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Some results on 4−remainder cordial labeling of graphs

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ABSTRACT

Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e(0) - \eta_o(1)| \leq 1$ where $\eta_e(0)$ and $\eta_o(1)$ respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with admits a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 4- remainder cordial labeling behavior of Prism, Crossed prism graph, Web graph, Triangular snake, $L_n \odot mK_1$, Durer graph, Dragon graph.

Keywords: prism, web graph, triangular snake, $L_n \odot mK_1$, durer graph, dragon graph

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1. Introduction

Graphs considered here are finite and simple. Graph labeling is used in several areas of science and technology like coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. The origin of graph labeling is graceful labeling which was introduced by Rosa (1967). The concept of cordial labeling was introduced by Cahit [1]. Motivated by this several authors [9, 10, 12, 13, 11] studied about cordial related labeling. Ponraj et al. [5, 4], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of path, cycle, star, bistar, complete graph, $S(K_{1,n})$, $S(B_{n,n})$, $S(W_n)$, P_n^2 , $P_n^2 \cup K_{1,n}$, $P_n^2 \cup B_{n,n}$, $P_n \cup B_{n,n}$, $P_n \cup K_{1,n}$, $K_{1,n} \cup S(K_{1,n})$, $K_{1,n} \cup S(B_{n,n})$, $S(K_{1,n}) \cup S(B_{n,n})$, and also the concept of k-remainder cordial labeling

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introduced in [8] and investigate the $k-$remainder cordial labeling behavior of Grid, Subdivision of crown, Subdivision of bistar, Book, Jelly fish, Subdivision of Jelly fish, Mondolian tent, Flower graph, Sunflower graph and Subdivision of Ladder graph, $L_n \odot K_1$, $L_n \odot 2K_1$, $L_n \odot K_2$ in [8, 6, 7]. In this paper we investigate the 4-remainder cordial labeling behavior of Prism, Crossed prism graph, Web graph, Triangular snake, $L_n \odot mK_1$, Durer graph, Dragon graph, etc,. Terms are not defined here follows from Harary [3] and Gallian [2].

2. Preliminary results

Definition 2.1. The *corona* of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Definition 2.2. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The *Product* of G and H, denoted by $G \times H$, has $V(G \times H) = \{(g, h)/g \in G; h \in H\}$ as the vertex set and $E(G \times H) = \{(g_1, h_1)(g_2, h_2)/g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)\}$.

Definition 2.3. A *Crossed prism* CP_n for positive even values of n is a graph with $V(CP_n) = V(C_n) \cup V(C_n^{'})$ and $E(CP_n) = E(C_n) \cup E(C_n^{'}) \cup \{u_iv_{i+1}, u_{i+1}v_i : i = 1, 3, \dots n-1\}$.

Definition 2.4. The *Web graph* WG_n is a graph consisting of r concentric copies of the cycle graph C_n, with corresponding vertices connected by "spokes".

Definition 2.5. A *Triangular snake* denoted by T_n is obtained from a path $v_1, v_2, \dots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $(1 \le i \le n-1)$.

Definition 2.6. The *Durer graph* denoted by DG_n is a graph consisting of $V(DG_n) = V(C_n) \cup \{u_i, v_i : 1 \le i \le n\})$ and $E(DG_n) = E(C_n) \cup \{u_iv_i : 1 \le i \le n\} \cup \{u_iu_{i+2} : 1 \le i \le n-2\}) \cup \{u_nu_2, u_{n-1}u_1\}$.

Definition 2.7. A *Dragon* is a graph formed by joining an end vertex of a path P_n to a vertex of the cycle C_m. It is denoted as $C_m@P_n$.

3. $k-$ Remainder cordial labeling

Definition 3.1. Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \dots, k\}$ where k is an integer $2 < k \le |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \ge f(v)$ or $f(v) \ge f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \le 1$, $i, j \in \{1, \dots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e(0) - \eta_o(1)| \le 1$ where $\eta_e(0)$ and $\eta_o(1)$ respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph.

First we investigate the $4-$remainder cordial labeling behavior of the prism.

