ARS COMBINATORIA 161 (2024) 175–185

Ars Combinatoria
www.combinatorialpress.com/ars

Combinatorial Press

Sharp Lower Bound for Randić index of Trees with Fixed Roman Domination Number

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ABSTRACT

Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E. The Randić index of graph G is the value $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$, where $d(u)$ and $d(v)$ refer to the degree of the vertices u and v. We obtain a lower bound for the Randić index of trees in terms of the order and the Roman domination number, and we characterize the extremal trees for this bound.

Keywords: Randić index, tree, Roman domination number

2020 Mathematics Subject Classification: 05C69, 05C35.

1. Introduction

Let $G = (V, E)$ be a simple, undirected and connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$, respectively. Here, uv represents an edge in the graph G that connects two vertices given by u and v. Moreover, $d(u)$ also known as the degree of vertex u, indicates the number of edges incident to u in graph G. Given that $d(u) = 1$, a vertex u in G is termed a pendant or leaf. In a graph G, the greatest vertex degree is expressed with the notation $\Delta(G)$ (or simply Δ).

The *open neighborhood* of each vertex $v \in V$ denotes the set $N(v) = \{u \in V | uv \in E\}$.

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Received 08 July 2024; accepted 20 December 2024; published 31 December 2024.

DOI: 10.61091/ars161-14

Meanwhile, the *closed neighborhood* denotes the set $N[v] = N(v) \cup \{v\}$. The cycle and the path on n vertices are expressed as C_n and P_n, respectively. Assume T is a tree. Then, the longest path between the two leaves defines a tree's diameter. Provided that $v_1, v_2, \ldots, v_d$ denotes a path in which the diameter is obtained, we may state that it resembles a diametrical path in T. To designate the forest generated by T via eliminating the vertices of $u_1, u_2, \ldots, u_k$ or the edges $e_1, e_2, \ldots, e_k$ in T, we employ $T - \{u_1, u_2, \ldots, u_k\}$ or $T - \{e_1, e_2, \ldots, e_k\}$. For other notations and terminologies which are not defined here, please refer to West [22].

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V | f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a $1 - 1$ correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. Thus, we will write $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a *Roman dominating function* (RDF) if $V_2 \succ V_0$, where $\succ$ means that the set V_2 dominates the set V_0, i.e., $V_0 \subseteq N[V_2] \setminus V_2$. The weight of f is $f(V) = \sum_{v \in V(G)} f(v) = 2n_2 + n_1$.

The *Roman domination number*, denoted $\gamma_R(G)$ (or γ_R for short), equals the minimum weight of an RDF of G, and we say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is an RDF and $f(V) = \gamma_R(G)$. For more details on Roman domination and its variants, please refer to [7, 6].

First, we have the following essential results.

Lemma 1.1. [7] *For $n \geq 3$, $\gamma_R(P_n) = \lceil \frac{2n}{3} \rceil$.*

Lemma 1.2. [8] *For a tree T with a pendant vertex v,*

$$\gamma_R(T) - 1 \leq \gamma_R(T - \{v\}) \leq \gamma_R(T).$$

Lemma 1.3. [8] *For a tree T and any vertex $v \in V(T)$, $\gamma_R(T) \leq n - d(v) + 1$.*

Graph theory has provided chemists with a variety of useful tools, such as topological indices. A topological index is a numeric quantity from the structural graph of a chemical compound [21]. Among many topological indices, the Randić index is the most widely used in applications to chemistry, especially in QSPR/QSAR investigations [15].

The *Randić* index was introduced by Randić [19] and is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where $d(u)$ and $d(v)$ denote the degrees of the vertices u, $v \in V(G)$ and uv denotes the edge connecting these two vertices.

The first Zagreb index M_1 and the second Zagreb index M_2 of graph G are defined as:

$$M_1(G) = \sum_{v \in V(G)} d^2(v),$$

and

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

The Zagreb indices has been studied extensively, see [5, 8, 17, 18] and references therein.

