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Conflict-free coloring games

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ABSTRACT

In 2003, the frequency assignment problem in a cellular network motivated Even et al. to introduce a new coloring problem: Conflict-Free coloring. Inspired by this problem and by the Gardner-Bodlaender's coloring game, in 2020, Chimelli and Dantas introduced the *Conflict-Free Closed Neighborhood k-coloring game (CFCN k-coloring game)*. The game starts with an uncolored graph G, $k \geq 2$ different colors, and two players, Alice and Bob, who alternately color the vertices of G. Both players can start the game and respect the following legal coloring rule: for every vertex v, if the closed neighborhood $N[v]$ of v is fully colored then there exists a color that was used only once in $N[v]$. Alice wins if she ends up with a Conflict-Free Closed Neighborhood k-coloring of G, otherwise, Bob wins if he prevents it from happening. In this paper, we introduce the game for open neighborhoods, the *Conflict-Free Open Neighborhood k-coloring game (CFON k-coloring game)*, and study both games on graph classes determining the least number of colors needed for Alice to win the game.

Keywords: conflict-free coloring, coloring game, combinatorial games

2020 Mathematics Subject Classification: 05C15, 05C57.

1. Introduction

We consider undirected, finite and simple *graphs* $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. We say that vertices $u, v \in V(G)$ are *adjacent* if uv is an edge of G. The *open neighborhood* $N(v)$ of a vertex v is the set of vertices that are adjacent to v, and the *closed neighborhood* $N[v]$ of vertex v is the union $N(v) \cup \{v\}$. A *vertex k-coloring* of a graph G is a function $c : V(G) \to \{c_1, c_2, ..., c_k\}$, where $\{c_1, c_2, ..., c_k\}$ represents a set of k different colors.

In 2003, Even et al. [10] introduced the *Conflict-Free coloring* inspired by the Cellular Network

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problem: n base stations establish a link via radio frequencies, interference occurs if one particular mobile device establishes a link with two or more base stations that have the same radio frequency. So, every base station must contact a mobile device with a unique radio frequency. A solution for this problem can be obtained by assigning n different frequencies to the n base stations but, since having a lot of different frequencies is expensive, it is important trying to minimize their quantity, in a way that there is no interference.

The Conflict-Free coloring in graphs is obtained by modeling base stations as vertices, interference constraints as edges, and radio frequencies as colors, ensuring that every mobile device (neighborhood) has at least one uniquely colored base station. Therefore, the Conflict-Free coloring of a graph G consists of assigning different colors to the vertices of G such that, for every vertex v, there exists a vertex v' in the neighborhood of v, such that the color of v' differs from the color of every other vertex in the neighborhood of v.

Formally, given a graph G, a vertex k-coloring is called a *Conflict-Free Closed Neighborhood k-coloring (CFCN k-coloring)* if for every vertex $v \in V(G)$, there exists a vertex v' in the closed neighborhood $N[v]$ such that $c(v') \neq c(w)$ for all $w \in N[v] \setminus \{v'\}$. The complexity of these colorings in graphs were studied in [1], and in [13], where the authors prove that the CFCN coloring problem is NP-complete. Moreover, in 2009, Cheilaris considered these colorings not only on graphs, but also on hypergraphs [5], a scenario also studied by Smorodinsky [18] and Cui & Hu [7].

Let G be a graph and $S \subseteq V(G)$, we say that S is *fully colored* if each vertex of S has a color assigned to it. A graph G together with a CFCN k-coloring is said to be *CFCN k-colored*. In Figure 1, we show an example of a graph G with a CFCN 2-coloring.

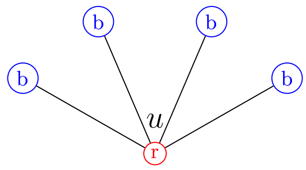


Fig. 1. Graph with a CFCN 2-coloring (where b is blue and r is red).

Another approach to examining this problem is to consider it from the perspective of combinatorial games [2]. Combinatorial games have been studied from different perspectives (see for example impartial games [8], coloring [11, 14, 16, 17, 19], domination [4], and labeling [15]).

In 1981, Martin Gardner [12] introduced a combinatorial game, called *coloring game*, which was studied later by Bodlaender [3]. The game consists of two players, Alice and Bob, who alternately color uncolored vertices of a graph using k colors, such that adjacent vertices have different colors (proper coloring). Alice wins if she obtains a proper k-coloring of the graph; otherwise, Bob wins. Inspired by the coloring game, in 2020, Chimelli and Dantas [6] proposed the combinatorial game called CFCN k-coloring game for complete graphs of small order.

The *CFCN k-coloring game* is a combinatorial game in which two players, Alice and Bob, alternately color the vertices of a graph as follows. In each turn, a player assigns one of the k colors to one arbitrary vertex v, in such a way that, after it, in every fully colored closed neighborhood to which v belongs, there exists a color that appears exactly once (*legal coloring*). Thus, for every $u \in V$, if $N[u]$ is fully colored, then there exists $u' \in N[u]$ such that $c(u') \neq c(w)$ for all $w \in N[u] \setminus \{u'\}$. Either Alice or Bob starts the game, they cannot skip turns, they play optimally, and they are constrained to use only legal colorings (moves) in each turn. Alice wins if it is guaranteed that she can obtain a CFCN k-coloring of G, otherwise Bob wins if he prevents it from happening.

We remark that, by *play optimally*, we mean that the players try to win with the fewest possible turns or, in case of knowing that it is not possible to win the game, delay the opponent's victory. For example, suppose Alice and Bob play the CFCN 2-coloring game on a star S_{n-1} graph (complete bipartite $K_{1,n-1}$, with $n \geq 3$). If Alice starts the game, then the best option for her is to color the central vertex (vertex of degree $n-1$) with a color c_1. She immediately wins the game because, by the legal coloring rule, in each turn, the vertices of degree one must be colored with a different color c_2. Else, if Bob starts the game (see Figure 2) then, to delay Alice's victory, he colors any vertex other than the central one. So, in the second turn, Alice colors the central vertex and wins.

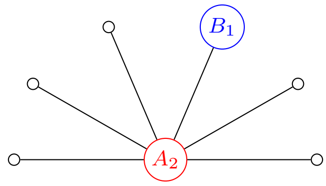


Fig. 2. Example of the CFCN 2-coloring game played on a star S_6, where Bob starts (B_1), Alice colors the central vertex (A_2) and wins (Alice colors with red and Bob with blue).

In the present work, we introduce the Conflict-Free Open Neighborhood k-coloring game (CFON k-coloring game), and contribute to this subject by studying the behavior of both games on some classic graph classes such as stars, complete graphs, paths, cycles.

In order to extend the results on complete graphs, we also analyze the behavior of this coloring on cliques in graphs studying windmill graphs and their generalization. Windmill graphs have been much studied since Erdös, Rényi and Sós [9] (1966) established that the only graphs with the property that every two vertices have exactly one neighbor in common are the friendship graphs, which are members of this larger family called windmill graphs.

