k}(-1)^j 2kj F_{(j-k)^2} = 2\cdot 5^{\frac{k-1}{2}} \prod_{j=1}^{k} F_{2j-1},$$

and for k is even,

$$\sum_{j=0}^{2k}(-1)^j 2kj L_{(j-k)^2} = 2\cdot 5^{\frac{k}{2}} \prod_{j=1}^{k} F_{2j-1}.$$

2.2. On-line weighted sums

Now, we will derive some identities for on-line weighted sums.

Theorem 2.3. *Let n and m be nonnegative integers. Then we have*

(i) $$\sum_{k=0}^{2n} 2nk_{U,m} U_{mk} = \begin{cases} 2U_{2mn} \prod_{k=1}^{n-1} V_{mk}^2, & m \text{ is even,} \\ 2U_{2mn} \prod_{k=1}^{n-1} V_{2mk}, & m \text{ is odd.} \end{cases}$$

(ii) $$\sum_{k=0}^{2n} 2nk_{U,m} U_{(2n-1)mk} = \begin{cases} 2U_{\frac{mn(2n-1)}{2}} V_{\frac{mn(2n-1)}{2}} \prod_{k=1}^{2n-1} V_{mk}, & m \text{ is even,} \\ 2\sum_{k=0}^{n} 2n-12k-1_{U,m} U_{(4k-2)mn}, & m \text{ is odd.} \end{cases}$$

(iii) $$\sum_{k=0}^{2n} 2nk_{U,m} V_{mk} = \begin{cases} \prod_{k=1}^{n} V_{mk}^2, & m \text{ is even,} \\ \prod_{k=1}^{n} V_{2mk}, & m \text{ is odd.} \end{cases}$$

(iv) $$\sum_{k=0}^{2n} (-1)^k 2nk_{U,m} U_{(2n-1)mk} = \begin{cases} 0, & m \text{ is even,} \\ 2\sum_{k=0}^{n-1} 2n-12k_{U,m} U_{4nmk}, & m \text{ is odd.} \end{cases}$$

(v) $$\sum_{k=0}^{2n} (-1)^k 2nk_{U,m} V_{mk} = \begin{cases} 0, & m \text{ is even,} \\ 4\prod_{k=1}^{n-1} V_{2mk}, & m \text{ is odd.} \end{cases}$$

Proof. In order to keep this paper within reasonable length, we restricted ourselves to a short selection. Thus we will only prove the first identity of Theorem 2.3. All the other verifications are very similar. To prove the first identity, we first translate everything into $q-$form. We see that the identity

$$\sum_{k=0}^{2n} 2nk_{U,m} U_{mk} = \begin{cases} 2U_{2mn} \prod_{k=1}^{n-1} V_{mk}^2, & m \text{ is even,} \\ 2U_{2mn} \prod_{k=1}^{n-1} V_{2mk}, & m \text{ is odd,} \end{cases}$$

will be

$$\sum_{k=0}^{2n} \left(1-q^{mk}\right) (-1)^{\frac{mk(2n-k+1)}{2}} q^{-\frac{mk(2n-k+1)}{2}} 2nk_{q^m}$$
$$= \begin{cases} 2(1-q^{2mn}) q^{-m\binom{n+1}{2}} \left(-q^m; q^m\right)_{n-1}^2, & m \text{ is even,} \\ 2(-q^m)^{-\binom{n+1}{2}} \left(1-q^{2mn}\right) \left(-q^{2m}; q^{2m}\right)_{n-1}, & m \text{ is odd.} \end{cases}$$

So we will prove this $q-$form. First we separate the sum into two sums, that is

$$\sum_{k=0}^{2n} \left(1-q^{mk}\right) (-1)^{\frac{mk(2n-k+1)}{2}} q^{-\frac{mk(2n-k+1)}{2}} 2nk_{q^m}$$

$$= \sum_{k=0}^{2n} (-1)^{\frac{mk(2n-k+1)}{2}} q^{\frac{mk(k-2n-1)}{2}} 2nk_{q^m} - \sum_{k=0}^{2n} (-1)^{\frac{mk(2n-k+1)}{2}} q^{\frac{mk(k-2n+1)}{2}} 2nk_{q^m}.$$