Theorem 3.2. *The prism C_nXP_2 is $4-$remainder cordial for all values of $n \in N$.*

Proof. Let $V(C_nXP_2) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(C_nXP_2) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\}$. Clearly the order and size of this C_nXP_2 are $2n$ and $3n$ respectively.

Case (i). n is even.

First we consider the vertices u_i. Assign the label 2 to the vertices $u_1, u_3, \ldots, u_{n-1}$ and 3 to the vertices $u_2, u_4, \ldots, u_n$. Then next assign the label 1 to the vertices $v_1, v_3, \ldots, v_{n-1}$ and assign the label 4 to the vertices $v_2, v_4, \ldots, v_n$.

Case (ii). n is odd.

As in case(i), assign the labels to the vertices u_i, and $v_i, (1 \leq i \leq n-1)$. Next finally assign the labels $3, 4$ respectively to the vertices u_n and v_n. The Table 1, given below establish that this labeling f is a $4-$ remainder cordial labeling of C_nXP_2.

Table 1.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$\eta_e(0)$	$\eta_o(1)$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$
n is odd	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

□

Next we investigate the $4-$remainder cordial labeling behavior of the n crossed prism.

Theorem 3.3. *The crossed prism CP_n is $4-$remainder cordial for all even integers $n \in 2N$.*

Proof. Let $C_n = u_1u_2 \ldots u_nu_1$ be a cycle and $C_n' = v_1v_2 \ldots v_nv_1$ be another cycle. Then crossed prism CP_n is a graph which is obtained from two cycles C_n and C_n' with vertex set $V(CP_n) = V(C_n) \cup V(C_n')$ and $E(CP_n) = E(C_n) \cup E(C_n') \cup \{u_iv_{i+1}, u_{i+1}v_i : i = 1, 3, \ldots n-1\}$. It is easy to verify that CP_nhas $2n$ vertices and $3n$ edges.

First we consider the vertices $u_i (1 \leq i \leq n)$ of the inner cycle C_n. Assign the label 2 to the vertices $u_1, u_3, \ldots, u_{n-1}$ and 3 to the vertices $u_2, u_4, \ldots, u_n$. Then next we consider the vertices $v_i (1 \leq i \leq n)$ of the outer cycle C_n'. Assign the label 4 to the vertices $v_1, v_3, \ldots, v_{n-1}$ and assign the label 1 to the vertices $v_2, v_4, \ldots, v_n$. Clearly $v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{n}{2}$, $\eta_e(0) = \eta_o(1) = \frac{3n}{2}$. For illustration, 4-remainder cordial labeling of CP_8 is shown in Figure 1. □

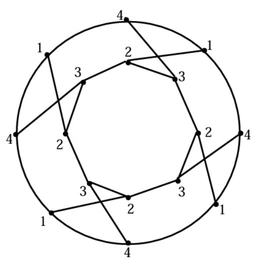

Fig. 1. 4-remainder cordial labeling of CP_8

Now we investigate the $4-$remainder cordial labeling behavior of the web graph WG_n.

Theorem 3.4. *The web graph WG_n is $4-$remainder cordial for all n.*

Proof. Let $V(WG_n) = V(C_nXP_2) \cup \{w_i : 1 \leq i \leq n\}$ and $E(WG_n) = E(C_nXP_2) \cup \{v_iw_i : 1 \leq i \leq n\}$. Then it is easy to verify that WG_n has $3n$ vertices and $4n$ edges.

Case (i). n is even.

First we consider the vertices u_1, v_1 and w_1. Assign the labels $3, 2$ and 3 respectively to the vertices u_1, v_1 and w_1. Next consider the vertices u_2, v_2 and w_2. Assign the labels $2, 3$ and 2 to the vertices u_2, v_2 and w_2 respectively. Next we move to the vertices u_3, v_3 and w_3 and assign the labels $3, 2$ and 3 respectively to the vertices u_3, v_3 and w_3. Next assign the labels $2, 3$ and 2 to the vertices u_4, v_4, w_4. That is assign the labels $3, 2$ and 3 respectively to the vertices u_{2i-1}, v_{2i-1} and w_{2i-1}, $(1 \leq i \leq \frac{n}{4})$ and assign the labels $2, 3$ and 2 to the vertices u_{2i}, v_{2i} and w_{2i}, $(1 \leq i \leq \frac{n}{4})$. In the same way assign the labels $1, 4$ and 1 to the vertices u_{2i-1}, v_{2i-1} and w_{2i-1}, $(\frac{n}{4} + 1 \leq i \leq \frac{n}{2})$ and assign the labels $4, 1$ and 4 respectively to the vertices u_{2i}, v_{2i} and w_{2i}, $(\frac{n}{4} + 2 \leq i \leq \frac{n}{2})$.