The *windmill graph* $W(n,p)$, $n \geq 2$, $p \geq 2$, is a graph constructed by joining p copies of the complete graph K_n to a unique universal vertex u. We refer to Figure 3 for an example of the CFCN 2-coloring game played on a windmill graph $G = W(3,3)$, where Alice's turns are A_1 and A_3, and Bob's turns are B_2 and B_4. Alice starts the game, and Bob wins on the fourth turn since all colors are duplicated in the closed neighborhood of vertex u.

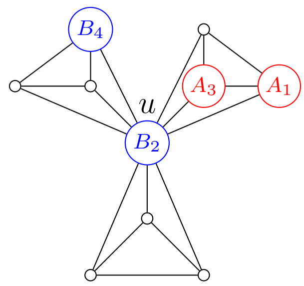


Fig. 3. Example of the CFCN 2-coloring game played on $G = W(3,3)$, where Alice starts (A_1) and Bob wins (Alice colors with red and Bob with blue).

This paper is organized as follows. In Section 2, we introduce the game and show definitions and notation. From Sections 3 to 7, we study the CFCN k-coloring game and the CFON k-coloring game on stars, complete graphs, paths, cycles, and windmill graphs and their generalizations. In each of the studied graph classes, we show strategies that determine the least number of colors necessary so that Alice wins the game. Finally, in Section 8, we present our final remarks.

2. Preliminaries

Let V' be a nonempty subset of V. The *subgraph* $H = (V', E')$ of G whose vertex set is V', and whose edge set E' is the set of edges of G that have both incident vertices in V' is called the subgraph of G *induced* by V', and it is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of G.

We recall that a vertex k-coloring of a graph G is called a *Conflict-Free Closed Neighborhood k-coloring (CFCN k-coloring)* if for every vertex $v \in V(G)$, there exists a vertex u in the closed neighborhood $N[v]$ such that $c(u) \neq c(w)$ for all $w \in N[v] \setminus \{u\}$. Similarly a vertex k-coloring is called a *Conflict-Free Open Neighborhood k-coloring (CFON k-coloring)* if for every vertex $v \in V(G)$, there exists a vertex u in the open neighborhood $N(v)$ such that $c(u) \neq c(w)$ for all $w \in N(v) \setminus \{u\}$.

We say that $N[v]$ (resp. $N(v)$) is *fully colored* if each vertex of $N[v]$ (resp. $N(v)$) has a color assigned to it. A graph G together with a CFCN k-coloring (resp. CFON k-coloring) is said to be *CFCN k-colored* (resp. *CFON k-colored*). A coloring of a vertex v is said to be a *legal coloring of v* if, after it, in every fully colored neighborhood in which v belongs, there exists a color that appears exactly once.

Formally, the *Closed* (resp. *Open*) *Neighborhood Conflict-Free Chromatic Number* of G, denoted by $\chi_{CN}(G)$ (resp. $\chi_{ON}(G)$), is the minimum number k of colors necessary for G to be CFCN k-colored (resp. CFON k-colored). Now, we are ready to present the rules of the game.

The *CFCN k-coloring game* (resp. *CFON k-coloring game*) is a combinatorial game in which two players, Alice and Bob, take turns (alternately) coloring the vertices of a graph as follows. In each turn, a player assigns one of the k colors to one arbitrary vertex v, in such a way that, after it, in every fully colored closed neighborhood in which v belongs, there exists a color that appears exactly once (*legal coloring*). Thus, for every $u \in V$, if $N[u]$ is fully colored, then there exists $u' \in N[u]$ such that $c(u') \neq c(w)$ for all $w \in N[u] \setminus \{u'\}$. Either Alice or Bob starts the game, they cannot skip turns, they play optimally (as explained in the Introduction section), and they are constrained to use only legal colorings (moves) in each turn. Alice wins if it is guaranteed that she can obtain a CFCN k-coloring (resp. CFON k-coloring) of G, otherwise Bob wins if he prevents it from happening.

We denote by $\chi^g_{CN}(G)$ (resp. $\chi^g_{ON}(G)$), the *Closed* (resp. *Open*) *Neighborhood Conflict-Free game Chromatic Number* of G, that is, the minimum number k of colors necessary for Alice to have a winning strategy for the CFCN (resp. CFON) k-coloring game on G, independent of who starts the game.

Next, we analyze the game on stars, complete graphs, paths, cycles, and windmill graphs and their generalization.

3. Stars

A *star graph* S_{n-1} is a tree on n vertices such that one vertex v_0 has degree $n-1$ (*central vertex*) and the other $n-1$ vertices have degree 1.

We consider CFCN k-coloring game and prove that $\chi^g_{CN}(S_{n-1}) = 2$; and we also study the CFON k-coloring game and prove that $\chi^g_{ON}(S_{n-1}) = \lceil \frac{n-1}{4} \rceil + 1$.

Theorem 3.1. *Alice wins the CFCN k-coloring game on a Star S_{n-1} with $n > 2$ vertices and $k \geq 2$ colors, independently of who starts playing.*

Proof. If Alice starts the game, then she colors the central vertex v_0 with a color $c(v_0)$ and wins.

Indeed, since $N[v] = \{v_0, v\}$ for each $v \in V(S_{n-1}) \setminus v_0$, by the legal coloring rule, no other vertex can be colored with $c(v_0)$.

Similarly, if Bob starts the game then, since he plays optimally, he does not color the central vertex in order to delay Alice's victory. However, in the second turn, Alice colors the central vertex and wins. □

Theorem 3.2. *Alice wins the CFON k-coloring game played on a star S_{n-1} with $n > 2$ vertices, when she starts the game, if and only if, $k > \lceil \frac{n-2}{4} \rceil$.*

Proof. Since $N(v_0) = V(S_{n-1}) \setminus \{v_0\}$ and $N(v) = v_0$ for every $v \neq v_0$, to obtain a CFON k-coloring for S_{n-1}, at least one color must appear exactly once in $N(v_0)$. We also remark that the colors chosen for $N(v_0)$ (resp. v_0) do not affect the decision for the color of the vertex v_0 (resp. vertices in $N(v_0)$).

In that case, to prevent a color from appearing only once in $N(v_0)$, Bob's strategy is to duplicate all k colors in $N(v_0)$. To delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated.

In addition, Alice starts the game by coloring the central vertex to ensure that duplication of the first two colors in $N(v_0)$ takes five turns instead of four (see Figure 4).

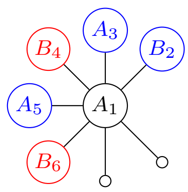


Fig. 4. CFON k-coloring game on S_{n-1}: if Alice starts (A_1) assigning any color to v_0, then the first two colors are duplicated in six turns (Alice's turns A_i, Bob's turns B_i; B_4, B_6 are red and B_2, A_3, A_5 are blue).

With the aforementioned strategies, it is immediate to demonstrate that, from the second turn, every $4t + 1$ turns Alice and Bob have used exactly $t + 1$ colors and each of them has been used at least twice.

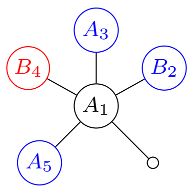


Fig. 5. CFON k-coloring game on S_5: if Alice starts (A_1) assigning any color to v_0, then, by the legal coloring rule, the last vertex must be colored blue (Alice's turns A_i, Bob's turns B_i; B_4 is red and B_2, A_3, A_5 are blue).