Numerous researchers have been interested in the connection between domination parameters and topological indices. Bermudo et al. [3] determined the lower and upper bounds of the Randić index of trees in terms of order and the domination number. More recently the upper and lower bounds of the Randić index for trees with a given total domination number are obtained and the corresponding extremal trees were characterized [10, 14]. In [1, 4, 20], authors investigated the sharp lower and upper bounds for the geometric-arithmetic, Zagreb, and Sombor index of a tree, respectively, in terms of the domination number. In [2, 17], bounds for the geometric-arithmetic index and Zagreb index of a tree, respectively, in terms of order and the total domination number were obtained. Very recently, Hasni et al. [9] obtained the upper bound for harmonic index of trees in terms of order and the total domination number.

Ahmad Jamri et al. [11] proposed a lower bound on the first Zagreb index of trees with a given Roman domination number and characterized all extremal trees. Furthermore, the upper bound for Zagreb indices of unicyclic and bicyclic graphs with a given Roman domination number is investigated. In [13], Ahmad Jamri et al. presented a lower bound on the second Zagreb index of trees with n vertices and Roman domination number. The upper bounds on the first and second Zagreb indices of trees with a given Roman domination number were studied in [8, 12].

This paper is a continuation of these studies. Namely, we present a new lower bound of the Randić index in terms of the order and the Roman domination number, and we characterize the extremal trees for that bound.

2. Main Results

We first show the following lemma to simplify the proof of the theorem, giving the lower bound of the Randić index in terms of the order of a tree and the given Roman domination number.

Lemma 2.1. *For $n > 2$, suppose that*

$$h(n,k) = \left(\sqrt{2}(n-k+1) + \frac{k-2}{2}\right)\left(\frac{1}{\sqrt{2n-k-2}} - \frac{1}{2n-k}\right) - \frac{\sqrt{2}}{\sqrt{2n-k-2}}.$$

Then $h(n,k+1) < h(n,k)$ and $h(n,k) < h(n+1,k)$ for any $2 \le k \le n-2$.

Proof. We show that $h(n,k+1) < h(n,k)$ for any $2 \le k \le n-2$. Since

$$\begin{aligned} h(n,k) &= \left(\sqrt{2}(n-k) + \frac{k-1}{2}\right)\left(\frac{1}{\sqrt{2n-k-3}} - \frac{1}{\sqrt{2n-k-1}}\right) - \frac{\sqrt{2}}{\sqrt{2n-k-3}} \\ &= \left(\sqrt{2}(n-k+1) + \frac{k-2}{2}\right)\left(\frac{1}{\sqrt{2n-k-3}} - \frac{1}{\sqrt{2n-k-1}}\right) - \frac{\sqrt{2}}{\sqrt{2n-k-3}} \end{aligned}$$

$$+\frac{1}{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)-\sqrt{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right).$$

Then $h(n,k+1)<h(n,k)$ if and only if

$$\begin{aligned}&\left(\sqrt{2}(n-k+1)+\frac{k-2}{2}\right)\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)-\frac{\sqrt{2}}{\sqrt{2n-k-3}}\\&\quad+\frac{1}{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)-\sqrt{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)\\&<\left(\sqrt{2}(n-k+1)+\frac{k-2}{2}\right)\left(\frac{1}{\sqrt{2n-k-2}}-\frac{1}{\sqrt{2n-k}}\right)-\frac{\sqrt{2}}{\sqrt{2n-k-2}}.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\left(\sqrt{2}(n-k+1)+\frac{k-2}{2}\right)\Bigg[\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)\\&-\left(\frac{1}{\sqrt{2n-k-2}}-\frac{1}{\sqrt{2n-k}}\right)\Bigg]\\&<\sqrt{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)-\frac{1}{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)\\&\quad+\left(\frac{\sqrt{2}}{\sqrt{2n-k-3}}-\frac{\sqrt{2}}{\sqrt{2n-k-2}}\right).\end{aligned}$$

On the other hand,

$$\sqrt{2}(n-k+1)+\frac{k-2}{2}=(2n-k)-(2-\sqrt{2})n+(\frac{3}{2}-\sqrt{2})k+(\sqrt{2}-1)\leq 2n-k.$$