Case (ii). n is odd.

Subcase (i). $n \equiv 1 \pmod 4$

Assign the labels $4, 1$ and 4 respectively to the vertices u_1, v_1 and w_1. Next assign the labels $1, 4$ and 1 to the vertices u_2, v_2 and w_2 respectively. Next we move to the vertices u_3, v_3 and w_3 and assign the labels $4, 1$ and 4 respectively to the vertices u_3, v_3 and w_3. Therefore assign the labels $4, 1$ and 4 respectively to the vertices u_{2i-1}, v_{2i-1} and w_{2i-1}, $(1 \leq i \leq \frac{n-1}{4})$ and assign the labels $1, 4$ and 1 to the vertices u_{2i}, v_{2i} and w_{2i}, $(1 \leq i \leq \frac{n-1}{4})$. In the same manner assign the labels $2, 3$ and 2 to the vertices u_{2i-1}, v_{2i-1} and w_{2i-1}, $(\frac{n-1}{4} + 1 \leq i \leq \frac{n-1}{2})$ and assign the labels $3, 2$ and 3 respectively to the vertices u_{2i}, v_{2i} and w_{2i}, $(\frac{n-1}{4} + 2 \leq i \leq \frac{n-1}{2})$. Finally assign the labels $4, 3$ and 1 respectively to the vertices u_n, v_n and w_n.

Subcase (ii). $n \equiv 3 \pmod 4$

Assign the labels $1, 4$ and 1 respectively to the vertices u_1, v_1 and w_1. Next assign the labels $4, 1$ and 4 to the vertices u_2, v_2 and w_2 respectively. Then assign the labels $1, 4$ and 1 respectively to the vertices u_3, v_3 and w_3. Next we move to the vertices u_4, v_4 and w_4 and assign the labels $4, 1$ and 4 respectively to the vertices u_4, v_4 and w_4. We observe that assign the labels $1, 4$ and 1 respectively to the vertices u_{2i-1}, v_{2i-1} and w_{2i-1}, $(1 \leq i \leq \frac{n-1}{4})$ and assign the labels $4, 1$ and 4 to the vertices u_{2i}, v_{2i} and w_{2i}, $(1 \leq i \leq \frac{n-1}{4})$. In the similar manner assign the labels $2, 3$ and 2 to the vertices u_{2i-1}, v_{2i-1} and w_{2i-1}, $(\frac{n-1}{4} + 1 \leq i \leq \frac{n-1}{2})$ and assign the labels $3, 2$ and 3 respectively to the vertices u_{2i}, v_{2i} and w_{2i}, $(\frac{n-1}{4} + 2 \leq i \leq \frac{n-1}{2})$ respectively. Finally assign the labels $3, 4$ and 1 respectively to the vertices u_n, v_n and w_n.

The Table 2, shows that this vertex labeling f is a $4-$ remainder cordial labeling.

Table 2.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$\eta_e(0)$	$\eta_o(1)$
$n \equiv 0 \pmod 4$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$2n$	$2n$
$n \equiv 1 \pmod 4$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{n}{2}$	$2n$	$2n$
$n \equiv 2 \pmod 4$	$\frac{3n+2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$2n$	$2n$
$n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$2n$	$2n$

□

Next we investigate the triangular snake T_n.

Theorem 3.5. *The triangular snake T_n is $4-$remainder cordial for all n.*

Proof. Let $V(T_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n-1\}$ and $E(T_n) = \{u_iu_{i+1}, u_iv_i, v_iu_{i+1} : 1 \leq i \leq n-1\}$. We observe that the order and size of the T_n are $2n-1$ and $3n-3$ respectively.

Case (i). n is even.