If $|N(v_0)| \leq 5$, then it is clear that two colors are necessary for Alice to win (See Figure 5). If $|N(v_0)| > 5$, then there exists $t \in \mathbb{N}$ such that $4t+1 < |N(v_0)| \leq 4(t+1)+1$, and since $|N(v_0)| = n-1$, we have that $4t + 1 < n - 1 \leq 4(t + 1) + 1$.

Suppose that $k \leq \lceil \frac{n-2}{4} \rceil$. Thus, we have that

$$k \leq \left\lceil \frac{n-2}{4} \right\rceil \leq \left\lceil \frac{4(t+1)}{4} \right\rceil = t + 1.$$

Hence, by the time Alice and Bob have colored $4t+1$ vertices, they have already used all the k colors at least twice. Furthermore, since $4t+1 < |N(v_0)|$, the graph is not fully colored and Bob wins the game because there exists no available color to use only once.

Reciprocally, if $k > \lceil \frac{n-2}{4} \rceil$, since $4t < n-2 \leq 4(t+1)$, then $t < \lceil \frac{n-2}{4} \rceil \leq t+1$. Thus, $t+1 = \lceil \frac{n-2}{4} \rceil < k$. So, $k \geq t+2$ and, by the legal coloring rule, to duplicate $t+2$ colors, Bob needs at least $4(t+1)+2$ vertices in $N(v_0)$. Since $|N(v_0)| \leq 4(t+1)+1$, Alice wins the game. □

Theorem 3.3. *Alice wins the CFON k-coloring game played on a star S_{n-1} on $n > 2$ vertices, when Bob starts the game, if and only if, $k > \lceil \frac{n-1}{4} \rceil$.*

Proof. The proof is similar to that of Theorem 3.2 since to prevent a color from appearing only once, Bob's strategy is to duplicate all k colors in $N(v_0)$ and, to delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated. If Bob starts the game and colors the central vertex, then he duplicates the first two colors in the next four turns and reduces the number of turns by one to win the game (see Figure 6). If Bob starts the game and colors a vertex in $N(v_0)$, then it is the same as what we have in Theorem 3.2. □

Fig. 6. CFON k-coloring game on a star graph S_{n-1}: Bob starts coloring the central vertex, the first two colors are duplicated in the next four turns ((left) A_2, A_4 are red and B_3, B_5 are blue, (right) A_2, B_5 are red and B_3, A_4 are blue).

4. Complete graphs

A complete graph K_n with n vertices is a graph in which every pair of distinct vertices is joined by an edge. In the next results we prove that $\chi^g_{CN}(K_n) = \lceil \frac{n}{4} \rceil + 1$, for $n \geq 2$, and $\chi^g_{ON}(K_n) = \lceil \frac{n+7}{4} \rceil$, for $n \geq 7$.

In the context of CFCN coloring of complete graphs, a question arises regarding why Bob cannot simply replicate the color used by Alice in her preceding move. For example, if Alice begins and the graph contains an even number of vertices, this strategy results in an invalid coloring by the legal coloring rule. Moreover, with the strategy presented in the next result, Bob duplicates more colors in fewer turns.

Theorem 4.1. *Alice wins the CFCN k-coloring game on a complete graph K_n, $n \geq 2$, when she starts, if and only if $k > \lceil \frac{n}{4} \rceil$.*

Proof. Since $N[v] = V(K_n)$, for every vertex $v \in V(K_n)$, to obtain a CFCN k-coloring for K_n, we observe that: (i) at least one color must appear only once in $V(K_n)$; and (ii) until coloring the last vertex in K_n no neighborhood is fully colored.

To prevent one color from appearing exactly once, Bob's strategy is to duplicate all k colors in $V(K_n)$, and to delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated. Thus, since Alice starts the game with a color c_1 (and chooses this color in her next turn), in the second turn, Bob colors a vertex with a color $c_2 \neq c_1$, to maximize the number of duplicated colors in the first four turns. Hence, he obtains two duplicated colors in the first four turns, as shown in Figure 7.

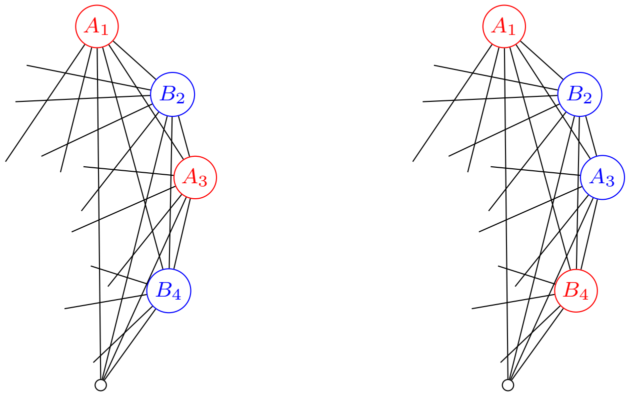


Fig. 7. Alice and Bob's first four turns on the CFCN k-coloring game on K_n, assuming that Alice started ((left) A_1, A_3 are red and B_2, B_4 are blue, (right) A_1, B_4 are red and B_2, A_3 are blue).

So, for every $4t$ turns Alice and Bob have used exactly $t+1$ colors and each of them has been used at least twice.

Suppose that $k \leq \lceil \frac{n}{4} \rceil$ and $n > 4$. Thus, there exists $t \in \mathbb{N}$ such that $4t < n \leq 4(t+1)$. Therefore,

$$k \leq \left\lceil \frac{n}{4} \right\rceil \leq \left\lceil \frac{4(t+1)}{4} \right\rceil = t+1,$$

and since $|V(K_n)| > 4t$, we have that, by the time Alice and Bob have colored $4t$ vertices, they have already used all the k colors at least twice, and the graph is not fully colored. Hence, Bob wins the game because there exists no color available to use only once.

Reciprocally, if $k > \lceil \frac{n}{4} \rceil$, since $4t < n \leq 4(t+1)$, we have that $t < \lceil \frac{n}{4} \rceil \leq t+1$, and so $\lceil \frac{n}{4} \rceil = t+1$. Thus, $k > \lceil \frac{n}{4} \rceil \geq t+2$ and Bob needs at least $4(t+1)+1$ vertices to duplicate $t+2$ colors. Hence, Alice wins since $|N(v)| \leq 4(t+1)$. □

Theorem 4.2. *Alice wins the CFCN k-coloring game on a complete graph K_n, $n \geq 2$, when Bob starts, if and only if $k > \lceil \frac{n-1}{4} \rceil$.*

Proof. The proof is similar to Theorem 4.1 since to prevent one color from appearing only once, Bob's strategy is to duplicate all k colors in $V(K_n)$ and to delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated. In that case, if Bob starts the game, Alice achieves that the first two colors are duplicated in the first five turns, delaying color duplication by one turn (see Figure 8).

□

Now, we analyze the *Conflict-Free Open Neighborhood k-coloring game*. We refer to Figure 9 and observe that the two unique ways of CFON k-coloring a complete graph K_n are the following:

- ($CFON_1$) all the k colors are chosen exactly twice when coloring all the vertices, or
- ($CFON_2$) at least two colors are chosen exactly once.

