The degree associated edge reconstruction number of bidegreed graphs is at most four

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ABSTRACT

The *degree* of an edge uv of a graph G is $d_G(u) + d_G(v) - 2$. The *degree associated edge reconstruction number* of a graph G (or $dern(\mathrm{G})$) is the minimum number of degree associated edge-deleted subgraphs that uniquely determines G. Graphs whose vertices all have one of two possible degrees d and $d + 1$ are called $(d, d+1)-$bidegreed graphs. It was proved, in a sequence of two papers [1, 17], that $dern(mK_{1,3}) = 4$ for $m > 1$, $dern(mK_{2,3}) = dern(rP_3) = 3$ for $m > 0$, $r > 1$ and $dern(G) = 1$ or 2 for all other bidegreed graphs G except the $(d, d+1)$-bidegreed graphs in which a vertex of degree $d + 1$ is adjacent to at least two vertices of degree d. In this paper, we prove that $dern(G) = 1$ or 2 for this exceptional bidegreed graphs G. Thus, $dern(G) \leq 4$ for all bidegreed graphs G.

Keywords: reconstruction, reconstruction number, isomorphism, bidegreed graphs

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. We shall mostly follow the graph theoretic terminology of [8]. A *vertex-deleted subgraph* or *card* $G - v$ of a graph (digraph) G is the unlabeled graph (digraph) obtained from G by deleting the vertex v and all edges (arcs) incident with v. The *deck* of a graph (digraph) G is its collection of cards. A graph (digraph) G is *reconstructible* if it can be uniquely determined from its deck. The well-known Reconstruction Conjecture (RC) due to Kelly [10] and Ulam [23] asserts that every graph with at least three vertices is reconstructible. The conjecture has been proved for many special classes, and many properties of G may be deduced from its deck. Nevertheless, the full conjecture remains open. Surveys of results

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on the RC and related problems include [5, 12]. Harary and Plantholt [9] defined the reconstruction number of a graph G, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph H, $H \not\cong G$. These cards thus uniquely identify G. Reconstruction numbers are known for only few classes of graphs [2].

An extension of the RC to digraphs is the *Digraph Reconstruction Conjecture* (DRC), proposed by Harary [7], which asserts that every digraph with at least seven vertices is reconstructible. The DRC was disproved by Stockmeyer [22] by exhibiting several infinite families of counter-examples and this made people doubt the RC itself. To overcome this, Ramachandran [19] introduced degree associated reconstruction for digraphs and proposed a new conjecture in 1981. It was proved [19] that the digraphs in all these counterexamples to the DRC obey the new conjecture, thereby protecting the RC from the threat posed by these digraph counterexamples.

The ordered triple (a, b, c) where a, b and c are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pair of arcs incident with v in a digraph D is called the *degree triple of v*. The *degree associated card* or *dacard* of a graph (digraph) is a pair (d, C) consisting of a card C and the degree (degree triple) d of the deleted vertex. The *dadeck* of a graph (digraph) is the multiset of all its dacards. A graph (digraph) is said to be *N-reconstructible* if it can be uniquely determined from its dadeck. The *new digraph reconstruction conjecture* [19] (NDRC) asserts that all digraphs are N-reconstructible. Ramachandran [20, 21] then studied the degree associated reconstruction number of graphs and digraphs in 2000. The *degree (degree triple) associated reconstruction number* of a graph (digraph) D is the size of the smallest collection of dacards of D that uniquely determines D. Articles [6], [15] and [4] are recent papers on the degree associated reconstruction number.

The *edge card, edge deck, edge reconstructible graphs* and *edge reconstruction number* are defined similarly with edge deletions instead of vertex deletions. The *edge reconstruction conjecture*, proposed by Harary [7], states that all graphs with at least 4 edges are edge reconstructible. Edge reconstruction numbers are known for trees [13] and disconnected graphs [3, 14]. The *degree* of an edge e, denoted by $d(e)$, is the number of edges adjacent to e. That is, if $e = uv$ is an edge, then $d(e) = d(u) + d(v) - 2$. The ordered pair $(d(e), G - e)$ is called a *degree associated edge card* or *da-ecard* of the graph G, where $d(e)$ (called the *degree* of e) is the number of edges adjacent to e in G. The *edeck* (*da-edeck*) of a graph G is its collection of ecards (da-ecards). The *degree associated edge reconstruction number* of a graph G, denoted by $dern(G)$, is the minimum number of da-ecards which can only belong to the da-edeck of G and not to the da-edeck of any other graph H, $H \not\cong G$. These da-ecards thus uniquely identify G. The $dern(G)$ has been determined only for few classes of graphs [11, 17, 16].

By an *m-vertex*, we mean a vertex with degree m. We call a set with size m by *m-set*. The *neighbourhood* of a vertex v in a graph G is written by $N_G(v)$ (or simply $N(v)$). An m-vertex u is called an *m-neighbour* of v if u is a neighbour of v; otherwise u is called an *m-nonneighbour* of v. Two vertices u and v of a graph G are said to be *bisimilar* if there is an automorphism on G interchanging u and v. A graph whose vertices have only one of two possible degrees is called a *bidegreed* (or *biregular*) graph. The degree sequence of a bidegreed graph G, represented by $[d^a, (d+i)^b]$ means that G contains exactly a vertices of degree d, and b vertices of degree $d + i$, where $i > 0$. A set of vertices $\{v_1, v_2, \ldots, v_k\}$ of a graph G is said to be a *module* of G if $N_G(v_i) - \{v_1, v_2, \ldots, v_k\} = N_G(v_j) - \{v_1, v_2, \ldots, v_k\}$ for all $i, j \in \{1, 2, \ldots, k\}$.

Myrvold et al. [18] have proved that bidegreed graphs are edge reconstructible (thus degree associated edge reconstructible). Monikandan and Sundar Raj [17] have shown that $dern(G) = 1$ for all regular graphs and few bidegreed graphs G (namely paths, wheels, bistars, and bidegreed

graphs with exactly one vertex of different degree which differs by at least three). On excluding these families of bidegreed graphs, Anu and Monikandan [1] have shown that that $dern(mK_{1,3}) = 4$ for $m > 1$, $dern(mK_{2,3}) = dern(rP_3) = 3$ for $m > 0$, $r > 1$ and $dern(G) = 1$ or 2 for all other bidegreed graphs G except the $(d, d+1)$-bidegreed graphs with a $(d+1)$-vertex adjacent to at least two d-vertices. In this paper, we prove that $dern(G) = 1$ or 2 for this exceptional bidegreed graphs G. Thus, $dern(G) \leq 4$ for all bidegreed graphs G.

2.   Bidegreed graphs with degrees d and $d+1$

An *extension* of a da-ecard $(d(e), F - e)$ of a graph F is a graph obtained from the da-ecard by adding a new edge which joins two nonadjacent vertices whose degree sum is $d(e)$ and it is denoted by $H(d(e), F - e)$ (or simply by H).

For a graph F, to prove $dern(F) = k$, we proceed as follows.

(i) First, find the da-edeck of F.

(ii) Determine next all possible extensions of every da-ecard of F.

(iii) Finally, show that all extensions other than F have at most $k - 1$ da-ecards in common with those of F, and that at least one extension has precisely $k - 1$ da-ecards in common with those of F.