Let m be even. Then

$$\begin{aligned} &\sum_{k=0}^{2n} (-1)^{\frac{mk(2n-k+1)}{2}} q^{\frac{mk(k-2n-1)}{2}} 2nk_{q^m} - \sum_{k=0}^{2n} (-1)^{\frac{mk(2n-k+1)}{2}} q^{\frac{mk(k-2n+1)}{2}} 2nk_{q^m} \\ &= \sum_{k=0}^{2n} q^{\frac{mk(k-2n-1)}{2}} 2nk_{q^m} - \sum_{k=0}^{2n} q^{\frac{mk(k-2n+1)}{2}} 2nk_{q^m} = L_1 - L_2. \end{aligned}$$

Here

$$L_1 = \sum_{k=0}^{2n} q^{\frac{mk(k-2n-1)}{2}} 2nk_{q^m} = \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} q^{-mk(n+1)} 2nk_{q^m}$$

and

$$L_2 = \sum_{k=0}^{2n} q^{\frac{mk(k-2n+1)}{2}} 2nk_{q^m}.$$

By Cauchy binomial theorem we can write

$$\begin{aligned} L_1 &= \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} q^{-mk(n+1)} 2nk_{q^m} = \prod_{k=1}^{2n} \left(1 + q^{m(k-n-1)}\right) \\ &= 2\left(1 + q^{-mn}\right) \prod_{k=1}^{n-1} \left(1 + q^{mk}\right)\left(1 + q^{-mk}\right) \\ &= 2\left(1 + q^{-mn}\right) \prod_{k=1}^{n-1} q^{-mk} \left(1 + q^{mk}\right)^2 \\ &= 2\left(1 + q^{-mn}\right) q^{-m\binom{n}{2}} \left(-q^m; q^m\right)_{n-1}^2, \end{aligned}$$

and

$$\begin{aligned} L_2 &= \sum_{k=0}^{2n} q^{\frac{mk(k-2n+1)}{2}} 2nk_{q^m} = \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} q^{-mnk} 2nk_{q^m} \\ &= \prod_{k=1}^{2n} \left(1 + q^{m(k-n)}\right) = 2\left(1 + q^{mn}\right) \prod_{k=1}^{n-1} \left(1 + q^{mk}\right)\left(1 + q^{-mk}\right) \\ &= 2\left(1 + q^{mn}\right) \prod_{k=1}^{n-1} q^{-mk} \left(1 + q^{mk}\right)^2 \\ &= 2\left(1 + q^{mn}\right) q^{-m\binom{n}{2}} \left(-q^m; q^m\right)_{n-1}^2. \end{aligned}$$

Hence for m is even, we get

$$\begin{aligned} L_1 - L_2 =& 2\left(1 + q^{-mn}\right) q^{-m\binom{n}{2}} \left(-q^m; q^m\right)_{n-1}^2 - 2\left(1 + q^{mn}\right) q^{-m\binom{n}{2}} \left(-q^m; q^m\right)_{n-1}^2 \\ =& 2(1 - q^{2mn}) q^{-m\binom{n+1}{2}} \left(-q^m; q^m\right)_{n-1}^2. \end{aligned}$$

Now let m be odd. Then the sum will separate into two sums as follows

$$\sum_{k=0}^{2n} \mathbf{i}^{k^2} q^{m\binom{k+1}{2}} q^{-m(n+1)k} \mathbf{i}^{-(2n+1)k} 2nk_{q^m} - \sum_{k=0}^{2n} \mathbf{i}^{k^2} q^{m\binom{k+1}{2}} q^{-mnk} \mathbf{i}^{-(2n+1)k} 2nk_{q^m} = S_1 - S_2.$$