Thus, it is enough to check that

$$\begin{aligned}&(2n-k)\left[\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)-\left(\frac{1}{\sqrt{2n-k-2}}-\frac{1}{\sqrt{2n-k}}\right)\right]\\&<(\sqrt{2}-\frac{1}{2})\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-1}}\right)+\sqrt{2}\left(\frac{1}{\sqrt{2n-k-3}}-\frac{1}{\sqrt{2n-k-2}}\right),\end{aligned}$$

which is obtained by considering the function

$$\begin{aligned}g(x)=&\left(\sqrt{2}-\frac{1}{2}\right)\left(\frac{1}{\sqrt{x-3}}-\frac{1}{\sqrt{x-1}}\right)+\sqrt{2}\left(\frac{1}{\sqrt{x-3}}-\frac{1}{\sqrt{x-2}}\right)\\&-x\left[\left(\frac{1}{\sqrt{x-3}}-\frac{1}{\sqrt{x-1}}\right)-\left(\frac{1}{\sqrt{x-2}}-\frac{1}{\sqrt{x}}\right)\right],\end{aligned}$$

and this fact that $g(x)$ is a positive function for any $x>2$. Now we prove that $h(n,k)<h(n+1,k)$. We have

$$h(n+1,k)=\left(\sqrt{2}(n-k+2)+\frac{k-1}{2}\right)\left(\frac{1}{\sqrt{2n-k}}-\frac{1}{\sqrt{2n-k+2}}\right)-\frac{\sqrt{2}}{\sqrt{2n-k}},$$

then $h(n,k) < h(n+1,k)$ if and only if

$$\left(\sqrt{2}(n-k+1)+\frac{k-2}{2}\right)\left(\frac{1}{\sqrt{2n-k-2}}-\frac{1}{\sqrt{2n-k}}\right)-\frac{\sqrt{2}}{\sqrt{2n-k-2}}$$
$$<\left(\sqrt{2}(n-k+1)+\frac{k-2}{2}\right)\left(\frac{1}{\sqrt{2n-k}}-\frac{1}{\sqrt{2n-k+2}}\right)-\frac{\sqrt{2}}{\sqrt{2n-k}}$$
$$+\sqrt{2}\left(\frac{1}{\sqrt{2n-k}}-\frac{1}{\sqrt{2n-k+2}}\right),$$

for any $n > k+2$. This inequality is equivalent to

$$\left(\sqrt{2}(n-k+1)+\frac{k-2}{2}\right)\left[\left(\frac{1}{\sqrt{2n-k-2}}-\frac{1}{\sqrt{2n-k}}\right)-\left(\frac{1}{\sqrt{2n-k}}-\frac{1}{\sqrt{2n-k+2}}\right)\right]$$
$$<\sqrt{2}\left(\frac{1}{\sqrt{2n-k}}-\frac{1}{\sqrt{2n-k+2}}\right)+\sqrt{2}\left(\frac{1}{\sqrt{2n-k-2}}-\frac{1}{\sqrt{2n-k}}\right).$$

Since $\sqrt{2}(n-k+1)+\frac{k-2}{2} \le 2n-k$, the function

$$g(x)=\sqrt{2}\left(\frac{1}{\sqrt{x-2}}-\frac{1}{\sqrt{x+2}}\right)-x\left(\frac{1}{\sqrt{x-2}}-\frac{2}{\sqrt{x}}+\frac{1}{\sqrt{x+2}}\right),$$

is a positive function for any $x \ge 2$. Consequently, the proof is completed. □

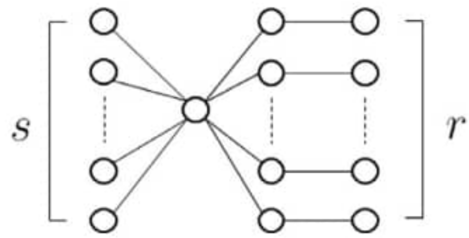

Fig. 1. The graph $T_{r,s}$

Lemma 2.2. *Let T be a tree shown in Figure 1 of order n and a Roman domination number γ_R. Then*

$$R(T)=\frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}}+\frac{\gamma_R-2}{2\sqrt{2}}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right).$$