For sake of clarity, we recall the following three results from [1].

Theorem 2.1. *([1]; Theorem 3) Let G be a bidegreed graph with degree sequence $[d^a, (d+1)^b]$. Then $dern(G) = 1$ if $a = 1$ or $b = 1$.*

Theorem 2.2. *([1]; Theorem 6) Let G be a bidegreed graph with degree sequence $[d^a, (d+1)^b]$, a, $b \geq 2$. Then $dern(G) = 1$ if and only if*
   (i) *two vertices of minimum degree in G are adjacent;*
   (ii) *all the $(d+1)$-vertices in G are mutually adjacent; or*
   (iii) *any two d-vertices in a da-ecard of associated degree $2d - 1$ or $2d$ are bisimilar.*

Theorem 2.3. *([1]; Theorem 7) If G is a bidegreed graph in which all the $(d+1)$-vertices are mutually nonadjacent, then $dern(G) = 1$, 2 or 3.*

Moreover, in Theorem 2.3, $dern(G)$ is equal to 3 only if $G \cong mK_{2,3}$ or $G \cong rP_3$ for $m > 0$ and $r > 1$.

Notation 2.4. For $i = 1, 2, ...$, by x_i, we mean any d-vertex in G; by y_i (or z_i) super scripted with $0, 1, 2$, or 3, we mean a $(d+1)$-vertex adjacent to exactly no, one, two or at least three d-vertices respectively.

Theorem 2.5. *If every $(d+1)$-vertex of a bidegreed graph G has exactly two d-neighbours, then $dern(G) = 1$ or 2.*

Proof. We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2 and 2.3 and hence $dern(G) \geq 2$. Let y_1^2 be a $(d+1)$-vertex and let x_1, x_2 and y_2^2 be three neighbours of y_1^2. Then y_1^2 has exactly $d - 1$ neighbours of degree $d + 1$. We proceed by two cases depending upon the number of these neighbours that are shared by the vertex x_1.

Case 1. Vertices y_1^2 and x_1 have exactly $d - 1$ common $(d+1)$-neighbours.

Consider the two da-ecards $(2d-1, G-f)$ and $(2d, G-g)$, where $f = y_1^2 x_2$ and $g = y_1^2 y_2^2$. By Case 1, vertex y_2^2 is adjacent to x_1 (Figure 1).

Hence in the da-ecard $(2d, G-g)$, the vertices y_1^2, x_1 is a pair of adjacent d-vertices with other d-neighbours (namely y_2^2 or x_2) and no three d-vertices are mutually adjacent.
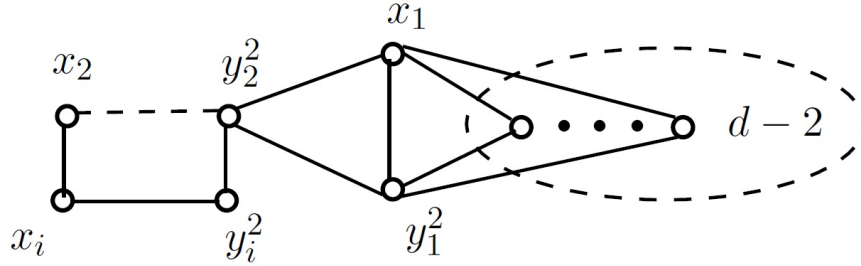

Fig. 1. Section of the da-ecard $G - f$

Now consider the extensions of the da-ecard $(2d-1, G-f)$. Note that in the da-ecard $(2d-1, G-f)$, the only adjacent d-vertices are y_1^2, x_1 and the only $(d-1)$-vertex is x_2. So, to get an extension H of the da-ecard $(2d-1, G-f)$, we have to join the $(d-1)$-vertex x_2 and any one of the d-vertices, say v by an edge in $(2d-1, G-f)$. Since $\{y_1^2, x_1\}$ is a module of $G-f$, it follows that $H \cong G$ when v is y_1^2 or x_1. So, we can assume that v is x_i for some $i \neq 1, 2$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d, H-e)$ of H. If e is $x_i y_j^2$ for some $j \neq 1, 2$, then, in the da-ecard $(2d, H-e)$, vertices y_1^2 and x_1 would be the only adjacent d-vertices such that none of them adjacent with other d-vertices and so $H - e \not\cong G - g$. Otherwise, e is $x_i y_2^2$ (if any) or $y_2^2 y_i^2$ for $i \neq 1$. Now in the da-ecard $(2d, H-e)$, vertex y_2^2 will become a d-vertex and the three d-vertices y_2^2, y_1^2 and x_1 will become mutually adjacent, which implies $H - e \not\cong G - g$. Hence every extension $H(2d, G-f)$ is either isomorphic to G or any $(2d, H-e)$ da-ecard of H would contain either two adjacent d-vertices having no other d-neighbours or three d-vertices are mutually adjacent and so $H - e$ is not isomorphic to $G - g$. Hence $dern(G) \leq 2$.
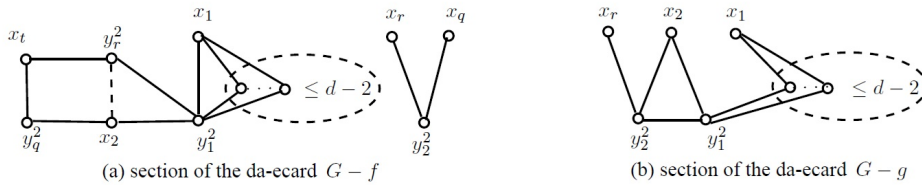

(a) section of the da-ecard $G - f$ (b) section of the da-ecard $G - g$

Fig. 2.

Case 2. Vertices x_1 and y_1^2 have at most $d - 2$ common $(d+1)$-neighbours.

Let y_2^2 be nonadjacent to x_1.

Case 2.1. Vertices x_2 and y_2^2 are nonadjacent.

Let x_r and x_q be the two d-neighbours of y_2^2. Now consider the two da-ecards $(2d, G-f)$ and $(2d-1, G-g)$, where $f = y_1^2 y_2^2$ and $g = y_1^2 x_1$. In $(2d-1, G-g)$, vertices y_1^2, x_2 is a pair of adjacent d-vertices and $(d-1)$-vertex nonadjacent to any d-vertices. In $(2d, G-f)$, exactly six d-vertices form a $2K_{1,2}$. So, to get an extension H of the da-ecard $(2d, G-f)$, we have to join two d-vertices, say

v_1, v_2 by an edge. If $\{v_1, v_2\} = \{y_1^2, y_2^2\}$ (Figure 2(a)), then $H \cong G$. Otherwise, since the vertices v_1 and v_2 will have degree $d+1$ in the extension $H(2d, G - f)$, every da-ecard $(2d-1, H - e)$ of H has either at least two pairs of adjacent d-vertices or the unique $(d-1)$-vertex is adjacent to a d-vertex, which implies $H - e \not\cong G - g$. Thus no da-ecard of H would be isomorphic to $(2d, G - g)$ and hence $dern(G) \leq 2$.

Case 2.2. Vertices x_2 and y_2^2 are adjacent.