So we will find the sums S_1 and S_2. Now

$$\begin{aligned}
S_1 &= \sum_{k=0}^{2n} \mathbf{i}^{k^2} \mathbf{i}^{-(2n+1)k} q^{m\binom{k+1}{2}} q^{-m(n+1)k} \binom{2n}{k}_{q^m} \\
&= \frac{1+\mathbf{i}}{2} \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} \left(\mathbf{i}^{-(2n+1)} q^{-m(n+1)}\right)^k \binom{2n}{k}_{q^m} \\
&\quad + \frac{1-\mathbf{i}}{2} \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} \left(-\mathbf{i}^{-(2n+1)} q^{-m(n+1)}\right)^k \binom{2n}{k}_{q^m} \\
&= \frac{1+\mathbf{i}}{2} \prod_{k=1}^{2n} \left(1 + \mathbf{i}^{-(2n+1)} q^{m(k-n-1)}\right) \\
&\quad + \frac{1-\mathbf{i}}{2} \prod_{k=1}^{2n} \left(1 - \mathbf{i}^{-(2n+1)} q^{m(k-n-1)}\right) \\
&= \mathbf{i}^{n^2} \left(1 + \mathbf{i}^{-(2n+1)} q^{-mn}\right) \prod_{k=1}^{n-1} \mathbf{i}^{-(2n+1)} q^{-mk} \left(1 + q^{2mk}\right) \\
&\quad + \mathbf{i}^{3n^2} \left(1 - \mathbf{i}^{-(2n+1)} q^{-mn}\right) \prod_{k=1}^{n-1} -\mathbf{i}^{-(2n+1)} q^{-mk} \left(1 + q^{2mk}\right) \\
&= \mathbf{i}^{-n^2+n+1} \left(1 + \mathbf{i}^{-(2n+1)} q^{-mn}\right) q^{-m\binom{n}{2}} \left(-q^{2m}; q^{2m}\right)_{n-1} \\
&\quad + \mathbf{i}^{n^2+3n-1} \left(1 - \mathbf{i}^{-(2n+1)} q^{-mn}\right) q^{-m\binom{n}{2}} \left(-q^{2m}; q^{2m}\right)_{n-1} \\
&= \mathbf{i}^{-n^2+n+1} q^{-m\binom{n}{2}} \left(-q^{2m}; q^{2m}\right)_{n-1} \\
&\quad \times \left[\left(1 + \mathbf{i}^{-(2n+1)} q^{-mn}\right) - \left(1 - \mathbf{i}^{-(2n+1)} q^{-mn}\right)\right] \\
&= 2(-1)^{-\binom{n+1}{2}} q^{-m\binom{n}{2}} q^{-mn} \left(-q^{2m}; q^{2m}\right)_{n-1}.
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \sum_{k=0}^{2n} \mathbf{i}^{k^2} q^{m\binom{k+1}{2}} q^{-mnk} \mathbf{i}^{-(2n+1)k} \binom{2n}{k}_{q^m} \\
&= \frac{1+\mathbf{i}}{2} \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} q^{-mnk} \mathbf{i}^{-(2n+1)k} \binom{2n}{k}_{q^m} \\
&\quad + \frac{1-\mathbf{i}}{2} \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} q^{-mnk} \mathbf{i}^{-(2n+1)k} \binom{2n}{k}_{q^m} \\
&= \frac{1+\mathbf{i}}{2} \prod_{k=1}^{2n} \left(1 + \mathbf{i}^{-(2n+1)} q^{m(k-n)}\right)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1-\mathbf{i}}{2}\prod_{k=1}^{2n}\left(1-\mathbf{i}^{-(2n+1)}q^{m(k-n)}\right) \\
= \;& \mathbf{i}^{n^2}\left(1+\mathbf{i}^{-(2n+1)}q^{mn}\right)\prod_{k=1}^{n-1}\mathbf{i}^{-(2n+1)}q^{-mk}\left(1+q^{2mk}\right) \\
& +\mathbf{i}^{3n^2}\left(1-\mathbf{i}^{-(2n+1)}q^{mn}\right)\prod_{k=1}^{n-1}-\mathbf{i}^{-(2n+1)}q^{-mk}\left(1+q^{2mk}\right) \\
= \;& \mathbf{i}^{-n^2+n+1}\left(1+\mathbf{i}^{-(2n+1)}q^{mn}\right)q^{-m\binom{n}{2}}\left(-q^{2m};q^{2m}\right)_{n-1} \\
& +\mathbf{i}^{n^2+3n-1}\left(1-\mathbf{i}^{-(2n+1)}q^{mn}\right)q^{-m\binom{n}{2}}\left(-q^{2m};q^{2m}\right)_{n-1} \\
= \;& \mathbf{i}^{-n^2+n+1}q^{-m\binom{n}{2}}\left(-q^{2m};q^{2m}\right)_{n-1} \\
& \times\left[\left(1+\mathbf{i}^{-(2n+1)}q^{mn}\right)-\left(1-\mathbf{i}^{-(2n+1)}q^{mn}\right)\right] \\
= \;& 2(-1)^{-\binom{n+1}{2}}q^{-m\binom{n}{2}}q^{mn}\left(-q^{2m};q^{2m}\right)_{n-1}.
\end{aligned}
$$

