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On total coloring of 1-planar graphs without 4-cycles

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ABSTRACT

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, we confirm the total-coloring conjecture for 1-planar graphs without 4-cycles with maximum degree $\Delta \geq 10$.

Keywords: planar graphs, graph coloring, planar graph theory, edge crossings

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1. Introduction

All graphs considered are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices that are adjacent to v in G. By $d(v) := |N_G(v)|$ denotes the *degree* of v in G. For planar graphs G, $F(G)$ denotes its face set, the degree of a face f, denoted by $d(f)$, is the length of a boundary walk around f in G. We call v a k-vertex, or a k^+-vertex, or a k^--vertex if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$ respectively and call f a k-face, or a k^+-face, or a k^--face if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$ respectively. Any undefined notation follows that of Bondy and Murty [2]. A $total-k-coloring$ of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total-k-coloring. It is clearly that $\chi''(G) \geq \Delta(G)+1$. Behzad and Vizing [1, 6] posed independently the conjecture, $\chi''(G) \leq \Delta(G)+2$ for any graph G, which is known as the total coloring conjecture.

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of a 1-planar graph was introduced by Ringel [4] in connection with the problem of simultaneous coloring of adjacent/incident vertices and faces of plane graphs. In [10], Zhang et al. proved that every 1-planar graph with maximum degree $\Delta(G) \geq 16$ is totally $(\Delta(G)+2)$-choosable,

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which implies that the total-coloring conjecture holds for 1-planar graphs with maximum degree at least 16. Later, Czap [3] proved (Without discharging method) that for every 1-planar graph G with $\Delta(G) \geq 10$ it holds $\chi''(G) \leq \Delta(G)+3$. Moreover, if $\chi(G) \leq 4$, then $\chi''(G) \leq \Delta(G)+2$. In the same paper, the author also verified that for every 1-planar graph G without adjacent triangles and with $\Delta(G) \geq 10$ it holds $\chi''(G) \leq \Delta(G)+3$. Moreover, if $\chi(G) \leq 4$, then $\chi''(G) \leq \Delta(G)+2$. Zhang and Hou [7] showed the following theorem which improve the lower bound for the maximum degree in the corollary of [10] to 13.

Recently, Sun and Wu [5] verified the total coloring $\chi''(G) \leq r+2$, for every 1-planar graph G if $\Delta(G) \geq 9$ and $g(G) \geq 4$ where $\Delta(G)$ is the maximum degree of G and $g(G)$ is the girth of G

Theorem 1.1. *Let G be a 1-planar graph with maximum degree $\Delta(G)$ and let r be an integer. If $\Delta \leq r$ and $r \geq 13$, then $\chi''(G) \leq r+2$.*

In this paper, we shall prove the following results:

Theorem 1.2. *Let G be a 1-planar graph without 4-cycles, with maximum degree $\Delta(G) \geq 10$. Then $\chi''(G) \leq \Delta(G)+2$.*

2. Preliminaries

Let G in this paper has been embedded on a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. The *associated plane graph* $G^{\times}$ of G is obtained by turning all crossings of G into new 4-vertices on a plane. For convenience, a vertex in $G^{\times}$ is called $false$ if it is not a vertex of G and *real* otherwise. A $false\ face$ means a face f in $G^{\times}$ that is incident with at least one $false\ vertex$; otherwise, f is a $normal\ face$. For a vertex $v \in V(G^{\times})$, we use $f_k(v)$ to denote the number of k-faces which are incident with v, $n_i(v)$ to denote the number of i-vertices which are adjacent to v, and $n_c(v)$ to denote the number of false vertices which are adjacent to v.

For convenience, we use $v_1, v_2, \cdots, v_d$ to denote the neighbors of a d-vertex v in $G^{\times}$ that occur around it in a clockwise order. By f_i denote the face incident with vv_i and vv_{i+1} in $G^{\times}$, where the addition on subscripts are taken modulo d.

Let G be a counterexample with $|E(G)|$ as small as possible to Theorem 1.2. By minimality of G we can assume that it is connected and that it has no total $(\Delta(G)+2)$-colorings. First we investigate some structural of properties of G. Here, we give some known lemmas.

Lemma 2.1. [10] *Let uv be an edge in G. If $\min\{d_G(u), d_G(v)\} \leq \lfloor \frac{\Delta+1}{2} \rfloor$, then $d_G(u)+d_G(v) \geq \Delta+3$.*

From this lemma, we deduce that $\delta(G) \geq 3$.

Lemma 2.2. [7] *Let V_i be the set of i-vertices in G. We have $|V_\Delta| > 2|V_3|$.*

Lemma 2.3. *Let G be a 1-planar graph without 4-cycles and let $G^{\times}$ be its associated plane graph. Then for every 5^+ vertex $v \in V(G)$, v is incident with at most $\lfloor \frac{4}{5} d_G(v) \rfloor$ 3-faces in $G^{\times}$.*

The proof is just similar to the one in [8], with only quite a little minor changes. So we omit it here and refer the reader to Lemma 4 of [8].

Lemma 2.4. [9] *Let G be a 1-plane graph and let $G^{\times}$ be its associated plane graph. Then the following hold:*

1) For any two false vertices u and v in $G^{\times}$, u and v are not adjacent in $G^{\times}$.