Let x_r and x_2 be the two d-neighbours of y_2^2. Consider the da-ecards $(2d-1, G-g)$ and $(2d, G-f)$, where $f = y_1^2 y_2^2$ and $g = y_1^2 x_1$. In $(2d, G - f)$, exactly five d-vertices together form a path P_5 (say w_1, w_2, w_3, w_4, w_5) such that each $(d+1)$-vertex has exactly two d-neighbours other than w_2 and w_4. In $(2d-1, G-g)$, the only adjacent d-vertices are y_1^2, x_2 and the only $(d-1)$-vertex is x_1. So, to get an extension H of the da-ecard $(2d-1, G-g)$, we have to join the $(d-1)$-vertex x_1 and any one vertex, say v among the d-vertices in $(2d-1, G-g)$ by an edge. If v is y_1^2 (Figure 2(b)), then $H \cong G$. Otherwise, v is x_i for some $i \neq 1$ so that it will become a $(d+1)$-vertex in H. Now every da-ecard $(2d, H - e)$ of H has either at least one $(d+1)$-vertex with exactly one d-neighbour or no five d-vertices together form a path P_5, which implies $H - e \not\cong G - f$. Thus no da-ecard of H would be isomorphic to $(2d, G - f)$ and hence $dern(G) \leq 2$. □

The next result is proved in [1].

Theorem 2.6. *([1]; Theorem 10) If every $(d+1)$-vertex of a bidegreed graph G has at most one d-neighbour, then $dern(G) = 2$.*

In view of Theorems 2.5 and 2.6, we can assume hereafter that G contains both a $(d+1)$-vertex (say y_r^2) adjacent to exactly two d-vertices (say x_r, x_q) and a $(d+1)$-vertex adjacent to at most one d-vertex.

Lemma 2.7. *Let G be a bidegreed graph in which vertices of type y_r^2, y_r^0 exist for some r and y_r^1 does not exist for any r. If y_r^2 and a d-neighbour of it have at most $d-2$ common $(d+1)$-neighbours, and y_r^2 is adjacent to some other y_q^2, then $dern(G) = 1$ or 2.*

Proof. We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2 and 2.3 and hence $dern(G) \geq 2$. Let x_r and x_q be two d-neighbours of y_r^2. Let y_r^2 and x_r have at most $d-2$ common $(d+1)$-neighbours.

Case 1. The vertex y_q^2 is nonadjacent to at least one of x_r and x_q.

Consider the two da-ecards $(2d, G - y_r^2 y_q^2)$ and $(2d, G - y^0 z^0)$ (or $(2d, G - y_k^2 y^0)$ where $y^0 \in N(y_k^2)$ for some k). In $G - y^0 z^0$ (or $G - y_k^2 y^0$), each d-vertex is nonadjacent to all other d-vertices (or there is a unique path of order three lying among d-vertices such that each d-vertex not in the path is nonadjacent to the other d-vertices). In $(2d, G - y_r^2 y_q^2)$, there is a unique path of order five lying among d-vertices (If $x_r \in N(y_q^2)$, then we obtain the path $x_q y_r^2 x_r y_q^2 x_p$, where $x_p \in N(y_q^2)$. If $x_q \in N(y_q^2)$, then we obtain the path $x_r y_r^2 x_q y_q^2 x_p$, where $x_p \in N(y_q^2)$) or there are only six d-vertices together forming $2K_{1,2}$ (when $x_r, x_q \notin N(y_q^2)$). So, to get an extension H of the da-ecard $(2d, G - y_r^2 y_q^2)$, we have to join two d-vertices by an edge e. If $e = y_r^2 y_q^2$, then $H \cong G$. Otherwise, e is $y_r^2 x_i$ or $y_q^2 x_i$ or $x_i x_j$. Now in the da-ecard $(2d, H - e)$ of the resulting extension H, either no three d-vertices together form a P_3 or three d-vertices together form a P_3 but at least one d-vertex not in the P_3 is adjacent to other d-vertices. Therefore no da-ecard of H would be isomorphic to $(2d, G - y^0 z^0)$ (or $(2d, G - y_k^2 y^0)$) and hence $dern(G) \leq 2$.

Case 2. The vertex y_q^2 is adjacent to both x_r and x_q.

Now if y_i^0 is nonadjacent to any other y_j^0 for all i and j, then we consider the two da-ecards $(2d-1, G - y_r^2 x_r)$ and $(2d, G - y_i^2 y^0)$ for some i. In the da-ecard $G - y_i^2 y^0$, exactly three d-vertices together form a path P_3 (say w_1, w_2, w_3) but no other d-vertices are adjacent, there is no $(d+1)$-vertex adjacent to exactly one d-vertex other than w_2 and there are d vertices of degree $d+1$ adjacent to exactly three d-vertices other than w_2. In the da-ecard $(2d-1, G - y_r^2 x_r)$, the only adjacent d-vertices are y_r^2, x_q and the only $(d-1)$-vertex is x_r. So, to get an extension H of the da-ecard $(2d-1, G - y_r^2 x_r)$, we have to join the $(d-1)$-vertex x_r and any one vertex, say v among the d-vertices by an edge in $(2d-1, G - y_r^2 x_r)$. If $v = y_r^2$, then clearly, $H \cong G$. Now, we can assume that v is x_i for some $i \neq q$ so that it will become a $(d+1)$-vertex in H. In the da-ecard $(2d, H - e)$, exactly three d-vertices together form a path P_3 (say z_1, z_2, z_3) and at least two d-vertices other than z_1, z_2 and z_3 are adjacent. If v is x_q, then, in the da-ecard $(2d, H - e)$, vertex x_q will become a $(d+1)$-vertex and at least $d+1$ vertices of degree $d+1$ adjacent to exactly three d-vertices other than z_2 or at least one $(d+1)$-vertex adjacent to exactly one d-vertex other than z_2. Therefore $H - e \ncong G - y_i^2 y^0$. Thus no da-ecard of H would be isomorphic to $(2d, G - y_i^2 y^0)$ and hence $dern(G) \leq 2$.

Otherwise, that is y^0 and z^0 are adjacent. If $N(x_r) \neq N(x_q)$, then we consider the two da-ecards $(2d, G - y_r^2 y_q^2)$ and $(2d, G - y^0 z^0)$. In the da-ecard $G - y^0 z^0$, there are $2d$ vertices of degree $d+1$ adjacent to exactly one or three d-vertices and each d-vertex is nonadjacent to all other d-vertices. In the da-ecard $(2d, G - y_r^2 y_q^2)$, there is a unique cycle, denoted by C_4^d, of order four lying among d-vertices (namely, $y_r^2 x_q y_q^2 x_r$). So, to get an extension H of the da-ecard $(2d, G - y_r^2 y_q^2)$, we have to join two d-vertices, say v_1, v_2 by an edge e. If $\{v_1, v_2\} = \{y_r^2, y_q^2\}$, then clearly, $H \cong G$. Otherwise, v_1, v_2 are equal to some $x_i's$, where $i \neq r$(or q), so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d, H - e)$ of H. In $(2d, H - e)$, at least two d-vertices are adjacent. If v_1, v_2 are equal to some $x_i's$, where $i = r, q$, then, since at least one $(d+1)$-neighbour of x_r is adjacent exactly one d-vertex, the da-ecard $(2d, H - e)$ has at least $2d + 1$ vertices of degree $d+1$ adjacent to exactly one or three d-vertices. Therefore no da-ecard of H would be isomorphic to $(2d, G - y^0 z^0)$ and hence $dern(G) \leq 2$.

