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The 2-burning Number of a Graph

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ABSTRACT

We study a discrete-time model for the spread of information in a graph, motivated by the idea that people believe a story when they learn of it from two different origins. Similar to the burning number, in this problem, information spreads in rounds and a new source can appear in each round. For a graph G, we are interested in $b_2(G)$, the minimum number of rounds until the information has spread to all vertices of graph G. We are also interested in finding $t_2(G)$, the minimum number of sources necessary so that the information spreads to all vertices of G in $b_2(G)$ rounds. In addition to general results, we find $b_2(G)$ and $t_2(G)$ for the classes of spiders and wheels and show that their behavior differs with respect to these two parameters. We also provide examples and prove upper bounds for these parameters for Cartesian products of graphs.

Keywords: Burning number, Graph burning, Discrete-time processes, Firefighter problem

2020 Mathematics Subject Classification: 05C40; 05C45; 05C75.

1. Introduction

The concept of graph burning, introduced by Bonato et al. [5] is a deterministic, discrete-time model for the spread of social contagion on a graph. A social network can be modeled by a graph in which the vertices represent people and the edges represent relationships. For example, on a graph whose vertices correspond to Facebook users, edges may represent users who are "Facebook friends". Information, such as gossip, rumors, or memes, can spread from vertex to vertex over time along edges of the graph and we model the process

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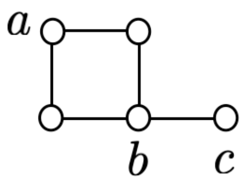

Fig. 1. A graph H with $b_2(H) = 3$.

using discrete time-steps called *rounds*. Such information may not stem from a single source vertex; there may be a number of sources that appear over time in the graph. Authors often refer to vertices as "unburned" (unaware of the information) and "burned" (aware of the information) since rumors can appear to spread swiftly like fire. We will instead use colors to depict these states: uncolored for unburned and blue for burned. The discrete-time spread of information also mimics the spread of fire in the Firefighter Problem [7], although the latter involves agents try to block the spread of fire on a graph.

In 2016, Bonato et al. [6] suggested a generalized burning process; and such a study was formally initialized in 2021 by Li et al. [8]. For a finite graph G, there are two possible states for a vertex: uncolored or blue. Initially, at round 0, all vertices are uncolored. At round j for $j \geq 1$, one or both of the following occur: (i) an uncolored vertex is selected as a source and colored blue, and (ii) every uncolored vertex that had at least r blue neighbors at round $j-1$ is colored blue.

If $r = 1$, the process describes the original burning model of [6], which has been studied extensively; see the survey [4]. One of the main questions surrounding the original burning model is how quickly the influence or contagion can propagate through the graph. For a finite graph G, Li et al. [8] denote by $b_r(G)$, the minimum number of rounds after which every vertex is colored blue and since G is finite, the parameter is well-defined. Li et al. [8] named the parameter, the *generalized burning number*, but because this label does not reference r and $b_r(G)$ can change for different values of r on a fixed graph G, we refer to the parameter as the *r-burning number* of graph G. For example, for the graph H in Figure 1 it is straightforward to show $b_2(H) = 3$ and this can be achieved by selecting a, b, c as sources in rounds $1, 2, 3$ respectively. If these same sources are selected in the order a, c, b, it requires four rounds for all vertices to turn blue.

The generalized burning process is related to r-neighbor bootstrap percolation, which is another way to model the spread of infection or information among vertices in a graph. In both processes, we color infected vertices blue, but in a generalized burning process there is at most one source vertex per round, whereas in bootstrap percolation, the source vertices are all selected at the start. More precisely, for a graph G and positive integer r, the process of r-neighbor bootstrap percolation is the following: during round 0, a set of vertices $A \subseteq V(G)$ turns blue; and during round $t > 0$, every vertex that has at least r blue neighbors at round $t-1$, becomes blue. For a set A in graph G, *percolation* of G occurs if every vertex of G is eventually blue and in this case, A is called a *percolating set*. Consequently, for a fixed positive integer r, the cardinality of a minimum size percolating set on graph G provides a lower bound for the r-burning number. We discuss the relationship of the r-burning number to r-neighbor bootstrap percolation further in

Section 5. For more background on r-neighbor bootstrap percolation, see [1, 9, 11].

In this paper, we focus on $b_2(G)$, the 2-burning number of graph G, motivated by the idea that people often believe a rumor when they hear it from two different people. Li et al. [8] determined the 2-burning number for some families of graphs including paths, cycles, and complete bipartite graphs. They also provided some preliminary bounds on $b_2(G)$ for a general graph G. We also study $t_2(G)$, the 2-burning source number of a graph G, which we define as the minimum number of sources so that all vertices of G are blue after $b_2(G)$ rounds. The parameter $t_2(G)$ provides a middle ground between the 2-burning number of G and the minimum size percolating set for 2-neighbor bootstrap percolation on G.

The rest of the paper is organized as follows. In Section 2 we provide the foundational definitions as well as results about subgraphs, coronas and joins. Section 3 focuses on two families of graphs: spiders and wheels. We find exact values for $b_2(G)$ and $t_2(G)$ when G is a spider or wheel and show that these parameters are within 1 for spiders but can be arbitrarily far apart for wheels. We turn to Cartesian products in Section 4 and provide examples and prove upper bounds for both the 2-burning number and the 2-burning source number. In our concluding section, we discuss connections between $b_2(G)$, $t_2(G)$, and the minimum size of an associated percolating set. We also pose a few open questions.

2. Definitions and preliminary results

We assume all graphs to be finite and connected.

2.1. Definitions

Algorithm 2-burning (below) gives a precise description of the 2-burning process on a graph with a specified sequence of sources. Note that the algorithm will always terminate since our graphs are finite. With the formal description of the 2-burning process given in Algorithm 2-burning, we can now introduce our fundamental definitions.

Definition 2.1. A 2-*burning sequence* s for a graph G is a sequence $(s_1, s_2, s_3, \ldots, s_m)$ of vertices of G for which all vertices are blue when Algorithm 2-burning terminates.

Definition 2.2. If $s = (s_1, s_2, s_3, \ldots, s_m)$ is a 2-burning sequence for graph G then the length of s, denoted by $\mathrm{len}(s)$, is m and $\mathrm{rd}(s)$ is the output of Algorithm 2-burning, that is, the number of rounds until all vertices of G are blue.

Returning to the graph in Figure 1, if $s = (a, b, c)$ then $\mathrm{len}(s) = 3$ and $\mathrm{rd}(s) = 3$; however, if $s' = (a, c, b)$ then $\mathrm{len}(s') = 3$ and $\mathrm{rd}(s') = 4$.

Definition 2.3. The 2-*burning number* for graph G, denoted by $b_2(G)$, is the minimum value of $\mathrm{rd}(s)$, taken over all 2-burning sequences for G. A 2-burning sequence that achieves this minimum is called *optimal*. The 2-*burning source number* for graph G is the

Algorithm 1: Algorithm 2-burning

Input: A graph G and a sequence $(s_1, s_2, \ldots, s_m)$ of vertices of G (called *sources*).
Output: Either the number of rounds until all vertices are blue, or a report that there will always be an uncolored vertex.

Initialize: At round 0, all vertices are uncolored.
while *Initialize $j = 1$ (or Let $j = 1$, or something equivalent) not all vertices are blue* **do**
  At round j:
  if *$j \leq m$ and s_j is uncolored* **then**
    Color the source vertex s_j blue.
  foreach *uncolored vertex v with at least two blue neighbors at round $j - 1$* **do**
    Color v blue.
  if *all vertices are blue* **then**
    Terminate and return j (the number of rounds until all vertices are blue).
  if *$j > m$ and no vertices turned blue in round j* **then**
    Terminate and report that there will always be an uncolored vertex.
  Increment j.

minimum length of an optimal 2-burning sequence for G and is denoted by $t_2(G)$.

For example, selecting any two distinct source vertices provides an optimal 2-burning sequence of minimum length for the complete graph K_n, but when $n \geq 3$, it takes one additional round for all vertices to turn blue. Thus $b_2(K_n) = 3$ and $t_2(K_n) = 2$ for $n \geq 3$. By definition, $t_2(G) \leq b_2(G)$ for all graphs G.