Assign the label 2 to the vertices $u_1, u_3, \ldots, u_{n-1}$ and 3 to the vertices $u_2, u_4, \ldots, u_n$. Then next consider the vertices v_i. Assign the label 1 to the vertices $v_1, v_3, \ldots, v_{n-1}$ and assign the label 4 to the vertices $v_2, v_4, \ldots, v_n$.

Case (ii). n is odd.

As in case(i), assign the labels to the vertices $u_i, ((1 \leq i \leq n-1))$, and $v_i, (1 \leq i \leq n-2)$. Finally assign the labels $2, 4$ respectively to the vertices u_n and v_{n-1}. The Table 3, establish that this labeling f is a 4− remainder cordial labeling of T_n.

Table 3.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$\eta_e(0)$	$\eta_o(1)$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{3n-2}{2}$	$\frac{3n-4}{2}$
n is odd	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{3n-3}{2}$	$\frac{3n-3}{2}$

□

Next we investigate the corona of L_n with mK_1.

Theorem 3.6. *$L_n \odot mK_1$ is 4−remainder cordial for all n.*

Proof. We denote the vertex set and edge set of L_n as follows. Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\})$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\}) \cup \{u_iv_i : 1 \leq i \leq n\}$.

Case (i). n is even and m is both even and odd.

Assign the label 2 to the vertices $u_1, u_3, \ldots, u_{n-1}$ and 3 to the vertices $u_2, u_4, \ldots, u_n$. Next assign the label 1 to the vertices $v_1, v_3, \ldots, v_{n-1}$ and assign the label 4 to the vertices $v_2, v_4, \ldots, v_n$. Now we consider the pendant vertices. Assign the label 1 to all the pendant vertices whose support receives the label 2 and assign the label 4 to all the pendant vertices whose support receives the label 3. Next assign the label 2 to all the pendant vertices whose support receives the label 1 and assign the label 3 to all the pendant vertices whose support receives the label 4.

Case (ii). n is odd and m is even.

Assign the labels to the vertices u_i, v_i, and all the pendant vertices adjacent to $u_i, v_i, (1 \leq i \leq n-1)$, as in case(i). Then next assign the label 3 to the vertices u_n and v_n. Next assign the label 1 to the $\frac{m}{2}+1$ pendant vertices which are adjacent to u_n and assign the label 3 to the remaining $\frac{m}{2}-1$ pendant vertices which are adjacent to u_n. Finally assign the label 2 to the $\frac{m}{2}$ pendant vertices which are adjacent to v_n and assign the label 4 to the remaining non-labeled $\frac{m}{2}$ pendant vertices which are adjacent to v_n.

Case (iii). n is odd and m is odd.

Assign the label 1 to the $\frac{m-1}{2}$ pendant vertices which are adjacent to u_n and assign the label 3 to the remaining $\frac{m-1}{2}$ pendant vertices which are adjacent to u_n.Finally assign the label 4 to the remaining pendant vertices adjacent to the vertex u_n. Next assign the label 2 to the $\frac{m+1}{2}$ pendant vertices which are adjacent to the vertex v_n and assign the label 4 to the remaining $\frac{m-1}{2}$ pendant vertices which are adjacent to the vertex v_n. Thus the tables [4, 5], shows that this vertex labeling f is a 4− remainder cordial labeling of $L_n \odot mK_1$.

Vertex condition of $L_n \odot mK_1$:

Table 4.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$
n is even	$\frac{2n(m+1)}{4}$	$\frac{2n(m+1)}{4}$	$\frac{2n(m+1)}{4}$	$\frac{2n(m+1)}{4}$
n is odd,m is even	$\frac{2n(m+1)+2}{4}$	$\frac{2n(m+1)-2}{4}$	$\frac{2n(m+1)+2}{4}$	$\frac{2n(m+1)-2}{4}$
n is odd,m is odd	$\frac{2n(m+1)}{4}$	$\frac{2n(m+1)}{4}$	$\frac{2n(m+1)}{4}$	$\frac{2n(m+1)}{4}$

Edge condition of $L_n \odot mK_1$:

Table 5.