Otherwise, that is $N(x_r) = N(x_q)$, we consider the two da-ecards $(2d-1, G - y_r^2 x_r)$ and $(2d, G - y^0 z^0)$. In the da-ecard $G - y^0 z^0$, the set consisting of y_r^2 and all d-neighbours of y_r^2 is a module and any two d-vertices are nonadjacent. In the da-ecard $(2d-1, G - y_r^2 x_r)$, the only adjacent d-vertices are y_r^2, x_q and the only $(d-1)$-vertex is x_r. So, to get an extension H of the da-ecard $(2d-1, G - y_r^2 x_r)$, we have to join the $(d-1)$-vertex x_r and any one vertex, say v among the d-vertices by an edge in $(2d-1, G - y_r^2 x_r)$. If $v = y_r^2$, then clearly, $H \cong G$. Now, we can assume that v is x_i for some $i \neq q$ so that it will become a $(d+1)$-vertex in H. In the da-ecard $(2d, H - e)$, at least two d-vertices are adjacent. If v is x_q, then, in the da-ecard $(2d, H - e)$, vertex x_q will become a $(d+1)$-vertex and it is adjacent to two d-vertices with distinct neighbourhoods. Therefore $H - e \ncong G - y^0 z^0$. Thus no da-ecard of H would be isomorphic to $(2d, G - y^0 z^0)$ and hence $dern(G) \leq 2$. $\square$

Lemma 2.8. *Let G be a bidegreed graph in which vertices of type y_r^2, y_r^1 and possibly y_r^0 exist. If y_r^2 and a d-neighbour of it have at most $d-2$ common $(d+1)$-neighbours, and y_r^2 is adjacent to some other y_q^2, then $dern(G) = 1$ or 2.*

Proof. We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2, 2.3 and Lemma 2.7 and hence $dern(G) \geq 2$. Let x_r and x_q be two d-neighbours of y_r^2. Let y_r^2 and x_r have at most $d-2$

common $(d+1)$-neighbours. Let y_r^2 be adjacent to y_q^2. Suppose y_q^2 is nonadjacent to at least one of x_q and x_r. Consider the da-ecards $(2d, G - y_r^2 y_q^2)$ and $(2d - 1, G - x_s y_s^1)$, where y_s^1 is adjacent to x_s. Clearly, all the d-vertices in $G - x_s y_s^1$ are mutually nonadjacent. Every extension $H(G - y_r^2 y_q^2)$ is either isomorphic to G or every da-ecard $(2d - 1, H - e)$ contains at least two adjacent d-vertices and so no da-ecard of H is isomorphic to $(2d - 1, G - x_s y_s^1)$.

Suppose that y_q^2 is adjacent to both x_q and x_r. We proceed by two cases.

Case 1. Vertex y_r^2 is adjacent to some y_s^1.

Denote the unique d-vertex adjacent to y_s^1 by x_i. If $x_i = x_r$ (or x_q), then we choose the two da-ecards A_j and B as below. Choose $A_1 = (2d - 1, G - y_s^1 x_i)$ and $B = (2d, G - y_r^2 y_q^2)$ when $(N(y_r^2) - \{x_i\} \cap N(y_s^1) - \{x_i\}) = \phi$; choose the da-ecards $A_2 = (2d - 1, G - y_q^2 x_q)$ and B when $N(y_r^2) \cap \{y_t^1\} \subseteq N(y_q^2)$ but $y_t^1 \notin N(x_i)$ for some t; choose the da-ecards $A_3 = (2d, G - y_r^2 y_s^1)$ and B when $N(y_r^2) \cap \{y_t^1\} \subseteq N(y_s^1)$ but $y_t^1 \notin N(x_i)$ for some t; or choose the da-ecards $A_4 = (2d - 1, G - y_t^1 x_t)$ and B when $\{y_t^1\} \subseteq N(y_r^2)$ and $\{y_t^1\} \notin N(x_r) \cup N(y_s^1) \cup N(y_q^2)$.

The da-ecard B contains a unique cycle C_4^d and all the d-vertices are nonadjacent to every d-vertex not in C_4^d. Hence every extension $H(A_i)$ is either isomorphic to G or any da-ecard $(2d, H(A_i) - e)$ has one of the following properties.

(i) It has a unique cycle C_4^d and at least one d-vertex has a d-neighbour which is not in C_4^d;

(ii) the number of copies of $C_4 + e$ containing exactly two d-vertices in $H(A_i) - e$ is fewer than that in B; or

(iii) it has exactly two adjacent d-vertices.

Hence no da-ecard of $H(A_i)$ is isomorphic to B.

Suppose $\{y_j^0, y_k^0\} \subseteq N(y_r^2)$. If y_j^0 is adjacent to y_k^0, then choose the two da-ecards $(2d, G - y_r^2 y_j^0)$ and $(2d - 1, G - y_s^1 x_i)$. Now every extension $H(G - y_r^2 y_j^0)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties.

(i) It has a $(d + 1)$-vertex, say y adjacent to exactly two d-vertices and to a $(d + 1)$-vertex having a d-neighbour not in $N(y)$; or

(ii) it has a $(d + 1)$-vertex, say w adjacent to three vertices of degree at most d and to a $(d + 1)$-vertex having a d-neighbour not in $N(w)$.

But the da-ecard $(2d - 1, G - y_s^1 x_i)$ satisfies none of the above two properties and so it is not a da-ecard of H.

If y_j^0 is not adjacent to y_k^0, then consider the two da-ecards $(2d - 1, G - y_s^1 x_i)$ and $(2d, G - y_r^2 y_j^0)$. In $G - y_s^1 x_i$, there is a unique path, say P_3^d of order 3 whose vertices are d-vertices having a common $(d + 1)$-neighbour. Now every extension $H(G - y_s^1 x_i)$ is either isomorphic to G or any da-ecard $(2d, H - e)$ has one of the following properties:

(i) The number of copies of $C_4 + e$ containing exactly two d-vertices in $H(G - y_s^1 x_i)$ is fewer than that in $G - y_s^1 x_i$; or

(ii) there is no $(d + 1)$-vertex as a common neighbour of the d-vertices in P_3^d.

Hence no da-ecard of H is isomorphic to $(2d - 1, G - y_s^1 x_i)$.

Suppose that $x_i \neq x_r$ and $x_i \neq x_q$. Consider the two da-ecards $(2d - 1, G - y_s^1 x_i)$ and $(2d, G - y_r^2 y_q^2)$. The da-ecard $G - y_r^2 y_q^2$ contains a unique cycle C_4^d and each $(d + 1)$-vertex is nonadjacent to all the d-vertices not in C_4^d. Hence every extension $H(G - y_s^1 x_i)$ is either isomorphic to G or any $(2d, H - e)$ da-ecard has one of the following properties.

(i) It contains a unique cycle C_4^d and at least one $(d + 1)$-vertex has a d-neighbour not in C_4^d; or

(ii) the number of copies of $C_4 + e$ containing exactly two d-vertices in $H(G - y_s^1 x_i) - e$ is fewer than that in $G - y_r^2 y_q^2$.