2.2. Subgraphs and vertices of degree one and two

For many real-valued functions defined on graphs (e.g., chromatic number, maximum degree, girth) the function value decreases for induced subgraphs. However, the same is not true for the 2-burning number. For example, the path P_7 is an induced subgraph of the path P_{12} and also of the wheel W_8. We will see in Theorems 2.8 and 3.3 that $b_2(P_7) = 5$, $b_2(P_{12}) = 7$ and $b_2(W_8) = 4$, and indeed these theorems show that the gap between the 2-burning number of a graph and that of an induced subgraph can be made arbitrarily large in either direction using paths and wheels. However, for *spanning* subgraphs, we can prove an inequality.

Lemma 2.4. *If G and H are connected graphs and H is a subgraph of G with $|V(G)| = |V(H)|$, then $b_2(G) \leq b_2(H)$ and $t_2(G) \leq t_2(H)$.*

Proof. Since the edges of H are present in G, any 2-burning sequence for H will also be a 2-burning sequence for G. Moreover, using any 2-burning sequence s for H, the number of rounds until all vertices are blue in H is at least as large as when s is used as a 2-burning sequence for G. Thus $b_2(G) \leq b_2(H)$ and $t_2(G) \leq t_2(H)$. □

We next consider the role of leaves (i.e., vertices of degree one) and vertices of degree two when creating a 2-burning sequence. Observe that there are only two ways for any vertex to turn blue, either it is a source or it has two blue neighbors. Since leaves have only one neighbor, they must be sources, which we record as an observation below. In Lemma 2.6 we show that for adjacent vertices of degree two, at least one must be a source.

Observation 2.5. *If v is a leaf of graph G and s is a 2-burning sequence for G then $v \in s$. Consequently, if graph G has k leaves, then $b_2(G) \geq k$.*

Lemma 2.6. *If G is a graph and v_1 and v_2 are adjacent vertices in G with $\deg(v_1) = \deg(v_2) = 2$, then at least one of v_1 and v_2 must be a source in any 2-burning sequence.*

Proof. Consider a 2-burning sequence for G and assume for a contradiction neither v_1 nor v_2 are sources. The vertex v_1 can only turn blue after its two neighbors are blue so v_1 turns blue after v_2. By symmetry, v_2 turns blue after v_1, which is a contradiction. □

The next lemma is helpful in identifying graphs G with $b_2(G) > t_2(G)$.

Lemma 2.7. *Let G be a graph and let s be a 2-burning sequence for G. If every vertex of s is adjacent to a degree two non-source vertex then $\mathrm{rd}(s) \geq \mathrm{len}(s) + 1$.*

Proof. By definition we know $\mathrm{rd}(s) \geq \mathrm{len}(s)$ so suppose for a contradiction that $\mathrm{rd}(s) = \mathrm{len}(s)$. Let v be the last source vertex in s. Then v has a degree two non-source neighbor x, and x cannot turn blue until at least one round after v turns blue, a contradiction. □

In [8] the authors find the 2-burning number for paths and cycles using direct arguments.

Theorem 2.8. [8] *If P_n is the path on n vertices and C_n is the cycle on n vertices then $b_2(P_n) = b_2(C_n) = \lceil \frac{n}{2} \rceil + 1$.*

The lower bound $b_2(C_n) \geq \lceil \frac{n}{2} \rceil + 1$ follows from combining the results in Lemmas 2.6 and 2.7 and equality holds because selecting every other vertex (when n is even) and one additional source vertex (when n is odd) yields an optimal 2-burning sequence of length $\lceil \frac{n}{2} \rceil$. The result for paths then follows using Lemma 2.4.

As above, C_n has an optimal 2-burning sequence of length $\lceil \frac{n}{2} \rceil$ for all $n \geq 3$. The path P_n has an optimal 2-burning sequence of length $\lceil \frac{n}{2} \rceil + 1$ when n is even, and of length $\lceil \frac{n}{2} \rceil$ when n is odd. We record this below.

Observation 2.9. *If P_n is the path on n vertices and C_n is the cycle on n vertices then $t_2(P_n) = \lceil \frac{n}{2} \rceil + 1$ and $t_2(C_n) = \lceil \frac{n}{2} \rceil$ when n is even and $t_2(P_n) = t_2(C_n) = \lceil \frac{n}{2} \rceil$ when n is odd.*

2.3. Coronas and joins

Observation 2.5 states that for any 2-burning sequence of a graph G, every leaf must be a source vertex. This motivates us to next consider a family of graphs with many leaves. For any graph G on $n \geq 2$ vertices, the corona $G \circ K_1$ is the graph on $2n$ vertices obtained by joining a leaf to each vertex of G.

Proposition 2.10. *For any graph G on $n \geq 1$ vertices, $b_2(G \circ K_1) = t_2(G \circ K_1) = n + 1$.*

Proof. Observation 2.5 implies that each of the n leaves of $G \circ K_1$ must be a source vertex in a 2-burning sequence for $G \circ K_1$. Each vertex of G is adjacent to only one leaf of $G \circ K_1$, thus there must also be a source vertex from among the vertices of G. Therefore, $n + 1 \leq t_2(G \circ K_1) \leq b_2(G \circ K_1)$.

It remains to show $b_2(G \circ K_1) \leq n + 1$. Let v_1 be any vertex of G and label the rest of the vertices $v_2, \ldots, v_n$, according to a breadth-first search rooted at v_1. Let T be the resulting breadth first search tree of G, so v_1 is the root of T and if v_i is the parent of v_j in T then $i < j$. Label the leaves of $G \circ K_1$ as $v'_1, v'_2, \ldots, v'_n$ so that v_i is adjacent to v'_i for $1 \leq i \leq n$ and let $s = (v_1, v'_2, v'_3, v'_4, \ldots, v'_n, v'_1)$.

When we run Algorithm 2-burning on graph $G \circ K_1$ and sequence s, we see that for $t : 2 \leq t \leq n$, vertex v'_t is blue at round t as is the parent of v_t in T. Thus v_t has two blue neighbors after round t and will be blue by round $t + 1$. Therefore, all vertices of $G \circ K_1$ are blue by round $n + 1$ and s is a 2-burning sequence of length $n + 1$ for $G \circ K_1$, proving that $b_2(G \circ K_1) \leq n + 1$. □

While the corona $G \circ K_1$ is a family of graphs with a high number of leaves, we next consider a family on the other end of the spectrum: graph joins. The *join* of graphs G and H is the graph denoted $G \vee H$ that has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

Consider graphs G and H with $|V(G)| \geq 2$ and $|V(H)| \geq 2$. If we select $s_1, s_2 \in V(G)$, then the sequence (s_1, s_2) is a 2-burning sequence for $G \vee H$ because all vertices in the subgraph induced by $V(H)$ turn blue in round 3 and subsequently, all vertices in the subgraph induced by $V(G)$ are blue by round 4. We record this in the following.

Observation 2.11. *Let G and H be graphs on $m \geq 2$ and $n \geq 2$ vertices, respectively. Then $b_2(G \vee H) \leq 4$ and $t_2(G \vee H) \leq 2$.*

When we apply Observation 2.11 to the graphs $G = \overline{K_n}$ and $H = \overline{K_m}$ we get an upper bound of 4 for the 2-burning number of the complete bipartite graph $K_{m,n}$. In [8] the authors proved directly that $b_2(K_{m,n}) = 4$ if $n, m \geq 4$ and $b_2(K_{m,n}) = 3$ if $m \in \{2, 3\}$ or $n \in \{2, 3\}$.

A graph G with universal vertex u is an example of a graph join: $G = K_1 \vee (G - u)$ and Observation 2.11 can be applied. However, for a graph G with universal vertex u, we can also bound the 2-burning number of G by the 1-burning number of $G - u$. We will use this idea later in the proof of Theorem 3.3, but state a more general result next. For a graph G, a set $D \subseteq V(G)$ is *dominating set* if every vertex of G is either in D or has a neighbor in D. We denote by $\gamma(G)$, the minimum cardinality of a dominating set on graph G.