2) If $d_G(u) = 3$ and v is a false vertex in $G^{\times}$, then either u and v are not adjacent in $G^{\times}$, or uv is not incident with two 3-faces.

3) Let v be a 3-vertex in G. If v is incident with two false 3-faces vv_1v_2 and vv_1v_3 in $G^{\times}$, then v_2 and v_3 are both false and v is incident with a 5^+-face in $G^{\times}$.

Lemma 2.5. *Let G be a 1-plane graph and let $G^{\times}$ be its associated plane graph. Then, every 5-face in $G^{\times}$ is incident with at most four 5^--vertices.*

The proof is just similar to the one in [7], with only quite a little minor changes. So we omit it here and refer the reader to Lemma 9 of [7].

Lemma 2.6. [10] *For each integer $3 \le k \le 5$, let $X_k = \{x \in V(G) | d_G(x) \le k\}$, $Y_k = \bigcup_{x \in X_k} N_G(x)$. If $X_k \neq \emptyset$, then there exists a bipartite subgraph $M_k = (X_k, Y_k)$ of G such that $d_{M_k}(x) = 1$ for any $x \in X_k$ and $d_{M_k}(y) \le k - 2$ for any $y \in Y_k$. We call y the k-master of x if $xy \in M_k$ and $x \in X_k$.*

By this lemma, we deduce that each k-vertex ($3 \le k \le 5$) has a j-master ($k \le j \le 5$).

Lemma 2.7. [7] *Let G be a 1-plane graph and let v be a vertex of G. If $d_G(v) = 3$, then, v cannot be contained in a triangle in G. If $d_G(v) = 4$ with $N_G(v) = v_1, v_2, v_3, v_4$, then, for any i,$(1 \le i \le 4)$, the edge vv_i can not be contained in two triangles.*

Lemma 2.8. *Let G be a 1-plane graph without 4-cycles and $G^{\times}$ be its associated plane graph. Let v be a vertex of G, then, there are no five consecutive 3-faces that are incident with v in $G^{\times}$. If v is incident with i consecutive 3-faces $f_1, f_2, \cdots, f_i$, $(3 \le i \le 4)$ in $G^{\times}$, then, there is at most a real small vertex among the neighbors of v on these consecutive 3-faces. Moreover, if v is incident with 4 consecutive 3-faces f_1, f_2, f_3, f_4, then v_1, v_3, v_5 are false vertices, v_2, v_4 are real vertices.*

The proof is just similar to the one in [8], with only quite a little minor changes. So we omit it here and refer the reader to Lemma 4 of [8].

3. The proof of Theorem 1.2

Then, we begin to prove the main result of the paper.

A vertex v in G is *small* if $d(v) \le 5$ and is *big* if $d(v) \ge 6$. Note that the degree of a false vertex in $G^{\times}$ is four, so every false vertex is small.

In the following, we apply the discharging method on associated 1-planar graph $G^{\times}$ of G and complete the proof by a contradiction. Since $G^{\times}$ is a plane graph, we have

$$\sum_{v \in V(G^{\times})} (d(v) - 6) + \sum_{f \in F(G^{\times})} (2d(f) - 6) = -12,$$

by the well-known Euler's formular. Now we define the initial charge function $ch(x)$ of $x \in V(G^{\times}) \cup F(G^{\times})$. Let $ch(v) = d(v) - 6$ if $x \in V(G^{\times})$ and $ch(f) = 2d(f) - 6$ if $x \in F(G^{\times})$. And we define suitable discharging rules below to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(G^{\times}) \cup F(G^{\times})$. Then we still have $\sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch'(x) = \sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch(x) = -12$, since any discharging procedure preserves the total charge of $G^\times$.

Our discharging rules are defined as follows.

R1. Each f in $G^\times$ where $d(f) \geq 4$ sends $\frac{2d(f)-6}{t(f)}$ to each small vertex incident with it, where $t(f)$ is the number of small vertices incident with the face f.

R2. Each 3-vertex in G receives $\frac{2}{9}$ from its i-master $(3 \leq i \leq 5)$.

R3. Each 4-vertex in G receives $\frac{6}{25}$ from its i-master $(4 \leq i \leq 5)$.

R4. Each Δ-vertex gives $\frac{1}{2}$ to a common pot from which each 3-vertex receives 1, if $|V_3| > 0$.

R5. Let w be a false vertex and w is incident with a 3-face f in $G^\times$, then each 8^+-neighbor of w on f sends $\frac{13}{50}$ to w.

R6. Let w be a real 4-vertex and w is incident with a normal 3-face f in $G^\times$, then each 8^+-neighbor of w on f sends $\frac{13}{50}$ to w.

R7. Let u be a real 4 -vertex, v is a false vertex in $G^\times$, $uv \in E(G^\times)$ and uv is incident with two 3-faces in $G^\times$, then v sends $\frac{13}{25}$ to u.

R8. If a false vertex v in $G^\times$ is incident with four 4^+-faces in $G^\times$, then v sends $\frac{5}{12}$ to each 4-neighbor of v.

R9. If a false vertex v in $G^\times$ is incident with exactly one 3-face f in $G^\times$, then v sends $\frac{1}{3}$ to its 3-neighbor on f.