Nature of n	$\eta_e(0)$	$\eta_o(1)$
n is even	$\frac{n(2m+3)-2}{2}$	$\frac{n(2m+3)-2}{2}$
n is odd,m is even	$\frac{n(2m+3)-1}{2}$	$\frac{n(2m+3)-3}{2}$
n is odd,m is odd	$\frac{n(2m+3)-1}{2}$	$\frac{n(2m+3)-3}{2}$

□

Next we investigate the Durer graph DG_n.

Theorem 3.7. *The Durer graph DG_n is 4−remainder cordial for all values of n.*

Proof. Let $C_n = v_1v_2\dots v_nv_1$ be the cycle. Let $V(DG_n) = V(C_n) \cup \{u_i, v_i : 1 \le i \le n\})$ and $E(DG_n) = E(C_n) \cup \{u_iv_i : 1 \le i \le n\} \cup \{u_iu_{i+2} : 1 \le i \le n-2\}) \cup \{u_nu_2, u_{n-1}u_1\}$. Then clearly the order and size of the Durer graph DG_n are $2n$ and $3n$ respectively.

Case (i). n is even and $n \ge 6$.

First we consider the vertices v_i of the cycle C_n. Assign the label 2 to the vertices $v_1, v_3, \dots, v_{n-1}$ and 3 to the vertices $v_2, v_4, \dots, v_n$. Next assign the labels to the vertices u_i. Assign the label 1 to the vertices $u_1, u_3, \dots, u_{n-1}$ and assign the label 4 to the vertices $u_2, u_4, \dots, u_n$.

Case (ii). n is odd and $n \ge 3$.

Assign the labels to the vertices $u_i, v_i, (1 \le i \le n-1)$, as in case(i). Finally assign the labels 3 and 4 respectively to the vertices v_n and u_n. The Table 6, establish that this vertex labeling f is a 4− remainder cordial labeling of DG_n.

Table 6.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$\eta_e(0)$	$\eta_o(1)$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$
n is odd	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

□

Next we investigate the dragon $C_m@P_n$.

Theorem 3.8. *The dragon graph $C_m@P_n$ is 4−remainder cordial for all $m \ge 3$ and n.*

Proof. Let $C_m = u_1u_2\dots u_mu_1$ be the cycle and $P_n = v_1v_2\dots v_n$ be the path. Identify u_1 with v_1. Clearly the order and size of the dragon $C_m@P_n$ are $m+n-1$ and $m+n-1$ respectively.

Case 1. $m \equiv 0 \pmod 4$

First we consider the vertices $u_1, u_2, \dots, u_m$ of the cycle C_m. Assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3 and u_4 respectively. Next assign the labels $1, 2, 3, 4$ respectively to the next four vertices u_5, u_6, u_7 and u_8. Then assign the labels $1, 2, 3, 4$ respectively to the next four vertices u_9, u_{10}, u_{11} and u_{12}. Proceeding like this until we reach the vertex u_m. Clearly the vertex u_m received the label 4 for this pattern.

Next we take the vertices $v_1, v_2, \dots, v_n$ of the path P_n. Assign the labels to the vertices $v_1, v_2, \dots, v_n$ by the following four subcases.

Subcase 1.1. $n \equiv 0 \pmod 4$

We fix the labels $2, 3, 4$ to the vertices v_2, v_3 and v_4 respectively. Assign the labels $1, 2, 3, 4$ respectively to the vertices v_5, v_6, v_7 and v_8. Next assign the labels $1, 2, 3, 4$ respectively to the next four vertices v_9, v_{10}, v_{11} and v_{12}. Then assign the labels $1, 2, 3, 4$ respectively to the next four vertices v_{13}, v_{14}, v_{15} and v_{16}. Continuing like this until we reach the vertex v_n. Clearly in this pattern the vertex v_n received the label 4.

Subcase 1.2. $n \equiv 1 \pmod 4$

Assign the labels to the vertices $v_i, (1 \le i \le n-1)$, as in subcase 1.1. Finally assign the label 1 to the vertices v_n.

Subcase 1.3. $n \equiv 2 \pmod 4$

Assign the labels to the vertices $v_i, (1 \le i \le n-2)$, as in subcase 1.1. Then finally assign the labels 3, and 1 respectively to the vertices v_{n-1}, and v_n.