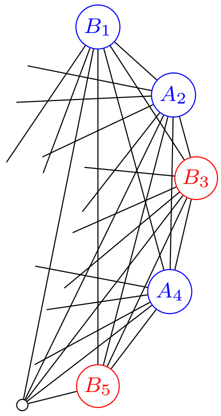


Fig. 8. Alice and Bob's first five turns on the CFCN k-coloring game on K_n, assuming that Bob started (B_3, B_5 are red and B_1, A_2, A_4 are blue).

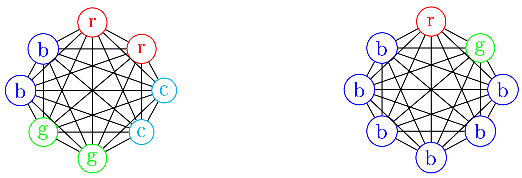


Fig. 9. Two ways to obtain a CFON k-coloring of K_8: ($CFON_1$) on the left, and ($CFON_2$) right (where b is blue, r is red, g is green and c cyan).

Theorem 4.3. *Alice wins the CFON k-coloring game on a complete graph K_n, when she starts, if and only if one of the following statements holds:*

- $n = 2$ *and* $k \geq 1$;
- $n = 4$ *and* $k \geq 2$;
- $n = 3, 5, 6$ *and* $k \geq 3$;
- $n \geq 7$ *and* $k \geq \lceil \frac{n+7}{4} \rceil$.

Proof. Since for all $v \in V(K_n)$ we have that $|N(v)| = n-1$ so, in the first $n-2$ turns, $N(v)$ is not fully colored, for all $v \in V(K_n)$.

If n is odd, only ($CFON_2$) can be used to obtain a CFON k-coloring of K_n. If n is even, both can be used: ($CFON_1$) requires $\frac{n}{2}$ colors to obtain a CFON k-coloring of K_n, and ($CFON_2$) only 3 colors. In Figure 9, we show a CFON k-coloring game of K_8 with $k = 4$ using ($CFON_1$), and $k = 3$ applying ($CFON_2$).

So, if $n \geq 8$, Alice reduces the number of colors needed to win with coloring ($CFON_2$). Since $n - 2 \geq 6$, in the first 5 turns, Alice chooses the same color with which she starts the game three times (no neighborhood is fully colored), discarding the possibility of each color appearing only two times to obtain a CFON k-coloring of K_n.

So, in both cases, to prevent a color from appearing only once, Bob's strategy is to duplicate all k colors in K_n and, to delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated.

We recall that, since Alice starts the game with a color c_1 (and chooses this color in her next turn), in the second turn, Bob colors a vertex with a color $c_2 \neq c_1$, to maximize the number of duplicated colors in the first four turns. Hence, he obtains two duplicated colors in the first four turns, With these strategies, for every $4t$ turns they have used exactly $t+1$ colors at least twice. Without loss of generality, we assume that Alice starts the game with color c_1 and chooses this color in her first $n-2$ turns.

If $n \geq 7$ and $k < \lceil \frac{n+7}{4} \rceil$, then there exists $t \in \mathbb{N}$ such that $4t < n-2 \leq 4(t+1)$, so $4t+2 < n \leq 4(t+1)+2$. Therefore,

$$k < \left\lceil \frac{n+7}{4} \right\rceil \leq \left\lceil \frac{4(t+1)+2+7}{4} \right\rceil = t+3.$$

Thus, $k \leq t+2$ and, since $|V(K_n) - 2| > 4t$, we have that: (i) by the time Alice and Bob have colored $4t$ vertices, they have already used all the $k-1$ colors at least twice; (ii) Alice chose color c_1 at least three times, and the graph is not fully colored. Hence, Bob wins the game, since there are no two colors that can appear only once in K_n.

Reciprocally, if $n \geq 7$ and $k \geq \lceil \frac{n+7}{4} \rceil$, we study four possible cases:

Case 1. Suppose that $n = 4t+3$ for some $t \in \mathbb{N}$. In the $(4t)$-th turn they have used exactly $t+1$ colors at least twice. Since in the $(4t+1)$-th turn Alice chooses the color c_1, in order to obtain a CFON k-coloring of K_n, in the $(4t+2)$-th and $(4t+3)$-th turns, by the legal coloring rule, they use two new colors that have not been chosen. Therefore, Alice wins if

$$k \geq t+3 = \left\lceil \frac{4t+10}{4} \right\rceil = \left\lceil \frac{4t+3+7}{4} \right\rceil = \left\lceil \frac{n+7}{4} \right\rceil.$$

Case 2. Suppose that $n = 4t+4$ for some $t \in \mathbb{N}$. In the $(4t)$-th turn they have used exactly $t+1$ colors at least twice. In the $(4t+1)$-th turn Alice chooses the color c_1. In the $(4t+2)$-th turn, Bob chooses a new color c_{t+2} (if there not exist such color, then Bob wins). In the turn $(4t+3)$-th, Alice chooses the color c_1 again. In the $(4t+4)$-th turn, by the legal coloring rule, Bob is forced to use a new color c_{t+3} (if such color does not exist, then Bob wins). Thus, Alice wins if

$$k \geq t+3 = \left\lceil \frac{4t+11}{4} \right\rceil = \left\lceil \frac{4t+4+7}{4} \right\rceil = \left\lceil \frac{n+7}{4} \right\rceil.$$

Case 3. If $n = 4t+5$ for some $t \in \mathbb{N}$ then, in the $(4t+3)$-th turn, Bob and Alice have used $t+2$ colors such that $t+1$ colors are used more than twice, and a single color is used only once. If in the $(4t+4)$-th turn Bob does not use a new color, then Alice does it in the final turn. If Bob uses a new color, then Alice just needs to use color c_1 again. In any case, Alice wins if

$$k \geq t+3 = \left\lceil \frac{4t+12}{4} \right\rceil = \left\lceil \frac{4t+5+7}{4} \right\rceil = \left\lceil \frac{n+7}{4} \right\rceil.$$

Case 4. If $n = 4t+6$ for some $t \in \mathbb{N}$ then, in the $(4t+4)$-th turn, Alice and Bob have used exactly $t+2$ colors at least twice. By the legal coloring rule, in the $(4t+5)$-th turn Alice used a new color c_{t+3} (if there is such color), and Bob is forced to use a color c_{t+4} never used before (if it exists). Hence, Alice wins if

$$k \geq t+4 = \left\lceil \frac{4t+14}{4} \right\rceil = \left\lceil \frac{4t+7+7}{4} \right\rceil = \left\lceil \frac{n+7}{4} \right\rceil.$$

□

Theorem 4.4. *Alice wins the CFON k-coloring game on a complete graph K_n, when Bob starts, if and only if one of the following statements holds:*
(a) $n = 2$ *and* $k \geq 1$*;*
(b) $n = 4$ *and* $k \geq 2$*;*
(c) $n = 3, 5, 6$ *and* $k \geq 3$*;*
(d) $n \geq 7$ *and* $k \geq \lceil \frac{n+6}{4} \rceil$.