R10. Let v be a 3-vertex and v is not incident with any 3-face in $G^\times$, then v sends $\frac{1}{6}$ to each false vertex which is adjacent to v.

R11. If a real 4-vertex v in $G^\times$ is incident with four 4^+-faces in $G^\times$, then v sends $\frac{11}{75}$ to each false vertex which is adjacent to v.

R12. If a false vertex u in $G^\times$ is adjacent to a 5-vertex v in $G^\times$, and uv is incident with 4^+-faces f_1 and f_2 which are adjacent in $G^\times$, then v sends $\frac{1}{6}$ to u.

In the following, we check that the final charge $ch'(x)$ on each vertex and face is nonnegative, and we also show the final charge of the common pot is nonnegative. This implies that $\sum_{x\in V(G^\times)\cup F(G^\times)} ch'(x) \geq 0$ for all $x \in V(G^\times) \cup F(G^\times)$, a contradiction. This completes the proof of Theorem 1.2.

First of all, by R4, the final charge of the common pot is at least $\frac{1}{2}|V_\Delta| - |V_3| > 0$ since $|V_\Delta| > 2|V_3|$ by Lemma 2.2. One can also check that the final charge of every face in $F(G^\times)$ is nonnegative by R1. Thus in the following we consider the vertices in $G^\times$.

Case 1. $d = 3$. By R2 and R4, v receives 1 from the common pot and $\frac{2}{9} \times 3 = \frac{2}{3}$ from its i-masters, where $3 \leq i \leq 5$. Since G is a 1-planar graph without 4-cycles, v is incident with at most two 3-faces in $G^\times$ by Lemma 2.4 and Lemma 2.7. Now, we consider three subcases.

Case 1.1. If v is not incident with any 3-face in $G^\times$, then f_1, f_2, f_3 are all 4^+-faces.

First, assume that v is incident with at least one 5^+-face, without loss of generality, assume that f_1, then v would receive at least 1 from f_1, and $\frac{2}{4} \times 2 = 1$ from f_2, f_3, by Lemma 2.5 and R1. By R10, v sends at most $\frac{1}{6} \times 3 = \frac{1}{2}$ to false vertices which are adjacent to v. Thus, $ch'(v) \geq -3 + 1 + \frac{2}{3} + 1 + 1 - \frac{1}{2} > 0$.

Second, assume that f_1, f_2, f_3 are all 4-faces. If v is adjacent to at least one real vertex in $G^\times$, say v_1, then $d(v_1) \geq 10$, thus f_1, f_3 sends at least $\frac{2}{3} \times 2 = \frac{4}{3}$ to v, and f_2 sends at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. By R10, v sends at most $\frac{1}{6} \times 3 = \frac{1}{2}$ to false vertices which are adjacent to v. Thus,$ch'(v) \geq -3 + 1 + \frac{2}{3} + \frac{4}{3} + \frac{1}{2} - \frac{1}{2} = 0$. Otherwise, v_1, v_2, v_3 are all false vertices. Let x_i be the fourth(undefined) vertices of the 4-faces f_i $(i = 1, 2, 3)$. It is easy to check that $x_1x_2, x_2x_3, x_3x_1 \in E(G)$ by the drawing of G. Since f_1, f_2, f_3 are all 4-faces, there are at least two big vertices among x_1, x_2, x_3 by Lemma 2.1, without loss of generality, assume that x_1, x_2, thus, f_1, f_2 send at least $\frac{2}{3} \times 2 = \frac{4}{3}$ to v, and f_3

sends at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. By R10, v sends at most $\frac{1}{6} \times 3 = \frac{1}{2}$ to false vertices which are adjacent to v. Thus, $ch'(v) \geq -3 + 1 + \frac{2}{3} + \frac{4}{3} + \frac{1}{2} - \frac{1}{2} = 0$.

Case 1.2. If v is incident with exactly one 3-face in $G^{\times}$, then without loss of generality assume that f_3 is a 3-face. Since no two false vertices are adjacent in $G^{\times}$ by Lemma 2.4, there is a real vertex, among v_1 and v_3, say v_1, then $d(v_1) \geq 10$.

Assume that v_2 is also a real vertex, then $d(v_2) \geq 10$. Thus, f_1 sends at least 1 to v, f_2 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v by R1, Thus,$ch'(v) \geq -3 + 1 + \frac{2}{3} + 1 + \frac{2}{3} > 0$. Otherwise, v_2 is a false vertex. Let x_i be the second neighbors of v_2 on f_i $(i = 1, 2)$, it is easy to check that $x_1x_2 \in E(G)$ by the drawing of G. Thus, at least one of x_1 and x_2 is big by Lemma 2.1. This implies that v receives at least $\min\{1 + \frac{1}{2}, \frac{2}{3} \times 2\} = \frac{4}{3}$, from f_1 and f_2 by R1. Therefore, $ch'(v) \geq -3 + 1 + \frac{2}{3} + \frac{4}{3} = 0$.

Case 1.3. If v is incident with exactly two 3-faces in $G^{\times}$, then without loss of generality assume that f_2 and f_3 are 3-faces. By Lemma 2.4 and Lemma 2.7, v_3 must be a real vertex, v_1 and v_2 are false vertices, and f_1 is a 5^+-face. Thus, f_1 sends at least $\frac{4}{5-1} = 1$ to v by Lemma 2.5 and R1. Since G is a 1-planar graph without 4-cycles, so, there is at least a vertex among v_1 and v_2 which is incident with exactly one 3-face, say v_2. Then, v_2 sends $\frac{1}{3}$ to v by R9. Thus, $ch'(v) \geq -3 + 1 + \frac{2}{3} + 1 + \frac{1}{3} = 0$.