Proposition 2.12. *Let G be a graph and $D \subseteq V(G)$ be a dominating set of cardinality $\gamma(G)$. Then $b_2(G) \leq \gamma(G) + b_1(G - D)$.*

Proof. Let G be a graph and $D \subseteq V(G)$ be a dominating set of cardinality $\gamma(G)$. We choose the vertices of D (in any order) to be the first $\gamma(G)$ source vertices for G. At the end of round $\gamma(G)$, every vertex in $G - D$ is adjacent to a blue vertex. Thus, the process reduces to the 1-burning process on $G - D$. □

3. The classes of spiders and wheels

In this section, we consider two classes of graphs: spiders and wheels. For each, we are able to find the exact value for the 2-burning number and the 2-burning source number. These classes provide an interesting contrast between $b_2(G)$ and $t_2(G)$. For spiders these quantities differ by at most 1, while for wheels they can be arbitrarily far apart.

3.1. Spider Graphs

The spider graph $S_{n_1,n_2,..,n_r}$ consists of a central vertex v^* of degree r, and paths of lengths $n_1, n_2, \ldots, n_r$ emanating from v^*. Thus if we remove v^* from $S_{n_1,n_2,..,n_r}$, the resulting graph consists of the paths $P_{n_1}, P_{n_2}, .., P_{n_r}$. Figure 2 shows four spider graphs. Note that when $r = 1$ or $r = 2$ the spider graph is a path graph, thus we assume $r \geq 3$.

Interestingly, although determining the 1-burning number of a spider graph is NP-hard [3], we next determine the 2-burning number exactly.

Theorem 3.1. *Let G be the spider graph $S_{n_1,n_2,..,n_r}$ where $r \geq 3$ and $n = n_1+n_2+...+n_r+1$ and let k be the number of n_i that are odd. Then $b_2(G) = \lceil \frac{n}{2} \rceil + 1$ if $k \leq 2$ and $b_2(G) = \frac{n+k-1}{2}$ if $k \geq 3$.*

Proof. Let G be the spider graph $S_{n_1,n_2,..,n_r}$ and v^* the central vertex. As discussed above, $G - v^*$ is the graph whose components are the paths with n_i vertices for $1 \leq i \leq r$. We denote by H_i the ith component of $G - v^*$, which is a path with n_i vertices. Let x_i be the vertex of H_i that is adjacent to v^* and y_i be the vertex of H_i that is farthest from v^*. We construct 2-burning sequences for G using four cases that depend on the value of k. In each case below, when n_i is even we choose y_i and every other vertex of H_i starting at y_i to be a source. Thus we have $\frac{n_i}{2}$ sources from H_i for the even paths. We will select $\frac{n_i+1}{2}$ sources from H_i when n_i is odd but the particular sources chosen from these paths will vary in different cases.

First consider the case $k = 0$, which is illustrated for $S_{4,4,4,2,2}$ in Figure 2 (a). In this case, n is odd. In addition to the sources chosen from the H_i, we also select v^* as a source. So the number of sources is $1 + \frac{1}{2}(n_1 + n_2 + \cdots + n_r) = 1 + \frac{n-1}{2} = \lceil \frac{n}{2} \rceil$. Every non-source vertex of G is adjacent to two of these sources, so every vertex turns blue by round $\lceil \frac{n}{2} \rceil + 1$

and thus $b_2(G) \le \lceil \frac{n}{2} \rceil + 1$. In this case, the sources can be arranged in any order.

Next consider the case $k = 1$, which is illustrated for $S_{3,4,4,2}$ in Figure 2(b). In this case, n is even. Without loss of generality, let H_1 be the odd path. If $n_1 = 1$ let y_1 be a source, and if $n_1 \ge 3$, let y_1 and its neighbor z_1 be sources, as well as every other vertex starting at z_1. In addition to the sources chosen from the H_i, we also select v^* as a source. So the number of sources is $1 + \frac{1}{2}(1 + n_1 + n_2 + \cdots + n_r) = 1 + \frac{n}{2} = \lceil \frac{n}{2} \rceil + 1$. Arrange the sources so that v^* is the first and y_1 is the last. The remaining sources can appear in any order. Every non-source vertex is adjacent to two sources, so all vertices will turn blue and we have constructed a 2-burning sequence for G. Furthermore, the only neighbor of the last source y_1 is itself a source vertex, thus all vertices are blue by round $\lceil \frac{n}{2} \rceil + 1$, and hence $b_2(G) \le \lceil \frac{n}{2} \rceil + 1$.

Our third case is $k = 2$, which is illustrated for $S_{5,3,4,2}$ in Figure 2 (c). In this case, n is odd. Without loss of generality, let H_1 and H_2 be the odd paths, and for these paths (as well as the even ones) let y_i and and every other vertex starting at y_i be a source. This means that x_1 and x_2 will be chosen as sources. In this case we do *not* select v^* to be a source, but we do select x_1 and x_2 as the first two sources so v^* will turn blue in round 3. The number of sources is $\frac{1}{2}(2 + n_1 + n_2 + \cdots + n_r) = \frac{1}{2}(1 + n) = \lceil \frac{n}{2} \rceil$. Since $k = 2$ and $r \ge 3$, we know $n \ge 5$ so $\lceil \frac{n}{2} \rceil \ge 3$ and there are a sufficient number of rounds for v^* to turn blue. After all the sources turn blue, every non-source is adjacent to two blue vertices, so in one additional round, all the vertices become blue. Thus $b_2(G) \le \lceil \frac{n}{2} \rceil + 1$.

Finally, we consider the case $k \ge 3$, which is illustrated for $S_{5,3,3,3,3,3,4}$ in Figure 2(d). Without loss of generality, assume that n_i is odd for $1 \le i \le 3$. For the odd path H_1, if $n_1 = 1$ let y_1 be a source, and if $n_1 \ge 3$, let y_1 and its neighbor z_1 be sources, as well as every other vertex starting at z_1. For the remaining odd paths (as well as the even ones), let y_i and every other vertex starting at y_i be a source. This means that x_2 and x_3 will be sources. In this case we do *not* select v^* to be a source, but we do select x_2 and x_3 as the first two sources so v^* will turn blue in round 3 and we select y_1 to be the last source. The remaining sources can appear in any order. There are $\frac{n_i}{2}$ sources for each H_i when n_i is even and $\frac{1+n_i}{2}$ sources for each H_i when n_i is odd. Thus the number of source vertices is $\frac{1}{2}(k + n_1 + n_2 + \cdots + n_m) = \frac{1}{2}(n + k - 1)$. Note that this number is an integer since k and n have opposite parity. Every non-source vertex is either adjacent to two sources or to v^* and one source, so all vertices will turn blue and we have constructed a 2-burning sequence for G. Furthermore, the only neighbor of the last source y_1 is itself a source vertex, thus all vertices are blue by round $\frac{1}{2}(n + k - 1)$, hence $b_2(G) \le \frac{1}{2}(n + k - 1)$.

It remains to show that $b_2(G) \ge \lceil \frac{n}{2} \rceil + 1$ for $k \le 2$ and $b_2(G) \ge \frac{1}{2}(n + k - 1)$ for $k \ge 3$. We begin by calculating the minimum number of source vertices in H_i in a 2-burning sequence. Of the vertices of H_i, vertex y_i is a leaf of G and the remaining $n_i - 1$ vertices have degree 2 in G. By Lemma 2.6, at least $\frac{n_i - 2}{2}$ of these degree two vertices must be sources and by Observation 2.5, the leaf y_i must also be a source. Thus there are at least $\frac{n_i}{2}$ vertices in H_i that are sources. Summing over all i, we get at least $\lceil \frac{n-1}{2} \rceil$ sources from $G - v^*$ in any 2-burning sequence of G.