Therefore the sum that we want to evaluate will be

$$
\begin{aligned}
S_1-S_2 \;=\;& 2(-1)^{-\binom{n+1}{2}}q^{-m\binom{n}{2}}q^{-mn}\left(-q^{2m};q^{2m}\right)_{n-1} \\
& {}_2(-1)^{-\binom{n+1}{2}}q^{-m\binom{n}{2}}q^{mn}\left(-q^{2m};q^{2m}\right)_{n-1} \\
=\;& 2(-1)^{-\binom{n+1}{2}}q^{-m\binom{n}{2}}\left(-q^{2m};q^{2m}\right)_{n-1}\left(q^{-mn}-q^{mn}\right) \\
=\;& 2(-q^m)^{-\binom{n+1}{2}}\left(1-q^{2mn}\right)\left(-q^{2m};q^{2m}\right)_{n-1}.
\end{aligned}
$$

Hence the proof is completed. □

Theorem 2.4. *Let n and m be nonnegative integers. Then*

(i) $$\sum_{k=0}^{2n+1} 2n+1k_{U,m}U_{2mk} = \begin{cases} 2U_{m(2n+1)}\prod_{k=1}^{n-1}V_{mn+2k}\prod_{k=1}^{n-1}V_{2mk}^2, & m \text{ is even,} \\ 2U_{m(2n+1)}V_{m(2n+1)}\prod_{k=1}^{n-1}V_{2mk}, & m \text{ is odd.} \end{cases}$$

(ii) $$\sum_{k=0}^{2n+1} 2n+1k_{U,m}U_{2nmk} = \begin{cases} 2U_{\frac{m(2n+1)}{2}}V_{\frac{m(2n+1)}{2}}\prod_{k=1}^{2n+1}V_{mk}, & m \text{ is even,} \\ 2\sum_{k=0}^{n} 2n2k_{U,m}U_{(2n+1)2mk}, & m \text{ is odd.} \end{cases}$$

(iii) $$\sum_{k=0}^{2n+1} 2n+1k_{U,m}V_{2mk} = \begin{cases} 2V_{m(2n+1)}\prod_{k=0}^{1}V_{mn+mk}\prod_{k=1}^{n-1}V_{mk}^2, & m \text{ is even,} \\ 2(p^2+4)V_{m(2n+1)}^2\prod_{k=1}^{n-1}V_{2mk}, & m \text{ is odd.} \end{cases}$$

(iv) $$\sum_{k=0}^{2n+1}(-1)^k 2n+1k_{U,m}U_{2nmk} = \begin{cases} 0, & m \text{ is even,} \\ -2\sum_{k=1}^{n} 2n2k-1_{U,m}U_{m(2n+1)(2k-1)}, & m \text{ is odd.} \end{cases}$$

(v) $$\sum_{k=0}^{2n+1}(-1)^k 2n+1k_{U,m}V_{2mk} = \begin{cases} 0, & m \text{ is even,} \\ -2V_mV_{m(2n+1)}\prod_{k=1}^{n-1}V_{2mk}, & m \text{ is odd.} \end{cases}$$

Proof. We will only prove the fifth identity of Theorem 2.4. All the other verifications are very similar. Again we translate everything into $q-$form. Then we observe that

$$\sum_{k=0}^{2n+1}(-1)^k 2n+1k_{U,m}V_{2mk} = \begin{cases} 0, & m \text{ is even}, \\ -2V_mV_{m(2n+1)}\prod_{k=1}^{n-1}V_{2mk}, & m \text{ is odd}, \end{cases}$$

can be written in $q-$form as

$$\sum_{k=0}^{2n+1}(-1)^k(-q)^{-\frac{mk(2n-k+3)}{2}}(1+q^{2mk})2n+1k_{q^m}$$
$$= \begin{cases} 0, & m \text{ is even}, \\ 2\mathbf{i}^{n^2+n}q^{-m\binom{n}{2}}q^{-m(n+1)}(1+q^m)(1+q^{m(2n+1)})\left(-q^{2m};q^{2m}\right)_{n-1}, & m \text{ is odd}. \end{cases}$$