Case 2. $d = 4$ and v is a real vertex, then v has one 4-master and one 5-master. So v receives totally $\frac{6}{25} \times 2 = \frac{12}{25}$ from its masters by R3. Since G is a 1-planar graph without 4-cycles, v is incident with at most three 3-faces in $G^{\times}$ by Lemma 2.7.

If v is incident with exactly one 3-face in $G^{\times}$, say f_1, then there is at most one false vertex among v_1 and v_2 by Lemma 2.4. Suppose that v_1 is a false vertex, then, $d(v_2) \geq 9$ by Lemma 2.1, thus, v receives at least $\frac{2}{4-1} = \frac{2}{3}$ from f_2, receives $\frac{1}{2} \times 2 = 1$ from f_3 and f_4 by R1. Therefore, $ch'(v) \geq -2 + \frac{12}{25} + \frac{2}{3} + 1 = \frac{11}{75}$.

If v is not incident with any 3-face, then v is incident with four 4^+-faces.

First, assume that v is incident with at least one 5^+-face, say f_1, then, v receives at least 1 from f_1 by Lemma 2.5 and R1, receives $\frac{1}{2} \times 3 = \frac{3}{2}$ from f_2, f_3 and f_4 by R1. v sends at most $\frac{11}{75} \times 4 = \frac{44}{75}$ to false vertices that are adjacent to v by R11. Therefore, $ch'(v) \geq -2 + \frac{12}{25} + 1 + \frac{3}{2} - \frac{44}{75} > 0$.

Second, assume that v is incident with four 4-faces, if the neighbors of v are all false vertices, then, let x_i be the fourth(undefined) vertices of the 4-faces f_i $(i = 1, 2, 3, 4)$. It is easy to check that $x_1x_2, x_3x_4 \in E(G)$ by the drawing of G. Thus, at least one of x_1 and x_2 is big, similarly to x_3 and x_4 by Lemma 2.1. This implies that v receives at least $\frac{1}{2} \times 2 + \frac{2}{3} \times 2 = \frac{7}{3}$ from f_1, f_2,f_3 and f_4 by R1. v sends at most $\frac{11}{75} \times 4 = \frac{44}{75}$ to false vertices that are adjacent to v by R11. Therefore, $ch'(v) \geq -2 + \frac{12}{25} + \frac{7}{3} - \frac{44}{75} = \frac{17}{75}$. Otherwise, v is adjacent to at least one real vertex. Thus, v receives at least $\frac{1}{2} \times 4 = 2$ from f_1, f_2,f_3 and f_4 by R1. v sends at most $\frac{11}{75} \times 3 = \frac{33}{75}$ to false vertices that are adjacent to v by R11. Therefore, $ch'(v) \geq -2 + \frac{12}{25} + 2 - \frac{33}{75} = \frac{3}{75}$.

If v is incident with exactly three 3-faces, then without loss of generality assume that f_1 f_2 and f_4 are 3-faces. Since G is a 1-planar graph without 4-cycles, so, v is adjacent to exactly two false vertices by Lemma 2.4 and Lemma 2.7 in $G^{\times}$, and moreover f_3 is a 5^+-face. First, assume that two false vertices are not adjacent, say v_1 and v_3, then $d(v_2) \geq 9$, $d(v_4) \geq 9$ by Lemma 2.1. Then, f_3 sends at least 1 to v by R1, v_1 sends $\frac{13}{25}$ to v by R7. Thus, $ch'(v) \geq -2 + \frac{12}{25} + 1 + \frac{13}{25} = 0$.

Second, assume that two false vertices are v_3 and v_4, then, $d(v_1) \geq 9$, $d(v_2) \geq 9$ by Lemma 2.1. Thus, f_3 sends at least 1 to v by R1, v_1 and v_2 send $\frac{13}{50} \times 2 = \frac{13}{25}$ to v by R6. This implies that $ch'(v) \geq -2 + \frac{12}{25} + 1 + \frac{13}{25} = 0$.

If v is incident with exactly two 3-faces in $G^{\times}$, we consider four subcases.

Case 2.1. If v is not adjacent to any false vertex, then $v_i \geq 9$ $(i = 1, 2, 3, 4)$ by Lemma 2.1, and the two 3-faces that are incident with v have no common edge by Lemma 7, without loss of generality assume that f_2 and f_4 are 3-faces. Then, v receives a total of $\frac{2}{4-2}\times 2=2$ from f_1 and f_3, thus, $ch'(v)\geq -2+\frac{12}{25}+2>0$.