In the first case, $k = 0$, so n is odd and the number of sources from $G - v^*$ in any

2-burning sequence is at least $\lceil \frac{n}{2} \rceil - 1$. If none of the x_i are sources, then v^* must be a source, so there are at least $\lceil \frac{n}{2} \rceil$ sources. However, in this instance, every non-source has degree 2 and every source is adjacent to a non-source, so by Lemma 2.7, the 2-burning sequence requires $\lceil \frac{n}{2} \rceil + 1$ rounds. If exactly one of the x_i are sources, then again v^* must be a source, so there are at least $\lceil \frac{n}{2} \rceil + 1$ sources. Otherwise, at least two of the x_i are sources and again there are at least $\lceil \frac{n}{2} \rceil + 1$ sources. Thus any 2-burning sequence requires at least $\lceil \frac{n}{2} \rceil + 1$ rounds and $b_2(G) \geq \lceil \frac{n}{2} \rceil + 1$.

In the next case, $k = 1$, so n is even and the number of sources from $G - v^*$ in any 2-burning sequence of G is at least $\lceil \frac{n}{2} \rceil$. If only one x_i is a source then v^* must be a source, so in any case there are at least $\lceil \frac{n}{2} \rceil + 1$ sources. Thus any 2-burning sequence requires at least $\lceil \frac{n}{2} \rceil + 1$ rounds and $b_2(G) \geq \lceil \frac{n}{2} \rceil + 1$.

In the case $k = 2$ we again have n odd and we may assume that the odd paths are H_1 and H_2. In any 2-burning sequence there are at least $\frac{1+n_i}{2}$ sources from the odd path H_i for $i = 1, 2$, so the total number of sources from $G - v^*$ is it least $\frac{1}{2}(2 + n_1 + n_2 + \cdots + n_r) = \lceil \frac{1+n}{2} \rceil = \lceil \frac{n}{2} \rceil$. Suppose for a contradiction that there exists a 2-burning sequence s with $\text{len}(s) = \text{rd}(s) = \lceil \frac{n}{2} \rceil$. Consider the last source s_k of s (i.e., $k = \lceil \frac{n}{2} \rceil$). If $s_k = x_1$ or $s_k = x_2$ then v^* does not turn blue until round $\lceil \frac{n}{2} \rceil + 1$, a contradiction. Otherwise, s_k is adjacent to a degree 2 non-source vertex and that vertex does not turn blue until round $\lceil \frac{n}{2} \rceil + 1$, a contradiction. Thus any 2-burning sequence requires at least $\lceil \frac{n}{2} \rceil + 1$ rounds and $b_2(G) \geq \lceil \frac{n}{2} \rceil + 1$.

Finally, we consider the case $k \geq 3$. In any 2-burning sequence for G there are at least $\frac{n_i}{2}$ sources from each H_i when n_i is even and at least $\frac{1+n_i}{2}$ sources from each H_i when n_i is odd. Thus the total number of source vertices is at least $\frac{1}{2}(k + n_1 + n_2 + \cdots + n_r) = \frac{1}{2}(k + n - 1)$ and consequently $b_2(G) \geq \frac{1}{2}(k + n - 1)$. □

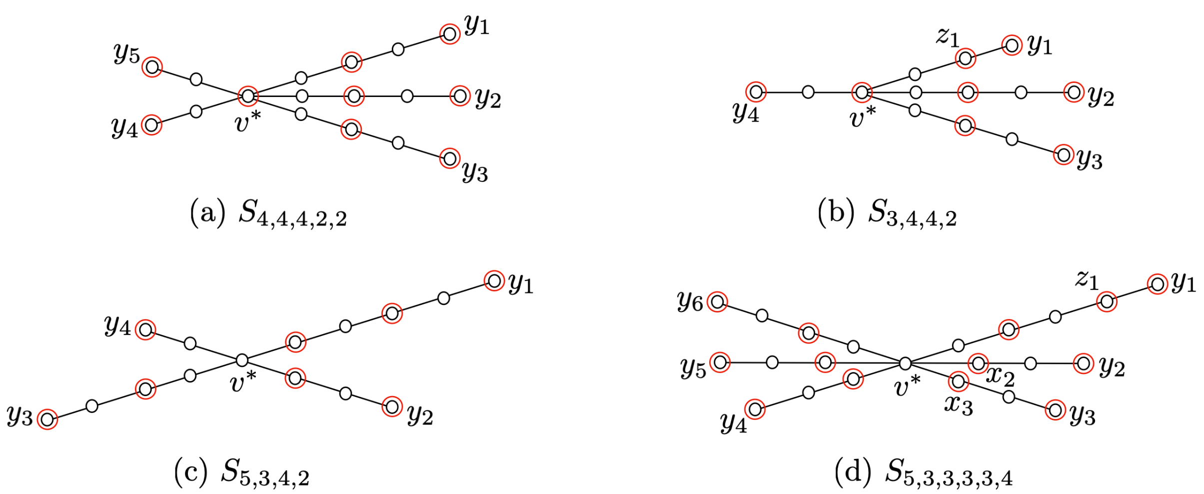

Fig. 2. Four spider graphs in which the circled vertices are sources in an optimal 2-burning sequence.

Corollary 3.2. Let G be the spider graph $S_{n_1,n_2,..,n_r}$ where $r \geq 3$ and $n = n_1 + n_2 + ... +$

n_r+1 and let k be the number of n_i that are odd. If $k=0$ or $k=2$ then $t_2(G)=b_2(G)-1$ and if $k=1$ or $k\geq 3$ then $t_2(G)=b_2(G)$.

Proof. In the proof of Theorem 3.1 we show that for $k=0$ and $k=2$, every 2-burning sequence s has $\text{len}(s)\geq\lceil\frac{n}{2}\rceil$ and there exists a 2-burning sequence of that length. Thus when $k=0$ or $k=2$ we have $t_2(G)=b_2(G)-1$. When $k=1$ the proof shows that $\text{len}(s)\geq\lceil\frac{n}{2}\rceil+1$ for any 2-burning sequence s, hence $t_2(G)=b_2(G)$. Finally, when $k\geq 3$ we show $\text{len}(s)\geq\frac{1}{2}(n+k-1)$ in the proof of Theorem 3.1, so $t_2(G)=b_2(G)$. □

3.2. Wheel Graphs

The *wheel graph* W_n is a graph formed from the cycle C_n by adding a central vertex that is adjacent to all the vertices of the cycle. By Proposition 2.12 we know, $b_2(W_{n+1})\leq 1+b_1(C_n)=1+\lceil\sqrt{n}\,\rceil$; the latter equality due to [6]. In Theorem 3.3 we determine $b_2(W_{n+1})$ exactly, and Figure 3 shows the optimal 2-burning sequences for W_{30} and W_{22} that are constructed in the proof.

Theorem 3.3. *If $n\geq 5$, then $b_2(W_n)=\lceil\sqrt{n+6}\,\rceil$ and $b_2(W_4)=3$.*

Proof. Let v^* be the central vertex of W_n and label the vertices along the cycle consecutively as $1,2,3,\ldots,n$ where $n\geq 5$. We consider two possibilities for 2-burning sequences for W_n: those for which v^* is a source and those for which v^* is not a source. Note that it is not useful to choose v^* as a source in round 3 or later since it will be adjacent to the first two source vertices and will therefore turn blue at round 3. Additionally, note that selecting v^* as the source in round 2 is equivalent to selecting it as the source in round 1, so we consider the latter in Case 1.

Case 1: The central vertex v^* is the first source.

Observe that in this case, once the central vertex turns blue, each vertex in the cycle is adjacent to one blue vertex, namely v^*, so it will turn blue when it is chosen as a source or when it is adjacent to another blue vertex of the cycle. This reduces the problem to finding the 1-burning number for C_n, and adding one for the initial round of selecting v^* as a source. In [6], it is proven that $b_1(C_n)=\lceil\sqrt{n}\,\rceil$, hence in Case 1, the minimum number of rounds for all vertices to turn blue is $1+\lceil\sqrt{n}\,\rceil$.

Case 2: The central vertex v^* is not a source vertex.