Proof. The proof is similar to Theorem 4.3 but, in this case, the first two colors are duplicated in the first five turns. □

5. Paths

A *path* $P_n = (v_0, v_1, \ldots, v_{n-1})$ is a graph whose n vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if and only if they are consecutive in the sequence. We call the vertices $v_1, v_2, v_3, ..., v_{n-2}$ as internal vertices of P_n. In the next results, we prove that $\chi^g_{CN}(P_n) = 2$ and

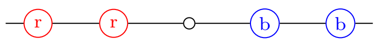
$$\chi^g_{ON}(P_n) = \begin{cases} 2 & , \text{ if } n \leq 7, \\ 3 & , \text{ if } n > 7. \end{cases}$$

Theorem 5.1. *Alice wins the CFCN k-coloring game on a path $P_n = (v_0, v_1, \ldots, v_{n-1})$ on $n \geq 2$ vertices with $k \geq 2$ colors, independently of who starts playing.*

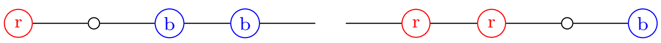
Proof.
Let P_n be a path with $n \geq 2$ and let c_1, c_2 and c_3 be three different colors. If $k = 2$ then, by the legal coloring rule, the only two strategies for Bob to win is by obtaining, on Alice's turn, the following colorings for subgraphs of P_n. We refer to Figure 10 for an illustration of Bob's strategies 1 and 2.

• (BS_1) coloring a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$, uniquely formed by internal vertices of P_n , such that vertices v_i, v_{i+1} are colored with a color c_1; vertices v_{i+3}, v_{i+4} are colored with a color c_2; and the vertex v_{i+2} is uncolored; or

• (BS_2) coloring the subgraph $P_4 = (v_0, v_1, v_2, v_3)$ (resp. $P_4 = (v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1})$), such that the vertex v_0 (resp. vertex v_{n-1}) is colored with a color c_1, and the vertices v_2 and v_3 (resp. vertices v_{n-4} and v_{n-3}) are colored with a color c_2, and vertex v_1 (resp. v_{n-2}) is uncolored.



(a) Bob's strategy BS_1.



(b) Bob's strategy BS_2.

Fig. 10. Winning strategies for Bob in the CFCN k-coloring game on P_n (where b is blue and r is red).

We prove that, Alice's strategy is to color any vertex adjacent to the vertex colored by Bob in the previous turn, with a color different from the one that Bob used. In case all the vertices adjacent to the vertex colored by Bob have already been colored, it is enough for Alice to choose any uncolored vertex w adjacent to a colored vertex v, and to color w with a color that is different to the one used in v.

If $n = 2, 3$, Alice wins, since the strategies (BS_1) and (BS_2) need at least four vertices.

For $n = 4$, without loss of generality, we analyze the game when Bob starts coloring v_0 or v_1. If Bob decides to start the game by coloring v_0 with a color c_1 then, on the second turn, Alice colors v_1 with a color c_2. Hence, no P_4 subgraph can be colored as in (BS_2). On the other hand, if Bob starts the game by coloring the vertex v_1 with a color c_1 then, on the next turn, Alice either colors the vertex v_0 or colors the vertex v_2 with a color c_2. Again, no P_4 subgraph can be colored as in (BS_2). In both cases, Alice wins.

For $n = 5, 6$, Bob's unique strategy is to consider subgraphs P_4 (BS_2). Indeed, strategy (BS_1) can not be used because P_5 or P_6 do not have five internal vertices. Without loss of generality, suppose that Bob tries to win by obtaining a subgraph $P_4 = (v_0, v_1, v_2, v_3)$ (it is analogous for $P_4 = (v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1})$). If Bob starts coloring any vertex v_i, with $0 \leq i \leq 2$ (resp. v_3) then, in the next turn, Alice colors an adjacent vertex (resp. vertex v_2) with a different color. Thus, no $P_4 = (v_0, v_1, v_2, v_3)$ subgraph can be colored as in (BS_2), and Alice wins.

Let $n \geq 7$. First, suppose that Bob tries to win obtaining a subgraph $P_5 = (v_i,\ v_{i+1},\ v_{i+2},\ v_{i+3},\ v_{i+4})$, that has the coloring (BS_1). Assume that it is Bob's turn and he looks for a subgraph P_5, that has only v_i colored with a color c_1 and v_{i+4} colored with c_2. We observe that if any internal vertex v_j, with $j \in \{i+1, i+2, i+3\}$, on the subgraph P_5 is already colored then, by Alice's strategy, no $P_5 = (v_i,\ v_{i+1},\ v_{i+2},\ v_{i+3},\ v_{i+4})$ subgraph can be colored as in (BS_1) (there exist two adjacent vertices of different colors in P_5), and Alice wins. Thus, in his turn, Bob tries to color v_{i+1} or v_{i+3}, with the same color of its adjacent vertex. In the next turn, Alice colors the vertex v_{i+2} and wins. If Bob tries to win by obtaining a subgraph P_4 with the coloring (BS_1), the proof that Alice wins is similar to the one used on the paths of order 5 or 6.

It may seem that since Alice wins when Bob starts the game, if she starts and follows the same strategy, then she wins. However, this is not the case. Assume that Bob follows strategy (BS_2) and tries to win on the subgraph $P_4 = (v_0, v_1, v_2, v_3)$. If Alice starts coloring vertex v_2 with a color c_1, then Bob colors v_3 with the same color c_1. Following her strategy, Alice colors v_4 with a different color. Now, Bob colors v_0 with color c_2, and he wins.

In the same way, if Bob finds a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$ such that v_i and v_{i+1} (resp. v_{i+3} and v_{i+4}) are colored with the same color c_1 and the remaining vertices in P_5 are uncolored (possible if Alice starts the game), then he colors v_{i+4} (resp. v_i) with a different color c_2. If in the next turn, Alice colors the adjacent vertex to v_{i+4} (resp. v_i) that is not in this subgraph P_5, then Bob colors the vertex v_{i+3} (resp. v_{i+1}) with the color c_2, and he wins.

In order to avoid that, Alice starts the game coloring v_1 or v_{n-2} of $P_n = (v_0,\ v_1,\ \ldots,\ v_{n-1})$ (preventing Bob from winning in subgraphs P_4 or P_5). From the third turn onward, she colors the adjacent vertices of those Bob colored (as in the case Bob started the game). Therefore, Alice wins the game with $k = 2$ colors.

If $k > 2$, then Alice colors either vertex v_{i+2} on subgraph P_5; or vertex v_1 (or v_{n-2}) on subgraph P_4 with a color c_3, winning the game. □

Now, we analyze the CFON k-coloring game on P_n. We show in Figure 11 the unique strategy for Bob to win the CFON k-coloring game in P_n.

• (BS_3) coloring a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$, such that vertex v_i is colored with a color c_1 and vertex v_{i+4} is colored with a color c_2, $c_1 \neq c_2$.

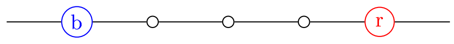


Fig. 11. The unique way for Bob to win the CFON k-coloring game in P_n (where b is blue and r is red).