Case 2.2. If v is adjacent to exactly one false vertex, without loss of generality assume that v_1, then $d(v_i)\geq 9$ $(i=2,3,4)$ by Lemma 2.1. First, assume that the two 3-faces that are incident with v have no common edge, say f_2 and f_4, then, v receives at least $\frac{2}{4-2}=1$ from f_3, receives at least $\frac{2}{4-1}=\frac{2}{3}$ from f_1 by R1, and v receives $\frac{13}{50}\times 2$ from v_2 and v_3 by R6. Thus, $ch'(v)\geq -2+\frac{12}{25}+1+\frac{2}{3}+\frac{13}{50}\times 2=\frac{2}{3}$. Second, assume that the two 3-faces that are incident with v have one common edge, since G has no 4-cycles, then, v_1 is incident with at least one 3-face. If v_1 is incident with exactly one 3-face, without loss of generality assume that f_1, then f_2 is a real 3-face in $G^{\times}$. By R6, v receives $\frac{13}{50}\times 2$ from v_2 and v_3, v receives at least $\frac{2}{4-2}=1$ from f_3 and receives at least $\frac{2}{4-1}=\frac{2}{3}$ from f_4 by R1. Thus, $ch'(v)\geq -2+\frac{12}{25}+1+\frac{2}{3}+\frac{13}{50}\times 2=\frac{2}{3}$. If v_1 is incident with two 3-faces, say f_1 and f_4, then, v receives at least $\frac{2}{4-2}\times 2=2$ from f_2 and f_3. Therefore, $ch'(v)\geq -2+\frac{12}{25}+2=\frac{12}{25}$.

Case 2.3. If v is adjacent to exactly two false vertices.

First, assume that two faces which are incident with v are not adjacent, say f_2 and f_4 are both 3-faces, then, f_1 and f_3 are both 4^+-faces. If two false vertices that are adjacent to v are incident with the same 4^+-face, say f_1, then, v_1 and v_2 are both false vertices, v_3 and v_4 are both big vertices. Since G has no 4-cycles, then, f_1 is a 5^+-face. It implies that f_1 sends at least 1 to v and f_3 sends at least $\frac{2}{4-2}=1$ to v by Lemma 2.5 and R1. Thus, $ch'(v)\geq -2+\frac{12}{25}+1+1=\frac{12}{25}$. Otherwise, two false vertices that are adjacent to v are incident with different 4^+-faces, say f_1 and f_3, since G has no 4-cycles, then, f_1 and f_3 are 5^+-faces. Thus, v receives at least $\frac{4}{5-1}\times 2=2$ from f_1 and f_3 by Lemma 2.5 and R1. Therefore, $ch'(v)\geq -2+\frac{12}{25}+2=\frac{12}{25}$.

Second, assume that two faces which are incident with v are adjacent, say f_1 and f_2 are both 3-faces, then, f_3 and f_4 are both 4^+-faces. If two false vertices that are adjacent to v are incident with the same 4^+-face, without loss of generality assume that v_1 and v_4 are both false vertices, then, $d(v_i)\geq 9(i=2,3)$ by Lemma 2.1. So, v receives $\frac{13}{50}\times 2=\frac{13}{25}$ from v_2 and v_3 by R6, v receives at least $\frac{2}{4}\times 2=1$ from f_3 and f_4 by R1, therefore, $ch'(v)\geq -2+\frac{12}{25}+\frac{13}{25}+1=0$. If two false vertices that are adjacent to v are incident with different 4^+-faces, say f_3 and f_4, then, v_1 and v_3 are both false vertices, and $d(v_i)\geq 9(i=2,4)$ by Lemma 2.1. Since G has no 4-cycles, then, f_3 and f_4 are all 5^+-faces. So, v receives at least $\frac{4}{5-1}\times 2=2$ from f_3 and f_4 by Lemma 2.5 and R1, Therefore, $ch'(v)\geq -2+\frac{12}{25}+2=\frac{12}{25}$. If two false vertices that are adjacent to v are v_2 and v_4, since G has no 4-cycles, there is at least one 5^+-face among f_3 and f_4, say f_3. Thus, f_3 sends at least $\frac{4}{5-1}=1$ to v, f_4 sends at least $\frac{2}{4-1}=\frac{2}{3}$ to v by Lemma 2.5 and R1. Therefore,$ch'(v)\geq -2+\frac{12}{25}+1+\frac{2}{3}=\frac{11}{75}$.

Case 2.4. If v is adjacent to exactly three false vertices, say v_1, v_2 and v_3, since G has exactly two 3-faces, so, f_3 and f_4 are all 3-faces by Lemma 2.4, f_1 and f_2 are all 4^+-faces. Since G has not 4-cycles, so, f_1 and f_2 are either all 4-faces, or all 5^+-faces, or there is at least one 6^+-face. First assume that there is one 6^+-face among f_1 and f_2, say f_1. Let x_i $(i=1,2)$ be the second(undefined) neighbors of v_2 on f_i, it is easy to check that $x_1x_2\in E(G)$ by the drawing of G. Thus, at least one of x_1 and x_2 is big by Lemma 2.1. Then, v receives at least $\min\{\frac{6}{6-1}+\frac{1}{2},\frac{6}{6}+\frac{2}{4-1}\}=\frac{5}{3}$ from f_1 and f_2 by R1.