Let $s=(s_1,s_2,\ldots,s_m)$ be a 2-burning sequence for which $s_i\neq v^*$ for $1\leq i\leq m$. Let $k=\text{rd}(s)$, so $3\leq m\leq k$. Apply Algorithm 2-burning with input W_n and sequence s. In round 3 the central vertex v^* turns blue. After that, each vertex with a blue neighbor on the cycle becomes blue in the next round. In round 4, the neighbors of s_1 on the cycle turn blue (if they are not already blue) and in round 5 their neighbors on the cycle turn blue, and this continues propagating outward along the cycle in both directions from s_1. Thus at each round, starting at round 4, vertex s_1 is responsible for at most two new blue vertices, so after k rounds, source s_1 turns at most $2(k-3)$ vertices of the cycle blue. The same is true for sources s_2 and s_3. For $j\geq 4$, source s_j is responsible for turning at most

2 of the cycle vertices blue in each round from $j+1$ to k, so it turns at most $2(k-j)$ vertices of the cycle blue in total. Let B be the set of blue vertices in the cycle after k rounds, thus $|B| \leq m + 3 \cdot 2(k-3) + \sum_{j=4}^{m} 2(k-j)$.

We simplify this sum to

$$\begin{aligned} |B| &\leq m + 6k - 18 + 2\sum_{j=4}^{m} k - 2\sum_{j=4}^{m} j \\ &= m + 6k - 18 + 2k(m-3) - (m+4)(m-3) \\ &= 2km - m^2 - 6 = m(2k-m) - 6. \end{aligned}$$

There are n vertices on the cycle, and all are blue after round k, thus $n \leq m(2k-m) - 6$. The product $m(2k-m)$ is maximized when $k = m$, so $n \leq k^2 - 6$ or equivalently $k \geq \sqrt{n+6}$. We know k is an integer, so in fact, $k \geq \lceil \sqrt{n+6}\, \rceil$. Hence when v^* is not a source vertex, every 2-burning sequence for W_n requires at least $\lceil \sqrt{n+6}\, \rceil$ rounds and at least m source vertices where $m(2k-m) \geq n+6$.

Next we show that the quantity $\lceil \sqrt{n+6}\, \rceil$ is also sufficient. Let $k = \lceil \sqrt{n+6}\, \rceil$ so $k^2 \geq n+6$. Choose m so that $2km - m^2 \geq n+6$, but $2kr - r^2 < n+6$ for $r < m$. That is, m is the minimum integer for which $m(2k-m) \geq n+6$. Note that m is well-defined since for $r = 0$ we know $0 < n+6$ and for $r = k$ we know $k^2 \geq n+6$. We construct a 2-burning sequence $s = (s_1, s_2, \ldots, s_m)$ and show that $\mathrm{rd}(s) \leq k$. This is illustrated in Figure 3 for $n = 30$ (where $k = 6$ and $m = 6$) and for $n = 26$ (where $k = 6$ and $m = 4$). Select the source vertices as follows. Let $s_1 = 1$, $s_2 = s_1 + 2(k-3) + 1$, and $s_3 = s_2 + 2(k-3) + 1$. For $4 \leq j \leq k$, let $s_j = s_{j-1} + (k-j+1) + (k-j) + 1$. One can check that $m \leq 3$ only when $n \leq 9$, and in each of those cases our sequence s is a 2-burning sequence for W_n. Thus we may assume $m \geq 4$.

By the end of round 3, the source vertices s_1, s_2, s_3 are blue as is the central vertex v^*. Once v^* is blue, each uncolored vertex with a blue neighbor on the cycle becomes blue in the next round. First consider vertices v with $s_1 < v < s_2$. By construction, there are $2(k-3)$ such vertices, so each is distance at most $k-3$ from one of s_1, s_2 along the cycle. Since s_1, s_2 and v^* are all blue by round 3, and there are k rounds total, vertex v will be blue after round k. The same is true for $s_2 < v < s_3$.

Next consider vertices v with $s_{j-1} < v < s_j$ for some j with $4 \leq j \leq m$. By construction, $s_j - s_{j-1} = 1 + (k-j+1) + (k-j)$, so there are there are $(k-j) + (k-(j-1))$ vertices strictly between s_j and s_{j-1} along the cycle. The $k-j$ vertices closest to s_j all turn blue by round k since s_j turns blue at round j and there are $k-j$ rounds remaining. Similarly, the $k-(j-1)$ vertices closest to s_{j-1} all turn blue by round k. Thus again, vertex v will be blue after round k.

Finally, we consider the vertices v with $s_m < v < n$. As before, the $k-3$ vertices closest to s_1 along the cycle are all blue after round k, so it suffices to consider v with $s_m < v \leq n-(k-3)$. Using our recursive formulas above, we can derive explicit formulas for the source vertices. In particular, $s_1 = 1$, $s_2 = 2 + 2(k-3)$, $s_3 = 3 + 4(k-3)$, and

$$s_m = m + 3(k-3) + (k-m) + 2\sum_{i=3}^{m-1}(k-i)$$

$$
\begin{aligned}
&= 4k - 9 + 2\sum_{i=3}^{m-1} k - 2\sum_{i=3}^{m-1} i \\
&= 4k - 9 + 2k(m-3) - (m+2)(m-3) \\
&= 2km - m^2 + m - 2k - 3.
\end{aligned}
$$

Hence, $s_m + (k-m) = 2km - m^2 - k - 3 \geq n + 6 - k - 3 = n - (k-3)$. Since s_m turns blue in round m and there are $k-m$ rounds remaining, the vertices v with $s_m < v \leq n - (k-3)$ will all be blue after round k. Thus s is a 2-burning sequence for W_n and $\mathrm{rd}(s) \leq k = \lceil \sqrt{n+6}\, \rceil$. Hence we have shown that when v^* is not a source, the minimum number of rounds for all vertices of W_n to turn blue is $\lceil \sqrt{n+6}\, \rceil$.

Combining the results of these cases, we conclude that $b_2(W_n) = \min\{1+\lceil \sqrt{n}\, \rceil, \lceil \sqrt{n+6}\, \rceil\}$. By inspection, $b_2(W_4) = 3$, and it is not hard to show that $\lceil \sqrt{n+6}\, \rceil \leq 1 + \lceil \sqrt{n}\, \rceil$ for $n \geq 5$, completing the proof. □

In our proof of Theorem 3.3 we found the minimum number of sources needed for an optimal 2-burning sequence for W_n. We record this in the following corollary.

Corollary 3.4. If $n \geq 5$ and $k = \lceil \sqrt{n+6}\, \rceil$ then $t_2(W_n)$ is the minimum integer $m \geq 3$ for which $m(2k-m) \geq n+6$.

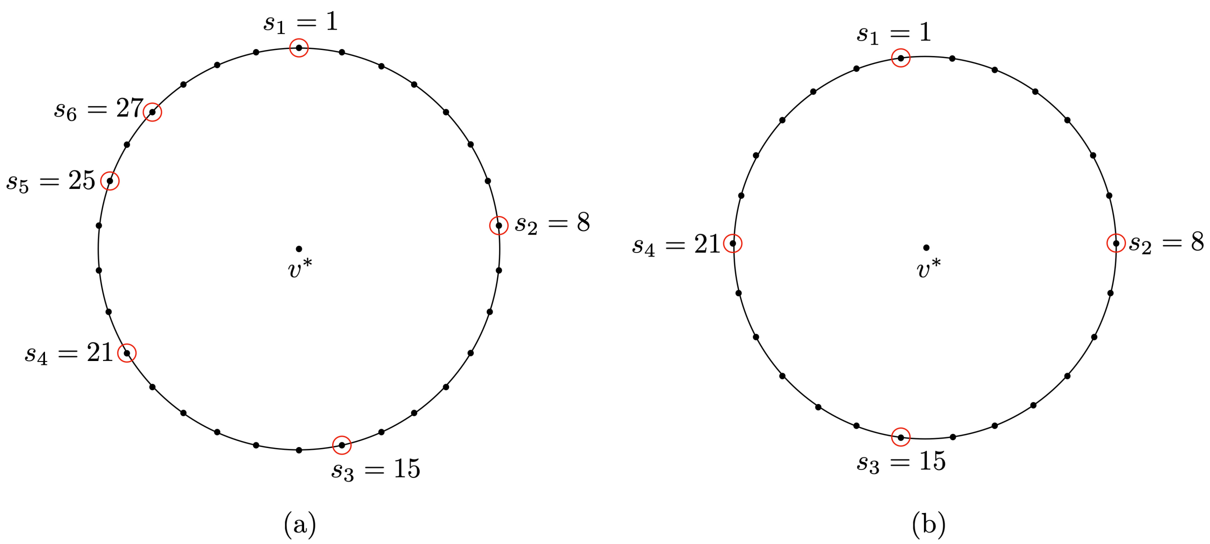

Fig. 3. The wheels W_{30} and W_{26} where the edges incident to v^* are omitted and only the sources in our optimal 2-burning sequences are labeled.