Proof. Assume that $T = T_{r,s}$ shown in Figure 1. In this tree, we have $n = s+2r+1$, $\gamma_R = 2(r+1)$. Since $r = \frac{\gamma_R}{2}-1$ and $s = n-\gamma_R+1$, we obtain

$$R(T_{r,s})=\frac{s}{\sqrt{r+s}}+\frac{r}{\sqrt{2}}+\frac{r}{\sqrt{2(r+s)}}=\frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}}+\frac{\gamma_R-2}{2\sqrt{2}}+\frac{\gamma_R-2}{2\sqrt{2n-\gamma_R}}$$
$$=\frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}}+\frac{\gamma_R-2}{2\sqrt{2}}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right).$$

□

Theorem 2.3. *Let T be a tree of order n and a Roman domination number γ_R. Then*

$$R(T) \geq \frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}} + \frac{\gamma_R-2}{2\sqrt{2}}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right),$$

with equality if and only if $T = T_{r,s}$ shown in Figure 1.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of graph T. The result is proved by induction on the number of vertices. To simplify the computations, we denote

$$f(n,\gamma_R) = \frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}} + \frac{\gamma_R-2}{2\sqrt{2}}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right).$$

For $n = 3$, $R(P_3) = \sqrt{2} = f(3,2)$. If $n = 4$, then $R(P_4) = \sqrt{2} + \frac{1}{2} = f(4,3)$ and $R(S_4) = \sqrt{3} = f(4,2)$. Therefore, we suppose that $n \geq 5$ and the result holds for any trees of order $n-1$. We will check if it is true for the tree with n vertices. Let $\Delta = 2$. Then $T \simeq P_n$. Using Lemma 1.1, $\gamma_R(P_n) = \frac{2n+r}{3}$ if $n \equiv r \ (mod\ 3)$. One can easily check that $R(P_n) = \sqrt{2} + \frac{n-3}{2} > f(n, \frac{2n+r}{3})$ for $n \geq 5$ and $r = 0, 1, 2$.

Now suppose that $\Delta \geq 3$ and let $v_1, v_2, \ldots, v_d$ be a diameter path in T. Suppose that $d(v_2) = i \geq 2$ and $N(v_2) = \{v_1, v_3, u_1, \ldots, u_{i-2}\}$, $d(v_3) = j \geq 2$ and $N(v_3) = \{v_2, v_4, w_1, \ldots, w_{j-2}\}$. We consider $T' = T - \{v_1\}$. Using Lemma 1.2, we study two following cases.

Case 1: Let $\gamma_R(T') = \gamma_R(T)$. In such a case, we get

$$\begin{aligned}
R(T) =& R(T') + \frac{1}{\sqrt{i}} + (i-2)\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i-1}}\right) + \frac{1}{\sqrt{ij}} - \frac{1}{\sqrt{(i-1)j}} \\
\geq& f(n-1,\gamma_R) + \frac{1}{\sqrt{i}} + (i-2)\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i-1}}\right) + \frac{1}{\sqrt{ij}} - \frac{1}{\sqrt{(i-1)j}} \\
=& f(n,\gamma_R) + \sqrt{2}(n-\gamma_R+1)\left(\frac{1}{\sqrt{2n-\gamma_R-2}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) - \frac{\sqrt{2}}{\sqrt{2n-\gamma_R-2}} \\
&+ \frac{\gamma_R-2}{2}\left(\frac{1}{\sqrt{2n-\gamma_R-2}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) + \frac{1}{\sqrt{i}} + (i-2)\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i-1}}\right) \\
&+ \frac{1}{\sqrt{ij}} - \frac{1}{\sqrt{(i-1)j}} \\
=& f(n,\gamma_R) + \left(\sqrt{2}(n-\gamma_R+1) + \frac{\gamma_R-2}{2}\right)\left(\frac{1}{\sqrt{2n-\gamma_R-2}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) \\
&- \frac{\sqrt{2}}{\sqrt{2n-\gamma_R-2}} + \frac{1}{\sqrt{i}} - (i-2)\left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{ij}} - \frac{1}{\sqrt{(i-1)j}}.
\end{aligned}$$