Therefore, $ch'(v)\geq -2+\frac{12}{25}+\frac{5}{3}=\frac{11}{75}$. Second, assume that f_1 and f_2 are all 5^+-faces, then, v receives at least $\frac{4}{5-1}\times 2=2$ from f_1 and f_2 by Lemma 2.5 and R1, thus, $ch'(v)\geq -2+\frac{12}{25}+2=\frac{12}{25}$. Third, assume that f_1 and f_2 are all 4-faces, then, v_2 is incident with four 4-faces, because otherwise, G has 4-cycles. By R8, v receives $\frac{5}{12}$ from v_2. Let x_i $(i=1,2)$ be the fourth(undefined) vertices of the 4-faces f_i, it is easy to check that $x_1x_2\in E(G)$ by the drawing of G. Thus, at least one of x_1 and x_2 is big by Lemma 2.1. This implies that v receives at least $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$ from f_1 and f_2 by R1.

Therefore, $ch'(v) \geq -2 + \frac{12}{25} + \frac{5}{12} + \frac{7}{6} = \frac{19}{300}$

Case 3. $d = 4$ and v is a false vertex, then, the neighbors of v are real vertices, and v is adjacent to at most two small vertices in G by Lemma 2.1. Since G has no 4-cycles, so, v is incident with at most two 3-faces, we consider three subcases.

Case 3.1. If v is not incident with any 3-face in $G^{\times}$, then v is incident with four 4^{+}-faces in $G^{\times}$.

Assume first that v has at least one 4-neighbor, say v_1, then, $d(v_3) \geq 9$, moreover, there is at least one big among v_2 and v_4 by Lemma 2.1, say v_2, thus, v would receive at least $\frac{2}{4-2} = 1$ from f_2, at least $\frac{2}{4-1} \times 2 = \frac{4}{3}$ from f_1 and f_3, at least $\frac{2}{4} = \frac{1}{2}$ from f_4 by R1, v sends at most $\frac{5}{12} \times 2 = \frac{5}{6}$ to its 4-neighbors by R8. Therefore, $ch'(v) \geq -2 + 1 + \frac{4}{3} + \frac{1}{2} - \frac{5}{6} = 0$. Otherwise, v does not have any 4-neighbors, then, v sends out nothing by R8, v would receive at least $\frac{2}{4} \times 4 = 2$, Thus,$ch'(v) \geq -2 + 2 = 0$.

Case 3.2. If v is incident with exactly one 3-face in $G^{\times}$, then without loss of generality assume that f_1 is the 3-face. There is at least one big among v_1 and v_2, say v_2.

Assume first that v_1 is a 3-vertex, then, both v_2 and v_3 are 10^{+}-vertices by Lemma 2.1. Thus, v would receive at least $\frac{2}{4-2} = 1$ from f_2, at least $\frac{2}{4-1} = \frac{2}{3}$ from f_3, at least $\frac{2}{4} = \frac{1}{2}$ from f_4 by R1, v would receive $\frac{13}{50}$ from v_2 by R5, v sends at most $\frac{1}{3}$ to v_1 by R9. Thus, $ch'(v) \geq -2+1+\frac{2}{3}+\frac{1}{2}+\frac{13}{50}-\frac{1}{3} = \frac{7}{75}$. Second, assume that $4 \leq d(v_1) \leq 7$, then, both v_2 and v_3 are 6^{+}-vertices by Lemma 2.1. So, v would receive at least $\frac{2}{4-2} = 1$ from f_2, at least $\frac{2}{4-1} = \frac{2}{3}$ from f_3, at least $\frac{2}{4} = \frac{1}{2}$ from f_4 by R1, v sends out nothing by R9. Therefore,$ch'(v) \geq -2 + 1 + \frac{2}{3} + \frac{1}{2} = \frac{1}{6}$. Third, assume that v_1 is a 8^{+}-vertex, then, v_1 sends $\frac{13}{50}$ to v by R5. There is at least a big vertex among v_2 and v_4, so, f_2 and f_4 send $\min\{\frac{2}{4-2} + \frac{1}{2}, \frac{2}{4-1} \times 2\} = \frac{4}{3}$ to v, f_3 sends $\frac{2}{4} = \frac{1}{2}$ to v by R1, v sends out nothing by R9. Therefore,$ch'(v) \geq -2 + \frac{13}{50} + \frac{4}{3} + \frac{1}{2} = \frac{7}{75}$.

Case 3.3. If v is incident with two 3-faces in $G^{\times}$, since G has no 4-cycles, then, the two 3-faces have a common edge, without loss of generality assume that f_3 and f_4 are 3-faces. There is a big vertex among v_1 and v_3, say v_1.

First assume that $6 \leq d(v_4) \leq 7$, then, both v_2 and v_3 are big by Lemma 2.1, moreover, f_1 and f_2 send at least $\frac{2}{4-2} \times 2 = 2$ to v by R1, v sends out nothing by R7. Thus, $ch'(v) \geq -2+2 = 0$. Second, assume that $d(v_4) \leq 5$, then, v_i $(i = 1, 2, 3)$ is 8^{+}-vertex by Lemma 2.1, moreover, f_1 and f_2 send at least $\frac{2}{4-2} \times 2 = 2$ to v by R1, v_1 and v_3 send $\frac{13}{50} \times 2 = \frac{13}{25}$ to v by R5, v sends at most $\frac{13}{25}$ to v_4 by R7. Thus,$ch'(v) \geq -2 + 2 + \frac{13}{25} - \frac{13}{25} = 0$.