The wheel graphs W_{30} and W_{26} and optimal 2-burning sequences for them are illustrated in Figure 3. For W_{30} the optimal 2-burning sequence constructed in the proof of Theorem 3.3 is $s = (1, 8, 15, 21, 25, 27)$ where $k = b_2(W_{30}) = 6$ and $m = t_2(W_{30}) = 6$. For W_{26} the optimal 2-burning sequence constructed in the proof of Theorem 3.3 is $s = (1, 8, 15, 21)$ where $k = b_2(W_{26}) = 6$ and $m = t_2(W_{26}) = 4$.

The next theorem shows that there exist wheel graphs G for which the 2-burning number is arbitrarily larger than the length of an optimal 2-burning sequence.

Theorem 3.5. *For any integer r, there exists an integer n for which $b_2(W_n)-t_2(W_n) \geq r$.*

Proof. Fix an integer r and choose an integer $k \geq 5$ such that $2(k-1) > r^2$. Let $n = (k-1)^2 - 5$, so $n+6 = (k-1)^2 + 1 = k^2 - 2(k-1)$, and thus $\lceil \sqrt{n+6}\,\rceil = k$. By Theorem 3.3 we know that $b_2(W_n) = k$ and by Corollary 3.4 we know that $t_2(W_n)$ is the smallest integer $m \geq 3$ for which $m(2k-m) \geq n+6$. Observe that for $j = k-r$ we have

$$j(2k-j) = (k-r)(k+r) = k^2 - r^2 > k^2 - 2(k-1) = n+6.$$

Hence $m \leq k-r$ and $b_2(W_n) - t_2(W_n) = k - m \geq r$ as desired. □

4. Cartesian products

Motivated by the interesting relationships that often emerge between graph parameters of input graphs versus their product, we study the 2-burning number and the 2-burning source number of Cartesian products. The *Cartesian product* of graphs G and H is the graph denoted $G \,\square\, H$ that has vertex set $\{(u,v) : u \in V(G), v \in V(H)\}$ and $(u_1,v_1)(u_2,v_2) \in E(G \,\square\, H)$ if and only if one of the following hold: (i) $u_1 = u_2$ and $v_1v_2 \in E(H)$ or (ii) $v_1 = v_2$ and $u_1u_2 \in E(G)$.

Recall that Observation 2.5 provided a lower bound for the 2-burning number of a graph based on the number of leaves; namely, if graph G has k leaves, then $b_2(G) \geq k$. Cartesian products of connected graphs have no leaves, so we begin by determining $b_2(K_m \,\square\, K_n)$, and use this to provide a lower bound for the 2-burning number of the Cartesian product of any pair of connected graphs on m and n vertices.

Theorem 4.1. *If $m \geq 5$ and $n = 3$, or $m \geq n \geq 4$, then $t_2(K_m \,\square\, K_n) = 2$ and $b_2(K_m \,\square\, K_n) = 5$.*

Proof. Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Partition $K_m \,\square\, K_n$ into m copies of K_n, labeled $H_1, H_2, \ldots, H_m$ where u_i is the first coordinate of each vertex in H_i. For the lower bound on $b_2(K_m \,\square\, K_n)$, without loss of generality, suppose the first source vertex is (u_1, v_1) in H_1. We consider two cases.

Case 1: The second source vertex is in H_1.

At the end of round 3, every vertex in H_1 has turned blue. Without loss of generality, the third source vertex is in H_2. So at the end of round 3, there are at least three uncolored vertices in H_2 and every vertex in H_3 is uncolored. Let (u_2, v_j) and (u_2, v_k) be distinct vertices of H_2 that are uncolored at the end of round 3, where $1 \leq j, k \leq n$ and $j \neq k$. Then at most one of $(u_3, v_j), (u_3, v_k)$ in H_3 turns blue during round 4, leaving at least one uncolored at the end of round 4.

Case 2: The second source vertex is not in H_1.

Without loss of generality, assume the second source is (u_2, v_j) in H_2. If $j = 1$ then during round 3, exactly one non-source vertex turns blue in H_i for $3 \leq i \leq m$, namely (u_i, v_1). Without loss of generality, assume (u_1, v_2) is the third source vertex. Then during round 4, exactly one non-source vertex turns blue in each H_i for $3 \leq i \leq m$, namely (u_i, v_2). Since $|V(H_i)| \geq 3$ for $i : 1 \leq i \leq m$ and at most one vertex from $H_1 \cup H_2$ is a source vertex in round 4, there is at least one uncolored vertex amongst the vertices of $H_1 \cup H_2$ at the end of round 4.

If $j \neq 1$, without loss of generality we may assume $j = 2$, so the first two sources are (u_1, v_1) and (u_2, v_2). Then the only non-source vertices to turn blue in round 3 are (u_1, v_2) and (u_2, v_1). During round 4, the remaining vertices of $H_1 \cup H_2$ turn blue; along with at most two non-source vertices in each of H_3, H_4. At the end of round 4, there are at most two source vertices in $H_3 \cup H_4$. If $m \geq n \geq 4$, this leaves at least two vertices in $H_3 \cup H_4$ uncolored by the end of round 4. If $m \geq 5$ and $n = 3$, observe that during round 4, exactly two non-source vertices in each of H_3, H_4, H_5 turn blue. Then at the end of round 4, there are at most two source vertices in $H_3 \cup H_4 \cup H_5$, leaving at least one uncolored vertex in $H_3 \cup H_4 \cup H_5$.

The above two cases show that $b_2(K_m \square K_n) \geq 5$. To see that $b_2(K_m \square K_n) = 5$, let $s = ((u_1, v_1), (u_2, v_2))$ and observe that s is a 2-burning sequence of $K_m \square K_n$ that results in every vertex blue by the end of round 5. It also shows that $t_2(K_m \square K_n) \leq 2$. Any graph with two or more vertices requires at least two sources, so $t_2(K_m \square K_n) = 2$. □

If G and H are graphs with $m = |V(G)|$ and $n = |V(H)|$ then $G \square H$ is a spanning subgraph of $K_m \square K_n$. We combine Lemma 2.4 and Theorem 4.1 to conclude the following.

Corollary 4.2. Let G and H be graphs on m and n vertices, respectively, where $m \geq 5$ and $n = 3$, or $m \geq n \geq 4$. Then $t_2(G \square H) \geq 2$ and $b_2(G \square H) \geq 5$.

In the next theorem we show that $b_2(G)$ is an additional lower bound for $b_2(G \square H)$.

Theorem 4.3. *Let G and H be connected graphs on m and n vertices, respectively, where $m \geq 5$ and $n = 3$, or $m \geq n \geq 4$. Then $b_2(G \square H) \geq \max\{5, b_2(G), b_2(H)\}$.*

Proof. Let $s = ((u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k))$ be an optimal 2-burning sequence for $G \square H$. We note that the first coordinates of vertices in sequence s may not all be distinct: for $i \neq j$, it is possible that $u_i = u_j$. Similarly, the second coordinates of vertices in s may not all be distinct.

For graph G, let $s_G = (u_1, u_2, \ldots, u_k)$. To complete the proof, we must show that when Algorithm 2-burning is applied to graph G and sequence s_G, it will terminate when all vertices of G are blue. The fact that the vertices in s_G are not necessarily all distinct causes no problem with this implementation: if source vertex u_j turns blue before round j, then no source vertex turns blue during round j.

We claim that any vertex $u_p \in V(G)$ turns blue by round r (via s_G) if there exists a vertex $(u_p, v_q) \in V(G \square H)$ that turns blue by round r (via s). By construction of s_G, the claim is true when u_p is in s_G. For a contradiction suppose the claim is false and let ℓ be minimum so that there exists $(u_p, v_q) \in V(G \square H)$ that turns blue by round ℓ (via s) but for which u_p is not blue (via s_G) by round ℓ. For (u_p, v_q) to turn blue during round ℓ, it must have at least two neighbors that are blue by the end of round $\ell - 1$. We consider two cases for neighbors of (u_p, v_q).