Let $n = i+2$. In this case, $T \simeq T_{1,s}$ shown in Figure 1. Thus, using Lemma 2.2, the result holds. So, we suppose that $n \geq i+3$. From Lemma 1.3, we have $\gamma_R \leq n-i+1$. Since $n \geq i+3$ and $\gamma_R \leq n-i+1$, by applying Lemma 2.1, we obtain

$$\left(\sqrt{2}(n-\gamma_R+1) + \frac{\gamma_R-2}{2}\right)\left(\frac{1}{\sqrt{2n-\gamma_R-2}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) - \frac{\sqrt{2}}{\sqrt{2n-\gamma_R-2}}$$

$$\geq \left(\sqrt{2}i + \frac{n-i-1}{2}\right)\left(\frac{1}{\sqrt{n+i-3}} - \frac{1}{\sqrt{n+i-1}}\right) - \frac{\sqrt{2}}{\sqrt{n+i-3}}$$
$$\geq \sqrt{2}(i+1)\left(\frac{1}{\sqrt{2i}} - \frac{1}{\sqrt{2i+2}}\right) - \frac{1}{\sqrt{i}}.$$

Therefore, we have

$$\begin{aligned}
R(T) \geq & f(n,\gamma_R) + \left(\sqrt{2}(n-\gamma_R+1) + \frac{\gamma_R-2}{2}\right)\left(\frac{1}{\sqrt{2n-\gamma_R-2}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) \\
& - \frac{\sqrt{2}}{\sqrt{2n-\gamma_R-2}} + \frac{1}{\sqrt{i}} - (i-2)\left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{ij}} - \frac{1}{\sqrt{(i-1)j}} \\
\geq & f(n,\gamma_R) + \sqrt{2}(i+1)\left(\frac{1}{\sqrt{2i}} - \frac{1}{\sqrt{2i+2}}\right) - \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{i}} - (i-2)\left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}}\right) \\
& + \frac{1}{\sqrt{2i}} - \frac{1}{\sqrt{2(i-1)}} \\
= & f(n,\gamma_R) + (i+1)\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}}\right) - (i-2)\left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{2i}} - \frac{1}{\sqrt{2(i-1)}} \\
> & f(n,\gamma_R),
\end{aligned}$$

for any $i \geq 2$. Hence, $R(T) > f(n,\gamma_R)$.

Case 2: Let $\gamma_R(T') = \gamma_R(T) - 1$. In such a case, we have $d(v_2) = 2$. If $n = j+3$, then $T \simeq T_{2,s}$ where $s \geq 1$. Hence using Lemma 2.2, the equality holds. Therefore, we suppose that $n \geq j+4$, for the tree $T' = T - \{v_1\}$, we get

$$\begin{aligned}
R(T) = & R(T') - \frac{1}{\sqrt{j}}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \\
\geq & f(n-1,\gamma_R-1) - \frac{1}{\sqrt{j}}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \\
= & f(n,\gamma_R) + \left(\sqrt{2}(n-\gamma_R+1) + \frac{\gamma_R-2}{2}\right)\left(\frac{1}{\sqrt{2n-\gamma_R-1}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) \\
& - \frac{1}{2\sqrt{2}}\left(1 + \frac{\sqrt{2}}{\sqrt{2n-\gamma_R-1}}\right) - \frac{1}{\sqrt{j}}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \\
= & f(n,\gamma_R) + \left(\sqrt{2}(n-\gamma_R+1) + \frac{\gamma_R-2}{2}\right)\left(\frac{1}{\sqrt{2n-\gamma_R-1}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) \\
& - \frac{1}{2\sqrt{2n-\gamma_R-1}} - \frac{1}{\sqrt{j}}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}.
\end{aligned}$$

Using Lemma 2.1 and since $\gamma_R \leq n-j+1$ and $n \geq j+4$, we get

$$\begin{aligned}
R(T) \geq & f(n,\gamma_R) + \left(\sqrt{2}(n-\gamma_R+1) + \frac{\gamma_R-2}{2}\right)\left(\frac{1}{\sqrt{2n-\gamma_R-1}} - \frac{1}{\sqrt{2n-\gamma_R}}\right) \\
& - \frac{1}{2\sqrt{2n-\gamma_R-1}} - \frac{1}{\sqrt{j}}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}
\end{aligned}$$