Let m be even. Then

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k(-q)^{-\frac{mk(2n-k+3)}{2}}(1+q^{2mk})2n+1k_{q^m}\\
&\quad=\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}(-q)^{-mk(n+1)}\mathbf{i}^{mk(1-k)}2n+1k_{q^m}\\
&\qquad+\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}(-q)^{-mk(n+1)}\mathbf{i}^{mk(1-k)}q^{2mk}2n+1k_{q^m}\\
&\quad=\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}q^{-mk(n+1)}2n+1k_{q^m}\\
&\qquad+\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}q^{mk(1-n)}2n+1k_{q^m}\\
&\quad=\prod_{k=0}^{2n}\left(1-q^{m(k-n-1)}\right)+\prod_{k=0}^{2n}\left(1-q^{m(k+1-n)}\right)\\
&\quad=0.
\end{aligned}$$

Now let m be odd. Then we have

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k(-q)^{-\frac{mk(2n-k+3)}{2}}(1+q^{2mk})2n+1k_{q^m}\\
&\quad=\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}(-q)^{-mk(n+1)}\mathbf{i}^{mk(1-k)}2n+1k_{q^m}\\
&\qquad+\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}(-q)^{-mk(n+1)}\mathbf{i}^{mk(1-k)}q^{2mk}2n+1k_{q^m}\\
&\quad=\sum_{k=0}^{2n+1}(-1)^k\mathbf{i}^{k^2}q^{m\binom{k}{2}}\left(\mathbf{i}^{2n+1}q^{-m(n+1)}\right)^k 2n+1k_{q^m}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{2n+1}(-1)^k\mathbf{i}^{k^2}q^{m\binom{k}{2}}\left(\mathbf{i}^{2n+1}q^{m(1-n)}\right)^k 2n+1k_{q^m}\\
&=S_1+S_2.
\end{aligned}
$$

Here

$$
\begin{aligned}
S_1=&\sum_{k=0}^{2n+1}(-1)^k\mathbf{i}^{k^2}q^{m\binom{k}{2}}\left(\mathbf{i}^{2n+1}q^{-m(n+1)}\right)^k 2n+1k_{q^m}\\
=&\frac{1+\mathbf{i}}{2}\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}\left(\mathbf{i}^{2n+1}q^{-m(n+1)}\right)^k 2n+1k_{q^m}\\
&+\frac{1-\mathbf{i}}{2}\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}\left(-\mathbf{i}^{2n+1}q^{-m(n+1)}\right)^k 2n+1k_{q^m}\\
=&\frac{1+\mathbf{i}}{2}\prod_{k=0}^{2n}\left(1-\mathbf{i}^{2n+1}q^{m(k-n-1)}\right)+\frac{1-i}{2}\prod_{k=0}^{2n}\left(1+\mathbf{i}^{2n+1}q^{m(k-n-1)}\right)\\
=&\mathbf{i}^{n^2}\left(1-\mathbf{i}^{2n+1}q^{-mn}\right)\left(1-\mathbf{i}^{2n+1}q^{-m(n+1)}\right)\\
&\times\prod_{k=1}^{n-1}-\mathbf{i}^{2n+1}q^{-mk}\left(1+q^{2mk}\right)\\
&+\mathbf{i}^{3n^2}\left(1+\mathbf{i}^{2n+1}q^{-mn}\right)\left(1+\mathbf{i}^{2n+1}q^{-m(n+1)}\right)\\
&\times\prod_{k=1}^{n-1}\mathbf{i}^{2n+1}q^{-mk}\left(1+q^{2mk}\right)\\
=&2\mathbf{i}^{n^2+n}q^{-m\binom{n}{2}}q^{-mn}\left(1+q^{-m}\right)\left(-q^{2m};q^{2m}\right)_{n-1}
\end{aligned}
$$