For the first case, assume (u_p, v_q) has a neighbor (u_p, v_c) that is blue by round $\ell - 1$. By the minimality of ℓ, since (u_p, v_c) is blue by round $\ell - 1$, vertex u_p in G is blue by round $\ell - 1$, a contradiction.

For the second case, assume (u_a, v_q) and (u_b, v_q) are distinct neighbors of (u_p, v_q) and are both blue by round $\ell - 1$. Then $u_a, u_b \in N_G(u_p)$ and since ℓ is minimum, u_a, u_b are both blue by round $\ell - 1$. Therefore, u_p is blue by the end of round ℓ, a contradiction. This proves our claim.

As a consequence, if s results in every vertex of $G \square H$ blue by round $b_2(G \square H)$, then s_G results in every vertex of G blue by round $b_2(G \square H)$. It follows that $b_2(G) \leq b_2(G \square H)$. A similar argument shows $b_2(H) \leq b_2(G \square H)$ and $5 \leq b_2(G \square H)$ follows from Corollary 4.2. □

The proof of Theorem 4.3 maps source vertices in $G \square H$ to source vertices in G to show $b_2(G) \leq b_2(G \square H)$. Unfortunately, this approach cannot be used to provide an analogous bound for $t_2(G \square H)$ based on $t_2(G)$. To see this, consider $P_4 \square P_3$, as shown in Figure 4. The sequence $s = \{(1,1), (2,2), (3,3), (4,2), (4,1)\}$ results in every vertex blue by the end of round 5 and Theorem 4.3 implies that $b_2(P_4 \square P_3) \geq 5$. Thus $b_2(P_4 \square P_3) = 5$. However, mapping the first three source vertices of s to P_4 results in sequence $s_{P_4} = \{1, 2, 3\}$, which is not a 2-burning sequence for P_4 as non-source vertex 4 has only one neighbor and cannot ever turn blue. We do not know if $t_2(G)$ forms a lower bound for $t_2(G \square H)$ in general.

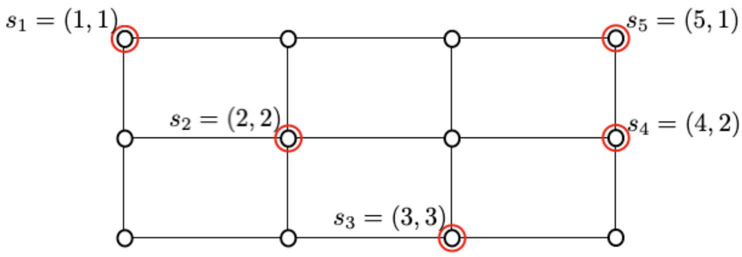

Fig. 4. The graph $P_4 \square P_3$ with 2-burning sequence s_1, s_2, s_3, s_4, s_5 labeled.

We next provide upper bounds for the 2-burning number and the 2-burning source number of the Cartesian product of graphs. Figure 5 illustrates the proof in the case $G = P_5$ and $H = P_4$.

Theorem 4.4. *If G and H are graphs then $t_2(G \square H) \leq t_2(G)t_2(H)$ and $b_2(G \square H) \leq t_2(G)t_2(H) + (b_2(H) - t_2(H)) + (b_2(G) - t_2(G))$.*

Proof. Let $m = |V(G)|$ and label the vertices of G as $g_1, g_2, \ldots, g_m$ such that $g = (g_1, g_2, \ldots, g_k)$ is an optimal 2-burning sequence for G with $\text{len}(g) = k = t_2(G)$. Let

$n = |V(H)|$ and label the vertices of H as $h_1, h_2, \ldots, h_n$ such that $h = (h_1, h_2, \ldots, h_\ell)$ is an optimal 2-burning sequence for H with $\mathrm{len}(h) = \ell = t_2(H)$. This is illustrated in Figure 5 where $G = P_5$, $H = P_4$, $m = 5$, $k = 3$, $n = 4$, and $\ell = 3$.

For $i : 1 \leq i \leq k$, let a_i be the sequence $(g_i, h_1), (g_i, h_2), \ldots, (g_i, h_\ell)$ and let s be the sequence $a_1, a_2, \ldots, a_k$. Thus s is a sequence of vertices in $G \square H$ and $\mathrm{len}(s) = k\ell = t_2(G)t_2(H)$. In Figure 5, the nine vertices of s are circled and labeled $s_1, \ldots, s_9$. We will show that s is a 2-burning sequence for $G \square H$ and calculate the number of rounds until all vertices are blue.

For each $i : 1 \leq i \leq k$, let $H_i = \{(g_i, h_1), (g_i, h_2), \ldots, (g_i, h_\ell), \ldots, (g_i, h_n)\}$ and note that H_i induces a copy of H in $G \square H$. The first ℓ vertices listed in H_i are blue after round $t_2(G)t_2(H)$ because they are part of s. Since $(h_1, h_2, \ldots, h_\ell)$ is a 2-burning sequence for H, after an additional $b_2(H) - t_2(H)$ rounds, the remaining vertices in H_i are blue. Thus, if $1 \leq i \leq k$ and $1 \leq j \leq n$ then the vertex (g_i, h_j) is blue after round $t_2(G)t_2(H) + b_2(H) - t_2(H)$.

It remains to consider vertices of the form (g_i, h_j) where $k + 1 \leq i \leq m$ and $1 \leq j \leq n$. For each $j : 1 \leq j \leq n$, let $G_j = \{(g_1, h_j), (g_2, h_j), \ldots, (g_k, h_j), \ldots, (g_m, h_j)\}$ and note that G_j induces a copy of G in $G \square H$. Since $(g_1, g_2, \ldots, g_k)$ is a 2-burning sequence for G of length $t_2(G)$, all vertices of G are blue $b_2(G) - t_2(G)$ rounds after g_k becomes blue. Similarly, since $(g_1, h_j), (g_2, h_j), \ldots, (g_k, h_j)$ is a 2-burning sequence of length $t_2(G)$ for the copy of G induced by G_j, all vertices of G_j are blue $b_2(G) - t_2(G)$ rounds after (g_k, h_j) becomes blue. We concluded above that (g_k, h_j) is blue after round $t_2(G)t_2(H) + b_2(H) - t_2(H)$ for $1 \leq j \leq n$, so all vertices in G_j are blue after round $t_2(G)t_2(H) + (b_2(H) - t_2(H)) + (b_2(G) - t_2(G))$. Thus sequence s is a 2-burning sequence for $G \square H$ of length $t_2(G)t_2(H)$ and $b_2(G \square H) \leq t_2(G)t_2(H) + (b_2(H) - t_2(H)) + (b_2(G) - t_2(G))$. □

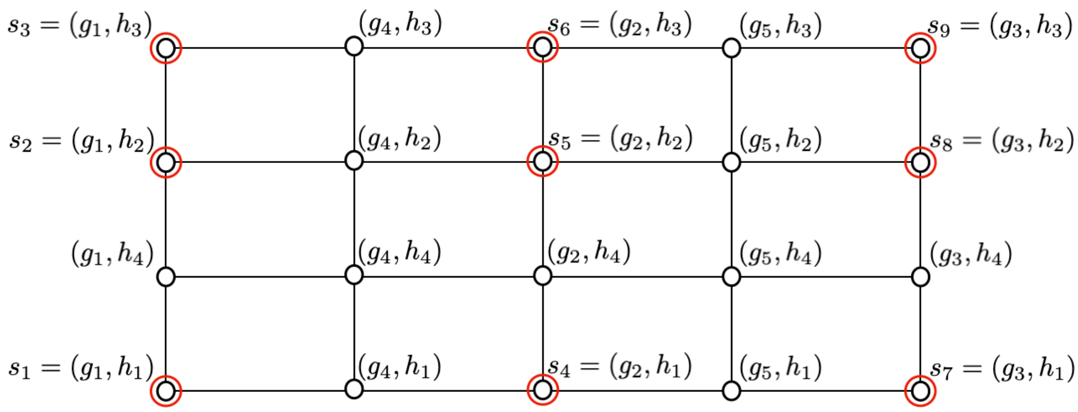

Fig. 5. The graph $P_5 \square P_4$ with vertices labeled as in the proof of Theorem 4.4 and with source vertices $s_1, s_2, \ldots, s_9$.