Third assume that v_4 is a 8^{+}-vertex. If there is at least one 5^{+}-face among f_1 and f_2, say f_1, then, f_1 sends at least $\frac{4}{5-1} = 1$ to v, f_2 sends at least $\frac{2}{4} = \frac{1}{2}$ to v by Lemma 2.5 and R1, v_4 sends $\frac{13}{50} \times 2 = \frac{13}{25}$ to v by R5. Thus, $ch'(v) \geq -2+1+\frac{1}{2}+\frac{13}{25} = \frac{1}{50}$. Otherwise, f_1 and f_2 are all 4-faces. Let $x_i (i = 1, 2)$ be the second(undefined) neighbors of v_2 on f_i, since G has no 4-cycles, then, both x_1 and x_2 are false vertices. Suppose that v_2 is a 6^{+}-vertex, then, f_1 sends at least $\frac{2}{4-2} = 1$ to v, f_2 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v by R1, v_4 sends $\frac{13}{50} \times 2 = \frac{13}{25}$ to v by R5. Thus, $ch'(v) \geq -2 + 1 + \frac{2}{3} + \frac{13}{25} = \frac{14}{75}$. Suppose that v_2 is a 3-vertex or a 4-vertex, since G has no 4-cycles, then, v_2 is not incident with any 3-faces. By R10 and R11, v_2 sends at least $\frac{11}{75}$ to v. Suppose that v_2 is a 5-vertex, by R12,v_2 sends $\frac{1}{6}$ to v. Thus, when v_2 is a 5^{-}-vertex, this implies that v_2 sends at least $\frac{11}{75}$ to v. And moreover, we consider the degree of v_3. If v_3 is a 5^{-}-vertex, then, v_1 is a 8^{+}-vertex by Lemma 2.1, v_1 sends $\frac{13}{50}$ to v, v_4 sends $\frac{13}{50} \times 2 = \frac{13}{25}$ to v by R5, f_1 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v, f_2 sends at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. Therefore, $ch'(v) \geq -2 + \frac{11}{75} + \frac{13}{50} + \frac{13}{25} + \frac{2}{3} + \frac{1}{2} = \frac{7}{75}$. If v_3 is a 6^{+}-vertex, since v_1 is big, then, each of f_1 and f_2 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v by R1, v_4 sends $\frac{13}{50} \times 2 = \frac{13}{25}$ to v by R5, therefore, $ch'(v) \geq -2 + \frac{11}{75} + \frac{2}{3} \times 2 + \frac{13}{25} = 0$.

Case 4. $d = 5$. v is incident with at most four 3-faces in $G^{\times}$ by Lemma 2.3. If v would send charges to a false vertex which adjacent to v by R12, then v is incident with at most three 3-faces in

$G^{\times}$. First assume that v is incident with exactly four 3-faces in $G^{\times}$, say f_1, f_2, f_3 and f_4, then v_1,v_3, and v_5 are false vertices, v_2 and v_4 are real vertices by Lemma 2.8. Since G has no 4-cycles, then, f_5 is a 6^+-face. Thus, f_5 sends 1 to v by R1, therefore, $ch'(v) \geq -1+1 = 0$. Second assume that v is incident with exactly three 3-faces in $G^{\times}$, then, v is incident with two 4^+-faces. If the two 4^+-faces are not adjacent, then, v sends out nothing by R12, v would receive at least $\frac{2}{4} \times 2 = 1$ from two 4^+-faces which are incident with v. Thus, $ch'(v) \geq -1+1 = 0$. If two 4^+-faces are adjacent, without loss of generality, assume that f_1 and f_2 are 4^+-faces. Moreover, if v_2 is a real vertex, then, v sends out nothing by R12, v would receive at least $\frac{2}{4} \times 2 = 1$ from f_1 and f_2 by R1. Thus, $ch'(v) \geq -1+1 = 0$. Otherwise, v_2 is a false vertex, then, v sends at most $\frac{1}{6}$ to v_2 by R12. Let x_i be the second(undefined) neighbors of v_2 on f_i $(i = 1, 2)$, it is easy to check that $x_1x_2 \in E(G)$ by the drawing of G. Thus, at least one of x_1 and x_2 is big by Lemma 2.1. This implies v would receive at least $\frac{2}{4} + \frac{2}{4-1} = \frac{7}{6}$ from f_1 and f_2 by R1, therefore, $ch'(v) \geq -1 + \frac{7}{6} - \frac{1}{6} = 0$. Third assume that v is incident with at most two 3-faces in $G^{\times}$, then, v is incident with at least three 4^+-faces. v sends at most $\frac{1}{6} \times 3 = \frac{1}{2}$ to false vertices which are adjacent to v by R12. v would receive at least $\frac{2}{4} \times 3 = \frac{3}{2}$ from 4^+-faces which are incident with v, thus, $ch'(v) \geq -1 + \frac{3}{2} - \frac{1}{2} = 0$.

Case 5. $6 \leq d \leq 7$. Then it is trivial that $ch'(v) = ch(v) \geq 0$.

Case 6. $8 \leq d \leq \Delta(G) - 2$. By Lemma 2.3, v is incident with at most $\lfloor \frac{4}{5}d \rfloor$ 3-faces in $G^{\times}$, then v sends at most $\lfloor \frac{4}{5}d \rfloor \times \frac{13}{50}$ to false vertices and real 4-vertices which are adjacent to v on 3-faces by R5 and R6. Thus, $ch'(v) \geq d - 6 - \lfloor \frac{4}{5}d \rfloor \times \frac{13}{50} \geq \frac{99d-750}{125} > 0$, since $d \geq 8$.