and

$$
\begin{aligned}
S_2=&\sum_{k=0}^{2n+1}(-1)^k\mathbf{i}^{k^2}q^{m\binom{k}{2}}\left(\mathbf{i}^{2n+1}q^{m(1-n)}\right)^k 2n+1k_{q^m}\\
=&\frac{1+\mathbf{i}}{2}\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}\left(\mathbf{i}^{2n+1}q^{m(1-n)}\right)^k 2n+1k_{q^m}\\
&+\frac{1-\mathbf{i}}{2}\sum_{k=0}^{2n+1}(-1)^k q^{m\binom{k}{2}}\left(-\mathbf{i}^{2n+1}q^{m(1-n)}\right)^k 2n+1k_{q^m}\\
=&\frac{1+\mathbf{i}}{2}\prod_{k=0}^{2n}\left(1-\mathbf{i}^{2n+1}q^{m(k-n+1)}\right)+\frac{1-i}{2}\prod_{k=0}^{2n}\left(1+\mathbf{i}^{2n+1}q^{m(k-n+1)}\right)\\
=&\mathbf{i}^{n^2}\left(1-\mathbf{i}^{2n+1}q^{mn}\right)\left(1-\mathbf{i}^{2n+1}q^{m(n+1)}\right)\prod_{k=1}^{n-1}-\mathbf{i}^{2n+1}q^{-mk}\left(1+q^{2mk}\right)\\
&+\mathbf{i}^{3n^2}\left(1+\mathbf{i}^{2n+1}q^{-mn}\right)\left(1+\mathbf{i}^{2n+1}q^{-m(n+1)}\right)\\
&\times\prod_{k=1}^{n-1}\mathbf{i}^{2n+1}q^{-mk}\left(1+q^{2mk}\right)
\end{aligned}
$$

$$=2\mathbf{i}^{n^2+n}q^{-m\binom{n}{2}}q^{mn}\left(1+q^{m}\right)\left(-q^{2m};q^{2m}\right)_{n-1}.$$

Thus the result follows as

$$\begin{aligned} S_1+S_2 &= 2\mathbf{i}^{n^2+n}q^{-m\binom{n}{2}}q^{-mn}\left(1+q^{-m}\right)\left(-q^{2m};q^{2m}\right)_{n-1}+ \\ & \quad 2\mathbf{i}^{n^2+n}q^{-m\binom{n}{2}}q^{mn}\left(1+q^{m}\right)\left(-q^{2m};q^{2m}\right)_{n-1} \end{aligned}$$

$$= 2\mathbf{i}^{n^2+n}q^{-m\binom{n}{2}}\left[q^{-mn}+q^{-m(n+1)}+q^{mn}+q^{m(n+1)}\right]\left(-q^{2m};q^{2m}\right)_{n-1}. \qquad \square$$

3. Outlines: Other Identities in $q-$form

We give here the complete list of $q-$binomial versions of the identities given in Section 2. Let n and m be both nonnegative integers. Identities given in Theorem 2.3 can be converted into $q-$forms as follows:

Identity (ii) can be converted as

$$\sum_{k=0}^{2n}(-q)^{-\frac{mk(4n-k-1)}{2}}\left(1-q^{mk(2n-1)}\right)2nk_{q^m}$$
$$=\begin{cases} 2(-q)^{-\frac{mn(2n-1)}{2}}\left(1-q^{mn(2n-1)}\right)\prod\limits_{k=1}^{2n-1}(-q)^{-\frac{mk}{2}}(1+q^{mk}), & m \text{ is even,} \\ 2\sum\limits_{k=0}^{n}(-q)^{-m(2k-1)(2n-k)}\left(1-q^{mn(4k-2)}\right)2n-12k-1_{q^m}, & m \text{ is odd.} \end{cases}$$

Identity (iii) can be converted as

$$\sum_{k=0}^{2n}(-q)^{-\frac{mk(2n-k+1)}{2}}(1+q^{mk})2nk_{q^m}=\begin{cases} \prod\limits_{k=1}^{n}q^{-mk}(1+q^{mk})^2, & m \text{ is even,} \\ \prod\limits_{k=1}^{n}(-1)^k q^{-mk}(1+q^{2mk}), & m \text{ is odd.} \end{cases}$$