Hence no da-ecard of $H(G - y_s^1 x_i)$ is isomorphic to $G - y_r^2 y_q^2$.

Case 2. Vertex y_r^2 is nonadjacent to any y_i^1.

If some y_i^2 is adjacent to x_r but not adjacent to y_r^2, then we consider the two da-ecards $(2d, G - y_r^2 y_q^2)$ and $(2d - 1, G - y_q^2 x_q)$. Clearly the da-ecard $G - y_q^2 x_q$ has a $K_{1,3} + e$ containing exactly two d-vertices such that all the neighbours of a $(d + 1)$-vertex in $K_{1,3} + e$ are nonadjacent to every d-vertex not in $K_{1,3} + e$. Now every extension $H(G - y_r^2 y_q^2)$ is either isomorphic to G or in any da-ecard $(2d - 1, H - e)$, at least one neighbour of a $(d + 1)$-vertex in any $K_{1,3} + e$ containing exactly two d-vertices, has a d-neighbour not lying in it and hence no da-ecard of H is isomorphic to $(2d - 1, G - y_q^2 x_q)$. So, we can take that $N(y_r^2) = \{x_r, x_q, y_1^2, ..., y_{r_1}^2, y_1^0, ..., y_s^0\}$ and $N(x_r) = \{y_r^2, y_1^2, ..., y_{r_1}^2, z_1^1, ..., z_s^1\}$.

If two adjacent z_i^1's (say z_1^1, z_2^1) are adjacent to x_r, then we consider the two da-ecards $(2d - 1, G - x_r z_1^1)$ and $(2d - 1, G - y_r^2 x_q)$. In $G - y_r^2 x_q$, exactly two d-vertices are adjacent and every uncommon $(d + 1)$-neighbour of these two d-vertices is nonadjacent to the other d-vertex. Now every extension $H(G - x_r z_1^1)$ is either isomorphic to G or in any da-ecard $(2d - 1, H - e)$, two d-vertices are adjacent and an uncommon $(d + 1)$-neighbour of these two d-vertices is adjacent to the other d-vertex and hence no da-ecard of H is isomorphic to $G - y_r^2 x_q$. Otherwise, if some y_i^0 (say y_s^0) is adjacent to both y_r^2 and y_q^2, then we consider the two da-ecards $(2d, G - y_r^2 y_q^2)$ and $(2d - 1, G - y_q^2 x_r)$. Now every extension $H(G - y_r^2 y_q^2)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties:

(i) It contains an induced subgraph $K_{1,3} + e$ with exactly two (nonadjacent) d-vertices having a common $(d + 1)$-neighbour lying outside $K_{1,3} + e$; or

(ii) it contains two adjacent d-vertices, one of which and the unique $(d - 1)$-vertex have a common $(d + 1)$-neighbour.

But the da-ecard $(2d - 1, G - y_q^2 x_r)$ satisfies none of the above two properties and hence it is not a da-ecard of H.

Otherwise, *if $z_j^1 \in N(y_i^0)$ or $z_j^1 \in N(z_i^1)$ for some $i \in \{1, 2, \cdots, s\}$ and j,* then consider the two da-ecards $(2d - 1, G - z_j^1 x_j)$ and $(2d - 1, G - y_r^2 x_r)$. In $G - y_r^2 x_r$, every $(d + 1)$-neighbour of the two adjacent d-vertices is not adjacent to any other d-vertex. Every extension $H(G - z_j^1 x_j)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard of the extension would contain at least one $(d + 1)$-neighbour of the two adjacent d-vertices is adjacent to the other d-vertex and hence no da-ecard of H is isomorphic to $(2d - 1, G - y_r^2 x_r)$. *Otherwise,* by proceeding as above, we will get, at some stage, a vertex y_p^0 that is adjacent to y_q^1 or y_q^0. If the first case happens (Figure 3), then by proceeding as above, we get $dern(G) \leq 2$. If the later case happens, that is, if y_p^0 is adjacent to some y_q^0 but not to any y_q^1, then consider the two da-ecards $(2d - 1, G - z_r^1 x_r)$ and $(2d - 1, G - y_r^2 x_r)$. In $G - y_r^2 x_r$, exactly two d-vertices are adjacent. Clearly, every extension $H(G - z_r^1 x_r)$ is either isomorphic to G or in any da-ecard $(2d - 1, H - e)$, all the d-vertices are mutually nonadjacent and hence no da-ecard of H is isomorphic to $(2d - 1, G - y_r^2 x_r)$.   □

Lemma 2.9. *Let G be a bidegreed graph in which vertices of type y_r^2, y_r^1 and possibly y_r^0 exist. If y_r^2 and a d-neighbour of it have at most $d - 2$ common $(d + 1)$-neighbours, and y_r^2 is not adjacent to any other vertex of type y_i^2, then $dern(G) = 1$ or 2.*

Proof. We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2 and 2.3 and hence $dern(G) \geq 2$. Let $N(y_r^2) = \{x_r, x_q, y_1^0, y_2^0, \cdots, y_{m_1}^0, z_1^1, z_2^1, \cdots, z_{n_1}^1\}$ and
$N(x_r) = \{y_1^1, y_2^1, \cdots, y_{m_2}^1, y_1^2, y_2^2, \cdots, y_{n_2}^2\}$. Suppose that $m_1 \neq m_2$ or $n_1 \neq n_2$.
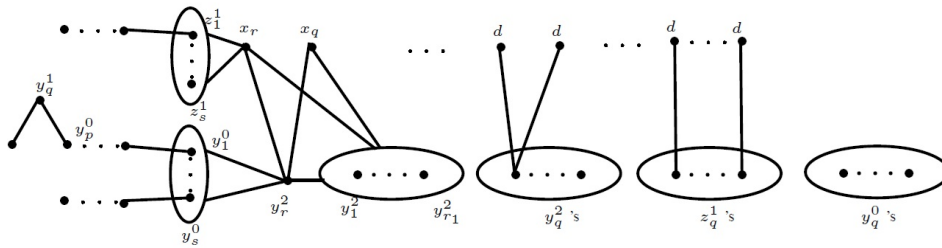
Fig. 3. Section of the graph G under Case 2

If $n_1 \neq n_2$ (or $n_1 = n_2 \neq 0$), then we consider the two da-ecards $(2d - 1, G - y_r^2 x_q)$ and $(2d - 1, G - z_{n_1}^1 x_{n_1})$. Now every extension $H(G - y_r^2 x_q)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties (α) and (β).

(α) Every $(d+1)$-vertex has at most two d-neighbours; or

(β) at most m vertices of degree $d + 1$ have exactly two d-neighbours such that

(i) the neighbourhood of each $(d+1)$-vertex stated in (β) can be partitioned into an m_1-set and an n_1-set such that every vertex in the m_1-set has no other d-neighbours and every vertex in the n_1-set has exactly one other d-neighbour, and

(ii) the neighbourhood of one vertex in every pair of d-vertices stated in (β) can be partitioned into an m_2-set and an n_2-set, where $m_1 \neq m_2$ and $n_1 \neq n_2$ such that every vertex in the m_2-set has no other d-neighbours and every vertex in the n_2-set has exactly one other d-neighbour.