Case 7. $d = \Delta(G) - 1$. By Lemma 2.3, v is incident with at most $\lfloor \frac{4}{5}d \rfloor$ 3-faces in $G^{\times}$. And by Lemma 2.1, we have $d(u) \geq 4$ if $uv \in E(G)$. So v can be a 4-master vertex of at most two vertices and a 5-master vertex of at most three vertices by Lemma 2.6.

Let $\Delta(G) = 10$, Then, $d = 9$. By Lemma 2.3, v is incident with at most seven 3-faces in $G^{\times}$. If v is incident with exactly seven 3-faces in $G^{\times}$, then, by Lemma 2.8, there are four consecutive 3-faces and another three consecutive 3-faces which are incident with v, and v is adjacent to at most two real small vertices. Thus, v sends at most $7 \times \frac{13}{50}$ to false vertices and real 4-vertices which are adjacent to v by R5 and R6. v sends at most $\frac{6}{25} \times 2 = \frac{12}{25}$ by R3. Therefore, $ch'(v) \geq 9 - 6 - 7 \times \frac{13}{50} - \frac{12}{25} = \frac{7}{10}$. If v is incident with at most six 3-faces in $G^{\times}$, then, v sends at most $6 \times \frac{13}{50}$ to false vertices and real 4-vertices which are adjacent to v by R5 and R6, v sends at most $\frac{6}{25} \times 2 + \frac{6}{25} \times 3 = \frac{6}{5}$ to real small vertices which are adjacent to it by R3. Thus, $ch'(v) \geq 9 - 6 - 6 \times \frac{13}{50} - \frac{6}{5} = \frac{6}{25}$.

Let $\Delta(G) \geq 11$, then, $d \geq 10$. By Lemma 2.3, v is incident with at most $\lfloor \frac{4}{5}d \rfloor$ 3-faces in $G^{\times}$. v sends at most $\lfloor \frac{4}{5}d \rfloor \times \frac{13}{50}$ to false vertices and real 4-vertices which are adjacent to v by R5 and R6, v sends out at most $\frac{6}{25} \times 2 + \frac{6}{25} \times 3 = \frac{6}{5}$ by R3. Thus $ch'(v) \geq \Delta(G) - 1 - 6 - \lfloor \frac{4}{5}(\Delta(G)-1) \rfloor \times \frac{13}{50} - \frac{6}{5} \geq \frac{99\Delta(G)-999}{125} \geq 0$, since $\Delta(G) \geq 11$.

Case 8. $d = \Delta(G)$. By Lemma 2.3, v is incident with at most $\lfloor \frac{4}{5}d \rfloor$ 3-faces in $G^{\times}$. And by Lemma 2.1, we have $d(u) \geq 3$ if $uv \in E(G)$. So v can be a 3-master vertex of at most one vertex, a 4-master vertex of at most two vertices and a 5-master vertex of at most three vertices by Lemma 2.6.

If $d = 10$, then v is incident with at most eight 3-faces. When v is incident with exactly eight 3-faces, there are two groups four consecutive 3-faces that are incident with v in $G^{\times}$, so, v is adjacent to at most two real small vertices by Lemma 2.8. Thus, v sends at most $\frac{6}{25} \times 2 = \frac{12}{25}$ to real small vertices that are adjacent to it by R2 and R3, v sends at most $8 \times \frac{13}{50}$ to false vertices and real 4-vertices that are adjacent to v by R5 and R6, and $\frac{1}{2}$ to the common pot by R4. Therefore, $ch'(v) \geq 10 - 6 - 8 \times \frac{13}{50} - \frac{12}{25} - \frac{1}{2} = \frac{47}{50} \geq 0$. When v is incident with at most seven 3-faces, v sends at most $7 \times \frac{13}{50}$ to false vertices and real 4-vertices that are adjacent to v by R5 and R6, v sends at most $\frac{6}{25} \times 5 + \frac{2}{9} = \frac{64}{45}$ to real small vertices that are adjacent to it by R2 and R3, v sends $\frac{1}{2}$ to the common pot by R4. Therefore, $ch'(v) \geq 10 - 6 - 7 \times \frac{13}{50} - \frac{64}{45} - \frac{1}{2} = \frac{58}{225} \geq 0$.

If $d \geq 11$, v sends at most $\lfloor \frac{4}{5} d \rfloor \times \frac{13}{50}$ to false vertices and real 4-vertices that are adjacent to v by R5 and R6, v sends at most $\frac{6}{25} \times 5 + \frac{2}{9} = \frac{64}{45}$ to real small vertices adjacent to it by R2, R3 and $\frac{1}{2}$ to the common pot by R4. Thus $ch'(v) \geq \Delta(G) - 6 - \lfloor \frac{4}{5} \Delta(G) \rfloor \times \frac{13}{50} - \frac{64}{45} - \frac{1}{2} \geq \frac{1782\Delta(G) - 17825}{2250} \geq 0$, since $\Delta(G) \geq 11$.

Therefore, we complete the proof of the Theorem.

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