Subcase 1.4. $n \equiv 3 \pmod 4$

Assign the labels to the vertices $v_i, (1 \le i \le n-3)$, as in subcase 1.1. Next assign the labels 1, 2, and 3 respectively to the vertices v_{n-2}, v_{n-1}, and v_n. Thus the Tables 7 and 8, shows that this vertex labeling f is a 4− remainder cordial labeling of the dragon with $m \equiv 0 \pmod 4$ and for all values of n. Vertex condition of the dragon with $m \equiv 0 \pmod 4$ and for all values of n:

Table 7.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$
$n \equiv 0 \pmod 4$	$\frac{m+n-4}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$
$n \equiv 1 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$
$n \equiv 2 \pmod 4$	$\frac{m+n-2}{4}$	$\frac{m+n-2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n-2}{4}$
$n \equiv 3 \pmod 4$	$\frac{m+n-3}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n-3}{4}$

Edge condition of the dragon with $m \equiv 0 \pmod 4$ and for all values of n:

Table 8.

Nature of $m \equiv 0 \pmod 4$ and n	$\eta_e(0)$	$\eta_o(1)$
$n \equiv 0 \pmod 4$	$\frac{m+n-2}{2}$	$\frac{m+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n-2}{2}$	$\frac{m+n}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$

Case 2. $m \equiv 1 \pmod 4$

As in case 1, assign the labels to the vertices $u_i, (1 \leq i \leq m-1)$. Finally assign the label 1 to the vertices u_m of the cycle C_m.

Next we consider the path vertices . Assign the labels to the vertices v_i for all $i = 1$ to n by the following four sub cases.

Subcase 2.1. $n \equiv 0 \pmod 4$

As in subcase 1.1, assign the labels to the vertices v_i for all $i = 1$ to n.

Subcase 2.2. $n \equiv 1 \pmod 4$

As in subcase 1.2, assign the labels to the vertices v_i for all $i = 1$ to n.

Subcase 2.3. $n \equiv 2 \pmod 4$

As in subcase 1.3, assign the labels to the vertices v_i for all $i = 1$ to n.

Subcase 2.4. $n \equiv 3 \pmod 4$

As in subcase 1.4, assign the labels to the vertices v_i for all $i = 1$ to n. Thus the following Tables 9 and 10, shows that this vertex labeling f is a $4-$ remainder cordial labeling of the dragon with $m \equiv 1 \pmod 4$ and for all values of n.

Vertex condition of the dragon with $m \equiv 1 \pmod 4$ and for all values of n:

Table 9.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$
$n \equiv 0 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$
$n \equiv 1 \pmod 4$	$\frac{m+n+2}{4}$	$\frac{m+n-2}{4}$	$\frac{m+n-2}{4}$	$\frac{m+n-2}{4}$
$n \equiv 2 \pmod 4$	$\frac{m+n+1}{4}$	$\frac{m+n-3}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n-3}{4}$
$n \equiv 3 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n-4}{4}$

Edge condition of the dragon with $m \equiv 1 \pmod 4$ and for all values of n:

Table 10.

Nature of $m \equiv 1 \pmod 4$ and n	$\eta_e(0)$	$\eta_o(1)$
$n \equiv 0 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n-2}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n-2}{2}$

Case 3. $m \equiv 2 \pmod 4$

Assign the labels to the vertices $u_i, (1 \leq i \leq m-2)$ as in case 1. Then finally assign the labels 2, and 1 respectively to the vertices u_{m-1} and u_m of the cycle C_m.

Next we consider the vertices $v_i, , (1 \leq i \leq n)$ of the path P_n.

Subcase 3.1. $n \equiv 0 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n)$ as in subcase 1.1.

Subcase 3.2. $n \equiv 1 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n-1)$ as in subcase 1.1. Next assign the label 3 to the end vertex v_n of the path P_n.

Subcase 3.3. $n \equiv 2 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n-1)$ as in subcase 3.2. Then next assign the label 1 to the last vertex v_n of the path P_n.