Theorem 5.2. *Bob wins the CFON k-coloring game on a path P_n, when Alice starts, if and only if $n > 7$ and $k = 2$.*

Proof. Let $P_n = (v_0, v_1, \ldots, v_{n-1})$ be a path with $n \geq 3$, let c_1, c_2 and c_3 be three different colors. If $k = 2$, the only way for Bob to win is considering, on Alice's turn, a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$, such that the vertex v_i is colored with a color c_1, and the vertex v_{i+4} with a color c_2. In that case, by the legal coloring rule, vertex v_{i+2} cannot be colored either with c_1, because of $N(v_{i+1})$; nor with c_2, because of $N(v_{i+3})$.

If $n \in \{2, 3, 4\}$ it is immediate that Alice wins the game. If $n = 5$ then Alice starts coloring the vertex v_2, and wins. If $n = 6$ then Alice starts coloring v_2 (resp. v_3) and, in the third turn, she colors v_3 (resp. v_2), and wins.

If $n = 7$, then Alice starts coloring the vertex v_3. By the legal coloring rule, Bob cannot color vertices v_1 and v_5 with different colors. Thus, in the next turn, Bob colors v_i, $i \in \{0, 2\}$ (resp. $i \in \{4, 6\}$). Alice colors v_{i+4} (resp. v_{i-4}) with the same color that Bob used. Alice wins because it is not possible to apply (BS_3) in one of the three subgraphs P_5.

If $n > 7$, independently of which vertex v_i (resp. v_{i+4}) Alice chooses to color in her first turn, Bob can always find a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}.v_{i+3}, v_{i+4})$, and color v_{i+4} (resp. v_i) with a different color.

Now assume $k > 2$. If $n \in \{1, 2, 3\}$ then it is immediate that Alice wins the game. If $n \geq 4$, then Alice colors vertex v_{i+2} of the subgraphs $P_5 = (v_i, v_{i+1}, v_{i+2}.v_{i+3}, v_{i+4})$, with a color c_3, winning the game. □

Theorem 5.3. *Bob wins the CFON k-coloring game on a path P_n, when he starts the game, if and only if $n > 8$ and $k = 2$.*

Proof. If $n \in \{2, 3, 4\}$ then it is immediate that Alice wins the game. If $n = 5$, then Alice colors vertex v_2, and wins. For $n \in \{6, 7, 8\}$, if Bob starts coloring a vertex v_j, then Alice colors either vertex v_{j-4}, if $j > 3$; or vertex v_{j+4}, if $j < 3$. In both cases, she prevents Bob to win the game.

If $n > 8$ and $k = 2$ then Bob starts coloring vertex v_4 with a color c_1. We note that there are two subgraphs P_5 such that v_4 is either the last or the first vertex. Now, independently of which vertex Alice colors, Bob wins the game.

Now, assume that $k > 2$. If $n \in \{1, 2, 3\}$, then it is immediate that Alice wins the game. If $n > 3$, then Alice colors the vertex v_{i+2} of all subgraphs P_5 with a color c_3, ensuring her victory in the game. □

6. Cycles

A *cycle* $C_n = (v_0, v_1, \ldots, v_n)$, with $n \geq 3$ and $v_0 = v_n$ is a graph whose n vertices can be arranged in a cyclic sequence in a way that two vertices are adjacent if and only if they are consecutive in the sequence. Next, we prove that $\chi^g_{CN}(C_n) = 2$ and $\chi^g_{ON}(C_n) = 3$.

The approach for cycles follows a similar pattern as with paths. However, Bob's victory is restricted on applying the strategy (BS_1). As previously, Alice adopts a strategy where she colors any vertex adjacent to Bob's previously colored vertex with a distinct color. In the event that all adjacent vertices to Bob's colored vertex have already been colored, Alice can ensure her advantage by selecting any uncolored vertex, denoted as w, that is adjacent to a previously colored vertex v, and coloring w with a different color than v.

Theorem 6.1. *Alice wins the CFCN k-coloring game on a cycle C_n with $k \geq 2$ colors, independently of who starts playing.*

Proof. Let $k = 2$. It is easy to see that Alice wins the game for $n \in \{3, 4\}$. Let $n = 5$ and consider that Alice starts the game coloring v_0 with color c_1. Independently of the vertex colored by Bob in the second turn, Alice colors an uncolored vertex v_2 or v_3 with color c_1.

Let $n \geq 6$. Assuming Bob is the first player, at some point in the game, he attempts to find a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$ in which v_i is colored with color c_1, v_{i+4} is colored with c_2 $(c_1 \neq c_2)$, and the remaining vertices of P_5 are uncolored. Similarly to paths, Alice's strategy is to color (on the subgraph P_5) either vertex v_{i+2} or the vertices v_{i+1} (resp. v_{i+3}) with a different color from that used for v_i (resp. v_{i+4}).

If Alice starts the game, her strategy has a slight variation. Consider the vertex v_i colored by Bob in his last turn as the reference vertex. If there exist two adjacent colored vertices with the same color in the clockwise direction of v_i, then Alice colors the vertex v_{i+1}. However, if these two vertices are in the counterclockwise direction, Alice colors the vertex v_{i-1}. Otherwise, if there exist no adjacent colored vertices with the same color in either the clockwise or counterclockwise direction, Alice simply colors any of the adjacent vertices.

This variant is used because, if Alice doesn't have a preference for coloring the adjacent vertex based on the direction of the adjacent colored vertices, it could potentially allow Bob to win the game. Indeed, suppose that Alice starts the game coloring a vertex v_i with a color c_1. Bob colors a vertex v_{i+1} with color c_1. Now, Alice colors v_{i+2} with a different color c_2. If there exist at least three uncolored vertices $\{v_{i-1}, v_{i-2}, v_{i-3}\}$ adjacent to v_i, then Bob tries (BS_1) on the subgraph $P_5 = (v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1})$, and colors v_{i-3} with a color c_2. With the variant strategy, Alice colors the adjacent vertex v_{i-2} with color c_1, and wins the game.

For $k > 2$, the idea is the same as in the Theorem 5.1. Therefore, Alice always wins the game. □

Since the results for the game on C_3 are addressed by Theorems 4.3 and 4.4, we now consider the CFON k-coloring game on C_n for $n \geq 4$ and $k \geq 2$.

Theorem 6.2. *Bob wins the CFON k-coloring game on a cycle C_n, $n > 4$, when Alice starts the game, if and only if $k = 2$.*

Proof. Again, the strategy applied to cycles C_n, with $n > 4$, is similar to that of paths. If $k = 2$, then Bob can only win by considering a subgraph $P_5 = (v_i,\ v_{i+1},\ v_{i+2},\ v_{i+3},\ v_{i+4})$, where the vertex v_i is colored with a color c_1 and the vertex v_{i+4} is colored with a different color c_2. The proof follows a similar approach to Theorem 5.2. Regardless of the vertex Alice chooses to color in the first turn, Bob can always find a subgraph $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$ where v_i is the vertex colored by Alice, and Bob can color v_{i+4} with another color, winning the game.

Conversely, if $k > 2$, Alice can always color the vertex v_{i+2} in all subgraphs P_5 with a color c_3, such that $c(v_i) \neq c(v_{i+4}) \neq c_3$. Therefore, she wins the game. □

Theorem 6.3. *Bob wins the CFON k-coloring game on a cycle C_n, $n > 4$, when he starts the game, if and only if $n \neq 8$ and $k = 2$.*

Proof. If $n = 8$ then, whenever Bob colors a vertex v_j, Alice colors the vertex $v_{j+4 \pmod n}$ with the same color used by Bob. Let $P_5 = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$ be a subgraph of C_n. If $k > 2$, then Alice can always color the uncolored vertices v_{i+2} of all subgraphs P_5 with a color c_3, such that $c(v_i) \neq c(v_{i+4}) \neq c_3$. In both cases, Alice wins the game.