But the da-ecard $(2d-1, G - z_{n_1}^1 x_{n_1})$ satisfies none of the above properties and hence it would not be a da-ecard of H. Otherwise, consider the two da-ecards $(2d - 1, G - y_r^2 x_q)$ and $(2d - 1, G - y_{m_2}^1 x_r)$. Now every extension $H(G - y_r^2 x_q)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties (α) and (β).

(α) At least one $(d+1)$-vertex is adjacent to a pair of vertices of degree d and $d - 1$ respectively, such that m_2 ($\neq m_1$) neighbours of this $(d+1)$-vertex have no other d-neighbours; or

(β) at most m vertices of degree $d + 1$ in the da-ecard are adjacent to two d-vertices such that

(i) the neighbourhood of each $(d+1)$-vertex stated in (β) can be partitioned into an m_1-set and an n_1-set such that every vertex in the m_1-set has no other d-neighbours and every vertex in the n_1-set has exactly one other d-neighbour, and

(ii) the neighbourhood of one vertex in each pair of d-vertices stated in (β) can be partitioned into an m_2-set and an n_2-set, where $m_1 \neq m_2$ and $n_1 \neq n_2$ such that every vertex in the m_2-set has no other d-neighbours and every vertex in the n_2-set has exactly one other d-neighbour.

But the da-ecard $(2d - 1, G - y_{m_2}^1 x_r)$ satisfies none of the above properties and hence it would not be a da-ecard of H.

Assume now that $m_1 = m_2$ and $n_1 = n_2 = 0$. We shall proceed by four cases as below.

Case 1. The set $\{y_1^0, y_2^0, \ldots, y_{m_1}^0\}$ is independent, but $\{y_1^1, y_2^1, \ldots, y_{m_2}^1\}$ is not.

Consider the two da-ecards $(2d - 1, G - y_r^2 x_q)$ and $(2d - 1, G - y_{m_2}^1 x_r)$. Now every extension $H(G - y_r^2 x_q)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties.

(α) At least one $(d+1)$-vertex is adjacent to a pair of vertices of degree d, $d - 1$ such that the set of all neighbours of this $(d + 1)$-vertex is not an independent set; or

(β) at most m vertices of degree $d + 1$ are adjacent to two d-vertices such that the set of all neighbours of one of them is an independent set where as the set of all neighbours of at least one of

them is not an independent set.

But the da-ecard $(2d - 1, G - y_{m_2}^1 x_r)$ satisfies none of the above two properties and hence it would not be a da-ecard of H.

Case 2. The set $\{y_1^1, y_2^1, \ldots, y_{m_2}^1\}$ is an independent set, but $\{y_1^0, y_2^0, \ldots, y_{m_1}^0\}$ is not.

In this case, consider the two da-ecards $(2d - 1, G - y_r^2 x_q)$ and $(2d - 1, G - y_{m_2}^1 x_r)$. Now every extension $H(G - y_r^2 x_q)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties.

(α) At least one $(d+1)$-vertex is adjacent to a pair of vertices of degree d, $d - 1$ such that the set of all neighbours of these $(d + 1)$-vertices is an independent set; or

(β) at most m vertices of degree $d + 1$ are adjacent to two d-vertices such that the set of all neighbours of one of them is not an independent set where as the set of all neighbours of at least one of them is an independent set.

But the da-ecard $(2d - 1, G - y_{m_2}^1 x_r)$ satisfies none of the above two properties and hence it would not be a da-ecard of H.

Case 3. Both the sets $\{y_1^0, y_2^0, \ldots, y_{m_1}^0\}$ and $\{y_1^1, y_2^1, \ldots, y_{m_2}^1\}$ are independent.

If $\{y_a^1\} \subseteq N(y_i^0)$ for some i and a (respectively $\{y_b^1\} \subseteq N(y_j^1)$ for some j and b, y_i^0 is adjacent to y_j^1, for some i and j), then consider the two da-ecards $(2d - 1, G - y_a^1 x_a)$ (respectively $(2d - 1, G - y_b^1 x_b)$, $(2d - 1, G - x_r y_j^1)$) and $(2d - 1, G - y_r^2 x_q)$. Now every extension $H(G - G - y_a^1 x_a)$ (respectively $H(G - y_b^1 x_b)$, $H(G - x_r y_j^1)$ is either isomorphic to G or any da-ecard $(2d - 1, H - e)$ has one of the following properties:

(α) At least one $(d + 1)$-neighbour of two adjacent d-vertices has another d-neighbour;

(β) there is a $(d + 1)$-vertex w adjacent to two d-vertices such that at least one neighbour of w is adjacent to a d-vertex; or

(γ) there is a $(d + 1)$-vertex adjacent to two d-vertices, say w_1, w_2 such that at least one neighbour of w_1 or w_2 has exactly one another d-neighbour.

Hence $H - e$ is not isomorphic to $(2d - 1, G - y_r^2 x_q)$. If at least one neighbour of y_i^0 is adjacent to y_a^1 for some a; then we consider the two da-ecards $(2d - 1, G - x_a y_a^1)$ (note that 'a' may be equal to r) and $(2d - 1, G - y_r^2 x_q)$. Now every extension $H(G - x_a y_a^1)$ is either isomorphic to G or any da-ecard $(2d - 1, H - e)$ has one of the following properties:

(α) At least one $(d + 1)$-neighbour of one of the adjacent d-vertices is adjacent to a neighbour of a d-vertex; or

(β) there is a $(d + 1)$-vertex (say w) adjacent to two d-vertices such that at least one neighbour of w is adjacent to a neighbour of a d-vertex.
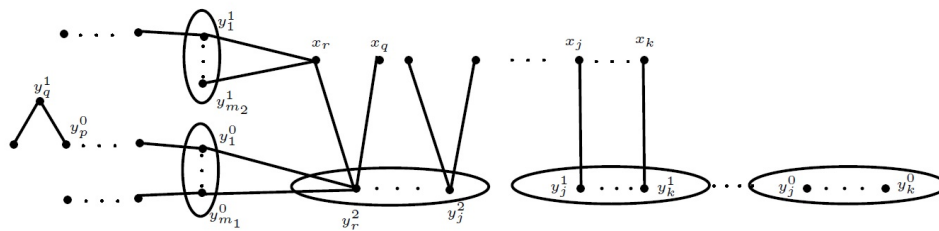

Fig. 4. Section of the graph G under Case 3

Hence $H - e$ is not isomorphic to $G - y_r^2 x_q$. Otherwise, by proceeding as just above, at some stage, we must get a vertex y_p^0 that is adjacent to y_q^1 or y_q^0. If the former holds (Figure 4), then, by

proceeding as above, we get $dern(G) \leq 2$. If the later holds, that is, if y_p^0 is adjacent to some y_q^0 but not to any y_q^1, then we consider the two da-ecards $(2d - 1, G - y_j^1 x_r)$ and $(2d - 1, G - y_r^2 x_q)$, where $1 \leq j \leq m_2$. In $G - y_r^2 x_q$, there are only two d-vertices are adjacent. Clearly, every extension $H(G - y_j^1 x_r)$ is either isomorphic to G or in any da-ecard $(2d - 1, H - e)$, all the d-vertices are mutually nonadjacent and hence no da-ecard of H is isomorphic to $(2d - 1, G - y_r^2 x_q)$.