Identity (iv) can be converted as

$$\sum_{k=0}^{2n}(-1)^k(-q)^{-\frac{mk(4n-k-1)}{2}}\left(1-q^{mk(2n-1)}\right)2nk_{q^m}.$$

The sum equals 0 for m is even. Otherwise, we have

$$2\sum_{k=0}^{n-1}(-q)^{-mk(4n-2k-1)}\left(1-q^{4mnk}\right)2n-12k_{q^m}.$$

Identity (v) can be converted as

$$\sum_{k=0}^{2n}(-1)^k(-q)^{-\frac{mk(2n-k+1)}{2}}(1+q^{mk})2nk_{q^m}=\begin{cases} 0, & m \text{ is even,} \\ 4\prod\limits_{k=1}^{n-1}(-1)^k q^{-mk}(1+q^{2mk}), & m \text{ is odd.} \end{cases}$$

Identities given in Theorem 2.4 can be converted into $q-$forms as follows:

Identity (i) can be converted as

$$\sum_{k=0}^{2n+1}(-q)^{-\frac{2kmn+3km-mk^2}{2}}\left(1-q^{2mk}\right)2n+1k_{q^m}.$$

If m is even, we have

$$2(-q)^{-\frac{m(2n+1)}{2}}\left(1-q^{m(2n+1)}\right)\prod_{k=1}^{n-1}(-q)^{-\frac{mn+2k}{2}}(1+q^{mn+2k})\prod_{k=1}^{n-1}q^{-2mk}(1+q^{2mk})^2,$$

and if m is odd

$$2(-q)^{-m(2n+1)}\left(1-q^{2m(2n+1)}\right)\prod_{k=1}^{n-1}(-q)^{-mk}(1+q^{2mk}).$$

Identity (ii) can be converted as

$$\sum_{k=0}^{2n+1}(-q)^{-\frac{mk(4n-k+1)}{2}}\left(1-q^{2mnk}\right)2n+1k_{q^m}$$
$$=\begin{cases}2(-q)^{-\frac{m(2n+1)}{2}}\left(1-q^{m(2n+1)}\right)\prod_{k=1}^{2n+1}(-q)^{-\frac{mk}{2}}(1+q^{mk}), & m \text{ is even},\\ 2\sum_{k=0}^{n}(-q)^{-mk(4n-2k+1)}\left(1-q^{2mk(2n+1)}\right)2n2k_{q^m}, & m \text{ is odd}.\end{cases}$$

Identity (iii) can be converted as

$$\sum_{k=0}^{2n+1}(-q)^{-\frac{mk(2n+3-k)}{2}}(1+q^{2mk})2n+1k_{q^m}.$$

If m is even we have

$$2q^{-(2mn+m)}(1+q^{mn})(1+q^{m(n+1)})(1+q^{m(2n+1)})\prod_{k=1}^{n-1}q^{-mk}(1+q^{mk})^2,$$

and if m is odd

$$-2(p^2+4)q^{-m(2n+1)}(1+q^{m(2n+1)})^2\prod_{k=1}^{n-1}(-q)^{-mk}(1+q^{2mk}).$$

Identity (iv) can be converted as

$$\sum_{k=0}^{2n+1}(-1)^k q^{-\frac{mk(4n+1-k)}{2}}\left(1-q^{2nmk}\right)2n+1k_{q^m}.$$

The sum equals 0 for m is even. Otherwise, we have

$$-2\sum_{k=1}^{n}(-q)^{-m(2k-1)(2n-k+1)}\left(1-q^{m(2n+1)(2k-1)}\right)2n2k-1_{q^m}.$$

4. Conclusion

In this paper we considered some sums which are called as on-line and non-line and we obtained some identities in $q-$form. For special $q-$values we obtain some Fibonomial sums identities and these results can be proved by using $q-$Zeilberger algorithm in Mathematica or Mapple program versions. Furthermore, sums with negative subscripts can be also considered in future studies. Our starting point for this case will be the following identity. The identity

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}\alpha^{k-kn}{n \brack k}_F=-1,$$

is obtained by using the identity

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\frac{k(k-1)}{2}}{n \brack k}_q=1.$$

So some results can also be obtained similar to this sum.

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Conflict of interest

The authors declare no conflict of interest.

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