Conversely, assume that $n \neq 8$ and $k = 2$. If $n = 6$ and Bob starts coloring the vertex v_j, Alice can not color the vertex $v_{j+4 \pmod n}$ (resp. $v_{j+2 \pmod n}$) with the same color, otherwise, the open neighborhood of $v_{j+5 \pmod n}$ (resp. $v_{j+1 \pmod n}$) would be monochromatic. Hence, Bob wins the game. In the cases $n \notin \{6, 8\}$, regardless of which vertex Alice chooses to color in the second turn, Bob can always find, in the third turn, a subgraph P_5 that includes either v_i or v_{i+4} as the vertex colored by Alice. □

7. Generalized windmill graph

The *windmill graph* $W(n, p)$ with $n \geq 2$, $p \geq 2$, is composed by the disjoint union of p complete graphs K_n of order n joined with a single vertex u called the *universal vertex*. We refer to Figure 12 for an example of the windmill graph $W(2, p)$, which is also known as *friendship* or *Dutch windmill* graph.

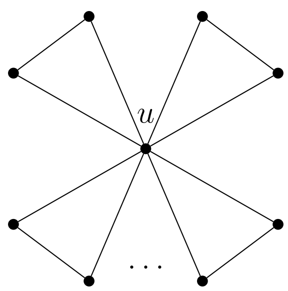


Fig. 12. Windmill $W(2, p)$ (friendship graph)

Similarly, a *generalized windmill graph* $W(N, p)$, with $N = \{n_1, n_2, \ldots, n_p\}$, $p \geq 2$, is composed by the disjoint union of p complete graphs K_{n_i} of order n_i, with $n_i \geq 2$, for $1 \leq i \leq p$, joined with a single universal vertex u. See an example of $W(\{3, 2, 4, 2\}, 4)$ in Figure 13.

In this section, we study the CFCN k-coloring game on generalized windmill graphs. Let $c : V(W(N, p)) \to \{c_1, c_2 \ldots, c_k\}$ be a k-vertex coloring of $W(N, p)$, where $\{c_1, c_2, ..., c_k\}$ represents a set of k different colors. Let $c^{-1}(c_i)$ be the subset of vertices assigned to color c_i, $i \in \{1, \ldots, k\}$ (color class). We observe that in a CFCN k-coloring of $W(N, p)$, one of the following statements holds:

(i) the color of the universal vertex appears only once;

(ii) the color of the universal vertex is duplicated but, for every $1 \leq i \leq p$, at least one color c_i appears exactly once in K_{n_i} and there exists $1 \leq j \leq p$ such that $|c^{-1}(c_j)| = 1$.

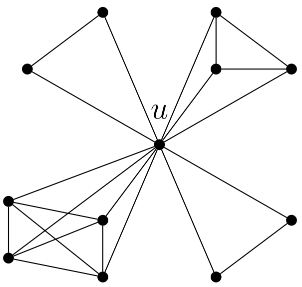


Fig. 13. Generalized windmill $W(\{3, 2, 4, 2\}, 4)$.

Again, Bob's strategy is to duplicate each of the k colors that are used and, to delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated.

Lemma 7.1. *If Alice tries to use as few colors as possible in the CFCN k-coloring game on $W(N, p)$, then Bob can always duplicate the color of the universal vertex u in the first 4 turns.*

Proof. We prove this result considering $W(2, 2)$, since $W(2, 2)$ is a induced subgraph of $W(N, p)$ for any arbitrary windmill graph $W(N, p)$, $N \geq 2$ and $p \geq 2$. We refer to Figures 14 and 15.

If Alice starts the game and colors vertex u with c_1, then Bob colors any vertex on $W(2, 2)$ with c_2. No matter which vertex Alice colors with c_1 (resp. c_2), in the fourth turn, Bob colors any vertex with c_2 (resp. c_1). If Alice starts the game and colors a vertex $v \neq u$ with c_1 then, in the next turn, Bob colors u with c_2. The case is similar if Bob starts the game. In any case, Bob duplicates the first two colors in the first four turns. $\square$

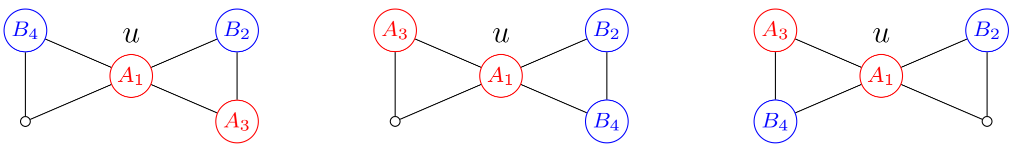


Fig. 14. Alice starts on the universal vertex, and Bob duplicates the color of the universal vertex of W(2, 2) in the first four turns (Alice colors with red and Bob with blue).

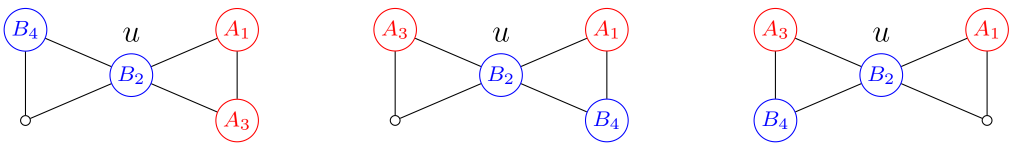


Fig. 15. Alice starts at a vertex different from the universal vertex, and Bob duplicates the color of the universal vertex of W(2, 2) in the first four turns (Alice colors with red and Bob with blue).

By Lemma 7.1, we have the following result:

Lemma 7.2. *Assume that Alice tries to use as few colors as possible and Bob duplicates the color of the universal vertex in the CFCN k-coloring game on $W(N,p)$. If there exist at least two uncolored vertices in $W(N,p)$, then Bob always duplicates a color that appeared only once.*

Proof. Without loss of generality, assume that there are only two uncolored vertices w and w', and that $c(v_i)$ appears only once. By Lemma 7.1, since the color of the universal vertex u is duplicated in the first four turns, then $v_i \in K_{n_i}$ and $w \neq w' \neq u$.