Case 4. Neither $\{y_1^0, y_2^0, \ldots, y_{m_1}^0\}$ nor $\{y_1^1, y_2^1, \ldots, y_{m_2}^1\}$ is an independent set.

Suppose that $\{y_a^1\} \subseteq N(y_j^1)$ for some j and a (or $N(y_j^1) = \{y_{r_1}^0, y_{r_2}^0, \ldots, y_{r_{d-1}}^0\}$ is an independent set or "$N(y_j^1) = \{y_{r_1}^0, y_{r_2}^0, \ldots, y_{r_{d-1}}^0\}$ is not an independent set and y_j^1 is adjacent to y_1^{0}"). The two da-ecards we considered here are $(2d - 1, G - x_r y_j^1)$ and $(2d - 1, G - y_r^2 x_q)$. Clearly, every extension $H(G - x_r y_j^1)$ is either isomorphic to G or any da-ecard $(2d - 1, H - e)$ has one of the following properties.

(α) Only two d-vertices are adjacent and a neigbour of exactly one of them is adjacent to a d-vertex;

(β) the set of all neighbours of each d-vertex is independent; or

(γ) at least one $(d + 1)$-vertex (say w) has two d-neighbours and each neighbour of w has a d-neighbour.

Hence $H - e$ is not isomorphic to $(2d - 1, G - y_r^2 x_q)$. For other remaining cases, proceeding as in the above two paragraphs, we get $dern(G) \leq 2$. This completes Case 4 and our assumption that $m_1 = m_2$ and $n_1 = n_2 = 0$.

Finally, we assume that $m_1 = m_2$ and $n_1 = n_2 \neq 0$. If $\{z_j^1\} \subseteq N(z_i^1)$, $1 \leq i, j \leq n_1$, then consider the two da-ecards $(2d, G - z_i^1 z_j^1)$ and $(2d - 1, G - y_r^2 x_q)$. Now every extension $H(G - z_i^1 z_j^1)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties:

(α) There is a $(d + 1)$-vertex adjacent to two d-vertices such that the set of all neighbours of at least one of them is not an independent set; or

(β) at least two d-vertices are adjacent.

But the da-ecard $(2d - 1, G - y_r^2 x_q)$ satisfies none of the above two properties and hence it would not be a da-ecard of H. Otherwise, let us assume that $\{z_j^1\} \nsubseteq N(z_i^1)$, $1 \leq i, j \leq n_1$ for all i, j. Suppose $\{y_k^2\} \subseteq N(z_i^1)$ for some k ($y_k^2 \neq y_r^2$). Then we consider the two da-ecards $(2d, G - y_r^2 z_i^1)$ and $(2d - 1, G - y_{n_2}^2 x_{n_2})$. Now every extension $H(G - y_r^2 z_i^1)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties.

(α) At least one $(d + 1)$-vertex is adjacent to three vertices of degree at most d; or

(β) there is a unique d-vertex adjacent to two distinct d-vertices.

But the da-ecard $(2d - 1, G - y_{n_2}^2 x_{n_2})$ satisfies none of the above two properties and hence it would not be a da-ecard of H. Suppose $\{y_k^2, y_k^1\} \nsubseteq N(z_i^1)$ for all k and for some i. Then consider the two da-ecards $(2d - 1, G - z_i^1 x_i)$ and $(2d - 1, G - y_r^2 x_q)$. Now every extension $H(G - z_i^1 x_i)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties.

(α) All the neighbours of the unique $(d - 1)$-vertex is nonadjacent to a d-vertex; or

(β) there are only two adjacent d-vertices and all the neighbours of at least one of them are nonadjacent to all other d-vertices.

But the da-ecard $(2d - 1, G - y_r^2 x_q)$ satisfies none of the above two properties and hence it would not be a da-ecard of H.

Now, let us take that $\{y_k^1\} \subseteq N(z_i^1)$ for some k. Suppose $\{y_k^1\} \nsubseteq N(y_j^2)$ for all k and $1 \leq j \leq n_1$. Consider the two da-ecards $(2d - 1, G - y_r^2 x_q)$ and $(2d - 1, G - z_i^1 x_i)$. Every extension $H(G - y_r^2 x_q)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard has one of the following properties.

(α) There is a $(d + 1)$-vertex (say w) adjacent to two d-vertices and a $(d + 1)$-neighbour of w is nonadjacent to both a d-vertex and a neighbour of a d-vertex; or

(β) there is a $(d+1)$-vertex adjacent to three d-vertices such that all the neighbours of at least one of these d-vertices are nonadjacent to all other d-vertices.

But the da-ecard $(2d-1, G-z_i^1 x_i)$ satisfies none of the above two properties and hence it would not be a da-ecard of H. So, we consider the case that $\{y_k^1\} \subseteq N(y_j^2)$ for some k.

If $\{y_k^2\} \subseteq N(x_i)$ for some k and $x_i \in N(z_i^1)$, then we consider the two da-ecards $(2d-1, G-y_k^2 x_k)$ and $(2d-1, G-y_k^1 x_s)$, where $k \neq i$. Every extension $H(G-y_k^2 x_k)$ is either isomorphic to G or any $(2d-1, H-e)$ da-ecard has one of the following properties.

(α) There are only two adjacent d-vertices; or

(β) at least one $(d+1)$-vertex (say w) has two d-neighbours such that a $(d+1)$-neighbour of w is adjacent to a $(d+1)$-vertex having two d-neighbours.

But the da-ecard $(2d-1, G-y_k^1 x_s)$ satisfies none of the above two properties and hence it would not be a da-ecard of H. Otherwise, consider the two da-ecards $(2d, G-y_r^2 z_i^1)$ and $(2d-1, G-z_i^1 x_i)$. Now every extension $H(G-y_r^2 z_i^1)$ is either isomorphic to G or any $(2d-1, H-e)$ da-ecard has one of the following properties.

(α) At least two d-vertices are adjacent ; or

(β) at least one $(d+1)$-vertex, say w, has two d-neighbours such that a $(d+1)$-neighbour of w is adjacent to a d-vertex in $N(b_k^2)$ for some k.

But the da-ecard $(2d-1, G-z_i^1 x_i)$ satisfies none of the above two properties and hence it would not be a da-ecard of H.
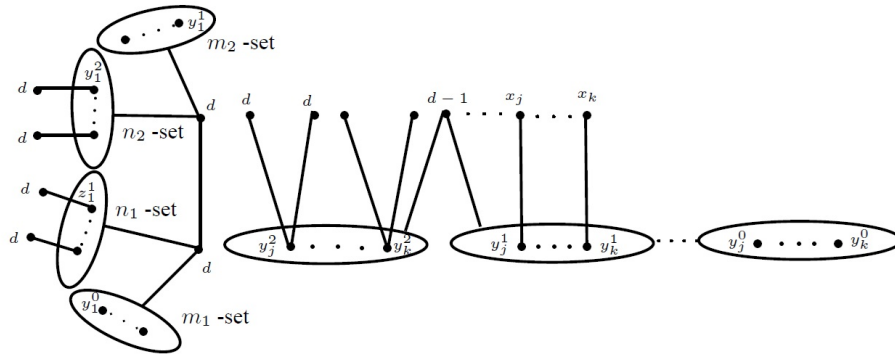

Fig. 5. Section of the da-ecard $(2d-1, G-y_r^2 x_q)$

Suppose that $y_k^1 \in N(x_i)$ for some i, k, where $1 \leq i \leq n_1$. If $\{z_i^1, y_k^2\} \subseteq N(x_i)$ for some i, k where $1 \leq i \leq n_1$, then we consider the da-ecards $(2d-1, G-x_i y_k^2)$ and $(2d-1, G-y_r^2 x_q)$. The da-ecard $G-y_r^2 x_q$ contains only one pair of adjacent d-vertices such that

(α) the neighbour of each of them can be partitioned into an m-set and an n-set; and

(β) every vertex in the m-set is adjacent to none of the other d-vertices and every vertex in the n-set has exactly one other d-neighbour (Figure 6).