Subcase 3.4. $n \equiv 3 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n-2)$ as in subcase 3.2. Finally assign the labels 4 and 1 to the vertices v_{n-1}, and v_n respectively. Thus the following Tables 11 and 12, establish that this vertex labeling f is a 4− remainder cordial labeling of the dragon with respect to $m \equiv 2 \pmod 4$ and for all values of n.

Vertex condition of the dragon with $m \equiv 2 \pmod 4$ and for all values of n:

Table 11.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$
$n \equiv 0 \pmod 4$	$\frac{m+n-2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n-2}{4}$	$\frac{m+n-2}{4}$
$n \equiv 1 \pmod 4$	$\frac{m+n-3}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n-3}{4}$
$n \equiv 2 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n-4}{4}$
$n \equiv 3 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$

Edge condition of the dragon with $m \equiv 2 \pmod 4$ and for all values of n:

Table 12.

Nature of $m \equiv 2 \pmod 4$ and n	$\eta_e(0)$	$\eta_o(1)$
$n \equiv 0 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n-2}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n-2}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$

Case 4. $m \equiv 3 \pmod 4$ and $n \equiv 0, 1 \pmod 4$

Assign the labels to the vertices $u_i, (1 \leq i \leq m-3)$, as in case 1. Then finally assign the labels $2, 3$, and 1 respectively to the vertices u_{m-2}, u_{m-1} and u_m of the cycle C_m.

Next assign the labels to the vertices $v_1, v_2, \ldots, v_n$ of the path P_n by the following two subcases.

Subcase 4.1. $n \equiv 0 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n)$, as in subcase 1.1.

Subcase 4.2. $n \equiv 1 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n)$, as in subcase 1.2. The following Tables 13, 14 establish that this vertex labeling f is a 4− remainder cordial labeling of the dragon with respect to $m \equiv 3 \pmod 4$ and for all values of n.

Vertex condition of the dragon with $m \equiv 3 \pmod 4$ and for all values of n:

Table 13.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$
$n \equiv 0 \pmod 4$	$\frac{m+n-3}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n-3}{4}$
$n \equiv 1 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n-4}{4}$

Edge condition of the dragon with $m \equiv 3 \pmod 4$ and for all values of n:

Table 14.

Nature of $m \equiv 3 \pmod 4$ and n	$\eta_e(0)$	$\eta_o(1)$
$n \equiv 0 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n-2}{2}$

Case 5. $m \equiv 3 \pmod 4$ and $n \equiv 2, 3 \pmod 4$

Assign the labels to the vertices $u_i, (1 \leq i \leq m-3)$, as in case 1. Then finally assign the labels $3, 2$, and 1 to the vertices u_{m-2}, u_{m-1} and u_m of the cycle C_m respectively.

Next we consider the vertices $v_1, v_2, \ldots, v_n$ of the path P_n. Assign the labels to the vertices $v_1, v_2, \ldots, v_n$ of the path P_n by the following the remaining subcases.

Subcase 5.1. $n \equiv 2 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n-2)$ as in subcase 1.1. Then assign the labels 4 and 1 respectively to the last two verticesv_{n-1} and v_n of the path P_n.

Subcase 5.2. $n \equiv 3 \pmod 4$

Assign the labels to the vertices $v_i, (1 \leq i \leq n-3)$ as in subcase 1.1. Finally assign the labels $4, 3$ and 1 to the vertices v_{n-2}, v_{n-1}, and v_n respectively. The following Tables 15 and 16 establish that this vertex labeling f is a $4-$ remainder cordial labeling of the dragon with $m \equiv 3 \pmod 4$ and for all values of n.

Vertex condition of the dragon with $m \equiv 3 \pmod 4$ and for all values of n:

Table 15.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$
$n \equiv 2 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n-1}{4}$
$n \equiv 3 \pmod 4$	$\frac{m+n-2}{4}$	$\frac{m+n-2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n-2}{4}$

Edge condition of the dragon with $m \equiv 3 \pmod 4$ and for all values of n:

Table 16.

Nature of $m \equiv 3 \pmod 4$ and n	$\eta_e(0)$	$\eta_o(1)$
$n \equiv 2 \pmod 4$	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n-2}{2}$	$\frac{m+n}{2}$

□

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