The next result illustrates that for some graphs G and H, the upper bound for $t_2(G \square H)$ given in Theorem 4.4 is exact.

Theorem 4.5. *The graph $C_4 \square C_4$ has $b_2(C_4 \square C_4) = 5$ and $t_2(C_4 \square C_4) = 4$.*

Proof. Let $V(C_4) = \{1, 2, 3, 4\}$ where vertex i is adjacent to vertex $i + 1 \pmod 4$. Corollary 4.2 provides the lower bound $b_2(C_4 \square C_4) \geq 5$; and it is straightforward to verify that $b_2(C_4 \square C_4) \leq 5$ by choosing source vertices $(1, 1), (2, 2), (3, 3), (4, 4)$. Thus $b_2(C_4 \square C_4) = 5$ and every optimal 2-burning sequence s has $\text{rd}(s) = 5$.

Theorem 4.4 provides the upper bound $t_2(C_4 \square C_4) \leq 4$. For a contradiction, suppose $t_2(C_4 \square C_4) \leq 3$ and fix an optimal 2-burning sequence s for $C_4 \square C_4$ with $\text{len}(s) \leq 3$. Since s is optimal, we know $\text{rd}(s) = 5$ and since there is no 2-burning sequence for $C_4 \square C_4$ with only 2 source vertices, we also know $\text{len}(s) = 3$. We will use the terms "rows" and "columns" as illustrated in Figure 6 where "row i" refers to the set $\{(i, 1), (i, 2), (i, 3), (i, 4)\}$; and "column i" refers to $\{(1, i), (2, i), (3, i), (4, i)\}$.

First consider the case in which there is at most one source vertex in each row and in each column. Without loss of generality we may assume that row 4 and column 4 contain no source vertices. The vertices $(1, 1)$ and $(1, 3)$ cannot both be blue after round 3 since at most one is a source and it would require a second source from row 1 to turn the other blue by round 3. Thus the earliest $(1, 1)$ and $(1, 3)$ are both blue is round 4, and therefore $(1, 4)$ cannot turn blue until round 5. Similarly, vertices $(3, 4)$, $(4, 1)$ and $(4, 3)$ cannot turn blue until round 5. So none of the neighbors of vertex $(4, 4)$ are blue until round 5, and consequently, vertex $(4, 4)$ is not blue at the end of round 5, a contradiction.

Otherwise, there exists a row or column containing two source vertices. Without loss of generality we may assume that column 2 has two source vertices. There must be a source vertex in column 4, or else no vertex in column 4 will ever turn blue. Without loss of generality we may assume that $(2, 4)$ is a source vertex. Any non-source vertex that turns blue in round 3 must have two source neighbors. Thus, at the end of round 3, the only vertices that are blue, are in row 2 and in column 2. Even if all seven of these vertices were blue at round 3 (see Figure 6), vertex $(4, 4)$ would not turn blue by the end of round 5, a contradiction. Thus $t_2(C_4 \square C_4) = 4$. $\square$

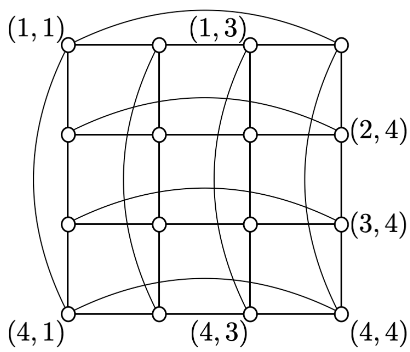

Fig. 6. The graph $C_4 \square C_4$ with some vertices labeled for reference, as in the proof of Theorem 4.5.

We know that $t_2(C_4) = 2$ and in Theorem 4.5 we show $t_2(C_4 \square C_4) = 4$, so $t_2(C_4 \square C_4) = t_2(C_4)t_2(C_4)$. Thus there exist graphs G and H for which the bound $t_2(G \square H) \leq$

$t_2(G)t_2(H)$ in Theorem 4.4 is exact. For the 2-burning number, Theorem 4.4 provides the bound $b_2(C_4 \square C_4) \leq 6$, but Theorem 4.5 shows that $b_2(C_4 \square C_4) = 5$. We do not know if there exist graphs G and H for which the bound on $b_2(G \square H)$ from Theorem 4.4 is exact.

5. Conclusion and Directions

Since the 2-burning numbers are known for P_n and C_n, it is natural to consider the 2-burning number of their products. Theorems 4.3 and 4.4 provide lower and upper bounds for 2-burning numbers, but the exact values remain unknown. For $P_n \square P_n$, we can gain insight and bounds on the 2-burning number from bootstrap percolation since $P_n \square P_n$ is one of the few graphs for which bootstrap percolation has been studied from an extremal perspective.

In Section 1 we defined the related problem of r-neighbor bootstrap percolation where rather than having a sequence of sources turning blue in successive rounds, there is a set of sources that all turn blue in round 0. Most work on bootstrap percolation has focused on the process when the source vertices are chosen independently at random with probability p, but some researchers have considered the problem of determining the minimum cardinality of a percolating set. Following [9], we denote by $m(G, r)$ the cardinality of a minimum percolating set on graph G following the r-neighbor bootstrap process. We denote by $\tau(G, r)$, the minimum number of rounds by which every vertex of G is blue, over all possible minimum percolating sets on G. Though we state the next observation for $r = 2$, we note that it holds for general r where the parameter $t_r(G)$ is defined analogously to $t_2(G)$.

Observation 5.1. *For a graph G on n vertices*

$$m(G, 2) \leq t_2(G) \leq b_2(G) \leq m(G, 2) + \tau(G, 2).$$

Observe that if $G = K_n$ and $n \geq 3$ then $m(K_n, 2) = 2 = t_2(K_n)$ and $b_2(K_n) = 3$, and $\tau(K_n, 2) = 1$; therefore the first and third inequalities are tight. We have seen other examples (e.g., paths and cycles on an even number of vertices) that make the middle inequality tight. Since $\tau(G, 2) > 0$ for all graphs G, there are no graphs for which equality holds throughout.

In Section 4 we considered Cartesian products and these have also been studied in the context of percolation. For 2-neighbor bootstrap percolation, the extremal problem of determining the smallest percolating set was first considered Pete [10], published in Hungarian; and later communicated by Balogh and Pete [2]. The vertices on the diagonal of the square grid $P_n \square P_n$ constitute a percolating set for this graph, thus $m(P_n \square P_n, 2) \leq n$. Indeed it is shown in [10] and [2] that $m(P_n \square P_n, 2) = n$. The same set of sources form a 2-burning sequence for $P_n \square P_n$ and using this sequence, all vertices turn blue by round $2n$. Thus, $n \leq b_2(P_n \square P_n) \leq 2n$. It would be interesting to determine the exact value of $b_2(P_n \square P_n)$ for general n.

Question 5.2. *Can we determine $b_2(P_n \square P_n)$ for all n?*

In this paper we have focused on $b_2(G)$, the 2-burning number, and our new parameter $t_2(G)$, the 2-burning source number. The quantities $b_2(G)$ and $t_2(G)$ are equal precisely when every optimal 2-burning sequence requires a source in each round. We found infinite families of graphs G for which $t_2(G) = b_2(G)$; including subsets of paths, cycles, and spiders. More generally, we would like to determine the set of graphs for which these parameters are equal.

Question 5.3. *Can we characterize all graphs G for which $t_2(G) = b_2(G)$?*

Finally, we state two questions relating to the discussions succeeding Theorems 4.3 and 4.5.

Question 5.4. *For arbitrary graphs G and H, are $t_2(G)$ and $t_2(H)$ lower bounds for $t_2(G \,\square\, H)$?*

Question 5.5. *Do there exist graphs G and H for which the bound on $b_2(G \,\square\, H)$ from Theorem 4.4 is exact?*

Conflict of interest

The authors declare no conflict of interest.

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