If $w, w' \in V(K_{n_j})$ (K_{n_j} is not fully colored), then one of them can be colored with $c(v_i)$. However, if they are in different K_{n_j}, then it is sufficient to color with $c(v_i)$ the uncolored vertex in K_{n_j} with $i \neq j$. □

To ensure that both players play optimally, we also need to verify the following result:

Lemma 7.3. *Assume that Bob duplicates the color of the universal vertex in the CFCN k-coloring game on $W(N,p)$. If there exist at least two uncolored vertices in $W(N,p)$, then Alice always chooses either the same color or any of the colors that have already been chosen at least twice.*

Proof. Without loss of generality, we assume that there are only two uncolored vertices. Furthermore, suppose that it is Alice's turn, she has chosen the same color c_1 in all previous turns, and, due to the legal coloring rule, she is now forced to select a new color. We claim that Alice chooses a duplicated color. If the two uncolored vertices are on the same K_{n_i}, then one of them can be colored with c_1 (K_{n_i} is not fully colored), a contradiction. Thus, one uncolored vertex is in K_{n_i} and the other one is in K_{n_j}, $1 \leq i, j \leq p$, $i \neq j$. If color c_1 cannot be chosen, by the legal coloring rule, either c_1 appears only once and the remaining colors appear at least twice in K_{n_i}, or every color appears at least twice in K_{n_i}. In any case, since Bob chooses a new color every time he achieves a color duplication, we have that the colors duplicated in some K_{n_i} are not duplicated in another K_{n_j}, for $1 \leq i, j \leq p$, $i \neq j$. So, there always exists a duplicated color in K_{n_i} (resp. K_{n_j}) that can be chosen by Alice in K_{n_j} (resp. K_{n_i}). □

The next two results show that

$$\chi^g_{CN}(W(N,p)) = \left\lceil \frac{|V(W(N,p))|+4}{4} \right\rceil.$$

Theorem 7.4. *Alice wins the CFCN k-coloring game on a generalized windmill graph $W(N,p)$, when she starts, if and only if*

$$k > \left\lceil \frac{|V(W(N,p))|}{4} \right\rceil.$$

In particular, since $|V(W(n,p))| = n \cdot p + 1$, she wins the CFCN k-coloring game on a windmill graph $W(n,p)$, when she starts, if and only if $k > \lceil \frac{n \cdot p+1}{4} \rceil$.

Proof. By Lemmas 7.2 and 7.3, before coloring the last vertex, for every $4t$ turns Alice and Bob have used exactly $t+1$ colors, and each of them has been used at least twice (the proof is by induction on t).

Since $|W(N,p)| > 4$, there exists $t \in \mathbb{N}^*$ such that $4t < |W(N,p)| \leq 4(t+1)$. If $k \leq \left\lceil \frac{|W(N,p)|}{4} \right\rceil$, then

$$k \leq \left\lceil \frac{|W(N,p)|}{4} \right\rceil \leq \left\lceil \frac{4(t+1)}{4} \right\rceil = t+1,$$

and we have that by the time Alice and Bob have colored $4t$ vertices, they have already used all the k colors at least twice. Furthermore, since $4t < |W(N,p)|$, the graph is not fully colored. Hence, Bob wins the game because there exists no color available to use only once.

Reciprocally, if $k > \lceil \frac{|W(N,p)|}{4} \rceil$, since $4t < |W(N,p)| \leq 4(t+1)$, then $t < \lceil \frac{|W(N,p)|}{4} \rceil \leq t+1$. Thus, $t+1 = \lceil \frac{|W(N,p)|}{4} \rceil < k$. So, $k \geq t+2$ and, to duplicate $t+2$ colors, Bob needs at least $4(t+1)+1$ vertices. Since $|W(N,p)| \leq 4(t+1)$, Alice wins the game. □

Now we show that, when Bob starts the CFCN k-coloring game on a generalized windmill graph $W(N,p)$, Alice requires one fewer color to win.

Theorem 7.5. *Alice wins the CFCN k-coloring game on a generalized windmill graph $W(N,p)$, when Bob starts, if and only if*

$$k > \left\lceil \frac{|W(N,p)|-1}{4} \right\rceil .$$

In particular, since $|V(W(n,p))| = n \cdot p + 1$, she wins the CFCN k-coloring game on a windmill graph $W(n,p)$, when Bob starts, if and only if $k > \lceil \frac{n \cdot p}{4} \rceil$.

Proof. The proof is similar to Theorem 7.4. Again, to prevent one color from appearing only once, Bob's strategy is to duplicate all k colors in $W(N,p)$, and to delay this duplication, Alice chooses as few colors as possible using colors that have already been duplicated. If Bob starts the game, then the first two colors are duplicated in the first five turns, delaying color duplication by one turn. □

8. Conclusion

In this work, we study the Conflict-Free k-coloring games on classic graph classes such as stars, complete graphs, paths, cycles, and windmill graphs and their generalization. In each of them, we show strategies that determine the least number of colors necessary for Alice to win the game.

We recall that the *Closed* (resp. *Open*) *Neighborhood Conflict-Free game Chromatic Number* of G, denoted by $\chi^g_{CN}(G)$ (resp. $\chi^g_{ON}(G)$), is the minimum number k of colors necessary for Alice to have a winning strategy for the CFCN (resp. CFON) k-coloring game on G, independently of who starts the game.

In Table 1, we show our results on the Conflict-free k-coloring game comparing $\chi_{CN}(G)$, $\chi^g_{CN}(G)$, $\chi_{ON}(G)$ and $\chi^g_{ON}(G)$. We remark that it may seem that if Alice wins when Bob starts the game, then she also wins when she starts the game. However, this is not always true. In fact, since Alice's general strategies are based on preventing Bob from achieving his general strategies, when Alice starts, her turn can eventually contribute to improve Bob's strategy to win the game. For example, in the CFCN 2-coloring game on the complete graph K_5, Alice wins when Bob starts the game, and loses when she starts the game (see Theorem 4.1 and Theorem 4.2). Sometimes it is possible to prevent that by improving Alice's strategies, as it occurs in the CFCN 2-coloring game on paths P_n and cycles C_n with $n \geq 5$, (see Theorem 5.1 and Theorem 6.1).

Finally, the general strategies lead us to pose the following question:

Table 1. Conflict-free k-coloring game

Graph	$\chi_{CN}(G)$	$\chi^g_{CN}(G)$	$\chi_{ON}(G)$	$\chi^g_{ON}(G)$	Theorems
S_{n-1}	2	2	2	$\lceil \frac{n-1}{4} \rceil + 1$	3.1, 3.2, 3.3
K_2	1	1	1	1	4.1, 4.2, 4.3, 4.4
K_4	2	2	2	2	4.1, 4.2, 4.3, 4.4
K_3, K_5, K_6	3	3	3	3	4.1, 4.2, 4.3, 4.4
K_n $(n \geq 7)$	2	$\lceil \frac{n}{4} \rceil + 1$	3	$\lceil \frac{n+7}{4} \rceil$	4.1, 4.2, 4.3, 4.4
P_n $(n \leq 7)$	2	2	2	2	5.1, 5.2, 5.3
P_n $(n > 7)$	2	2	2	3	5.1, 5.2, 5.3
C_n	2	2	2	3	6.1, 6.2, 6.3
$W(N,p)$	2	$\lceil \frac{\lvert V(W(N,p)) \rvert + 4}{9} \rceil$	-	-	7.4, 7.5

Question 8.1 (Monotonicity)**.** *Assume that Alice wins the CFCN k-coloring game (resp. CFON k-coloring game) on G. Does she win the CFCN (resp. CFON) $(k+1)$-coloring game on G when the same player starts the game?*

Author Contributions

All authors contributed equally to the research, analysis, and writing of this paper.

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Conflicts of Interest

All authors declare that they have no conflicts of interest.

Declaration of competing interest

There is no conflict of interest related to this work.

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