$$\geq f(n,\gamma_R)+\left(\sqrt{2}j+\frac{n-j-1}{2}\right)\left(\frac{1}{\sqrt{n+j-2}}-\frac{1}{\sqrt{n+j-1}}\right)$$
$$-\frac{1}{2\sqrt{n+j-2}}-\frac{1}{\sqrt{j}}\left(1-\frac{1}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}$$
$$\geq f(n,\gamma_R)+\left(\sqrt{2}j+1\right)\left(\frac{1}{\sqrt{2j+2}}-\frac{1}{\sqrt{2j+3}}\right)$$
$$-\frac{1}{2\sqrt{2j+2}}-\frac{1}{\sqrt{j}}\left(1-\frac{1}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}},$$

where for any $j \geq 2$, $R(T) > f(n, \gamma_R)$. □

Remark 2.4. Lu and Zhu in [16] obtained a lower bound for Randić index for any tree T with n vertices as $R(T) \geq \frac{3}{2} - \frac{1}{2(n-2)}$. Let

$$f(n,\gamma_R)=\frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}}+\frac{\gamma_R-2}{2\sqrt{2}}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right).$$

Since $\gamma_R \geq \frac{2n}{\Delta+1}$ [7] in which $\Delta \leq n-2$ is the maximum degree, we have $\gamma_R \geq \frac{2n}{n-1}$. Also, in [6], for any connected graph G of order $n \geq 3$ is proven that $2 \leq \gamma \leq \lfloor \frac{4n}{5} \rfloor \leq \frac{4n}{5}$. Hence from $\gamma_R \geq \frac{2n}{n-1}$ and $2 \leq \gamma \leq \frac{4n}{5}$, we get

$$f(n,\gamma_R)\geq\frac{\sqrt{2}(\frac{n}{5}+1)}{\sqrt{2n-\gamma_R}}+\frac{n}{\sqrt{2}(n-1)}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right)$$
$$\geq\frac{n+5}{5\sqrt{n-1}}+\frac{n}{\sqrt{2}(n-1)}\left(1+\frac{1}{\sqrt{n-1}}\right).$$

We consider $f(x)=\frac{x+5}{5\sqrt{x-1}}+\frac{x}{\sqrt{2}(x-1)}\left(1+\frac{1}{\sqrt{x-1}}\right)-\frac{3}{2}+\frac{1}{2(x-2)}$ which is a positive function. Therefore, we have

$$f(n,\gamma_R)\geq\frac{n+5}{5\sqrt{n-1}}+\frac{n}{\sqrt{2}(n-1)}\left(1+\frac{1}{\sqrt{n-1}}\right)$$
$$\geq\frac{3}{2}-\frac{1}{2(x-2)}.$$

Remark 2.5. Bermudo et al. [3] presented a lower bound in terms of order and domination number as follows

$$R(T)\geq\frac{n-2\gamma+1}{\sqrt{n-\gamma}}+\frac{\gamma-1}{\sqrt{2}}\left(1+\frac{1}{\sqrt{n-\gamma}}\right).$$

Since $\gamma \leq \gamma_R \leq 2\gamma$, we have

$$f(n,\gamma_R)=\frac{\sqrt{2}(n-\gamma_R+1)}{\sqrt{2n-\gamma_R}}+\frac{\gamma_R-2}{2\sqrt{2}}\left(1+\frac{\sqrt{2}}{\sqrt{2n-\gamma_R}}\right)$$

$$= \frac{n - \gamma_R + 1}{\sqrt{n - \frac{\gamma_R}{2}}} + \frac{\frac{\gamma_R}{2} - 1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{n - \frac{\gamma_R}{2}}}\right)$$
$$\geq \frac{n - 2\gamma + 1}{\sqrt{n - \gamma}} + \frac{\gamma - 1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{n - \gamma}}\right).$$

3. Conclusions

This paper is devoted to the investigation of the relationship between the Randić index and the Roman domination number of trees. More precisely, we established a lower bound for the Randić index of trees in terms of the order and the Roman domination number, and all trees attaining the equality are characterized. This result solves part of Problem 1 in [14] for the lower bound case. For the next study, researchers can work on the upper bound for the Randić index of trees in terms of the order and the Roman domination number. One can also work on the relationship between other degree-based topological indices, such as the geometric-arithmetic index, harmonic index and Sombor index, with Roman domination number.

Acknowledgements

The authors would like to express their sincere gratitude to the referees for their careful and insightful suggestions, which improved the paper.

Conflict of interest

The authors declare no conflict of interest.

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