Hence every extension $H(G-x_i y_k^2)$ is either isomorphic to G or any $(2d, H-e)$ da-ecard contains only one pair of adjacent d-vertices such that

(α) the neighbours of each of them can be partitioned into an m_1-set and an n_1-set, where $m_1 > m$ and $n_1 < n$; or

(β) every vertex in the m_1-set has no other d-neighbours and every vertex in the n_1-set has exactly one other d-neighbour.

Hence no da-ecard of H would be isomorphic to $(2d-1, G-y_r^2 x_q)$. Otherwise, we consider the two da-ecards $(2d, G-y_r^2 z_i^1)$ and $(2d-1, G-y_r^2 x_q)$. The da-ecard $G-y_r^2 x_q$ contains only one pair

of adjacent d-vertices such that

(α) the neighbour of exactly two (or one) of them can be partitioned into an m-set and an n-set;

(β) every vertex in the m-set has no other d-neighbours and every vertex in the n-set has exactly one other d-neighbour (say w); and

(γ) no neighbour of w has exactly one other d-neighbour.

Hence every extension $H(G - x_i y_k^2)$ is either isomorphic to G or any $(2d - 1, H - e)$ da-ecard contains only one pair of adjacent d-vertices such that

(α) the neighbour of exactly one (or none) of them can be partitioned into an m-set and an n-set;

(β) every vertex in the m-set has no other d-neighbours and every vertex in the n-set has exactly one other d-neighbour (say w_1); or

(γ) at least one neighbour of the vertex w_1 has exactly one other d-neighbour.

Hence no da-ecard of H would be isomorphic to $(2d - 1, G - y_r^2 x_q)$. Thus, in all the cases, we have shown that $dern(G) \leq 2$. $\qquad \square$

Theorem 2.10. *If every $(d + 1)$-vertex of a bidegreed graph G has at most two d-neighbours, then $dern(G) = 1$ or 2.*

Proof.   We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2 and 2.3 and hence $dern(G) \geq 2$. If the vertices y_r^2 and x_r have at most $d - 2$ common $(d + 1)$-neighbours, then $dern(G) \leq 2$ by Lemmas 2.7, 2.8 and 2.9 proved. So, we can assume that $\{y_r^2, x_r\}$ is a module of G.

Consider the two da-ecards $(2d - 1, G - y_r^2 x_q)$ and $(2d - 1, G - x_r w)$ (or $(2d - 1, G - x_p y_p^1)$ or $(2d, G - y_p^0 y_q^0)$, when $N(y_r^2) \setminus \{x_r, x_q, w\} = N(x_q) \setminus \{y_r^2, w\} = N(w) \setminus \{y_r^2, x_q, x_r\} \ \forall \ w$ hold), where w is a $(d + 1)$-vertex adjacent to x_r (Figure 6). In $(2d - 1, G - x_r w)$, there are no two adjacent d-vertices which form a module.
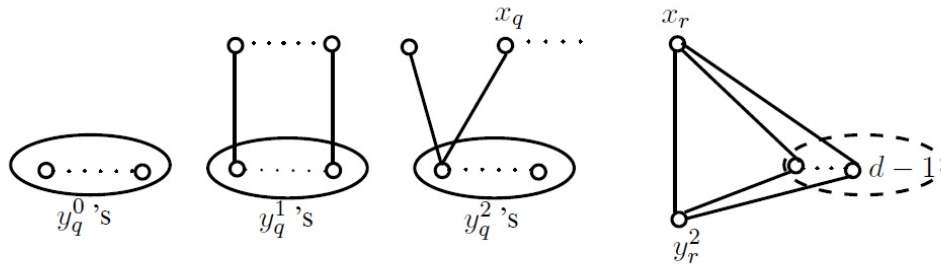

Fig. 6. Section of the da-ecard of $G - y_r^2 x_q$

Now consider the extensions of the da-ecard $(2d - 1, G - y_r^2 x_q)$. Note that in the da-ecard $(2d - 1, G - y_r^2 x_q)$, the only adjacent d-vertices are y_r^2, x_r and the only $(d - 1)$-vertex is x_q. So, to get an extension H of the da-ecard $(2d - 1, G - y_r^2 x_q)$, we have to join the $(d - 1)$-vertex x_q and any one vertex, say v among the d-vertices in $(2d - 1, G - y_r^2 x_q)$ by an edge. Since $\{y_r^2, x_r\}$ is a module of $G - y_r^2 x_q$, it follows that $H \cong G$ when v is y_r^2 or x_r. Thus we can assume that v is x_i for some $i \neq r, q$ so that it will become a $(d + 1)$-vertex in H. Now consider all possible da-ecards $(2d - 1, H - e)$ (or $(2d, H - e')$) of H. If e is $x_i y_j^2$ or $x_i y_j^1$, (or e' is $y_j^s y_k^t$, where $s, t = \{0, 1, 2\}$) then, in the da-ecard $(2d - 1, H - e)$(or $(2d, H - e')$), $\{y_r^2, x_r\}$ which form a module and so $H - e \not\cong G - g$ (or $H - e' \not\cong G - g'$). Therefore no da-ecard of H would be isomorphic to $(2d - 1, G - g)$ (or $(2d, G - g')$) and hence $dern(G) \leq 2$. $\quad \square$

Theorem 2.11. *If only one $(d+1)$-vertex of a bidegreed graph G has at least three d-neighbours, then $dern(G) = 1$ or 2.*

Proof. We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2 and 2.3 and hence $dern(G) \geq 2$. Let y_1^3 be adjacent to at least three d-vertices; let $x_1, x_2, \ldots, x_s$ be s d-neighbours of y_1^3, where $s \geq 3$.

Case 1. Each $(d+1)$-neighbour, say z other than y_1^3 of x_1 has no other d-neighbours.

Consider the da-ecards $(2d-1, G-f)$ and $(2d-1, G-g)$, where $f = x_1 y_1^3$ and $g = x_1 z$. In $(2d-1, G-g)$, each d-vertex is not adjacent to the $(d-1)$-vertex and all other d-vertices. In $(2d-1, G-f)$, the only $(d-1)$-vertex is x_1 and the above s (≥ 3) d-vertices together form $K_{1,s-1}$. So, to get an extension H of the da-ecard $(2d-1, G-f)$, we have to join the $(d-1)$-vertex x_1 and any one vertex, say v among the d-vertices in $(2d-1, G-f)$ by an edge. If $v = y_1^3$, then clearly, $H \cong G$. Otherwise, $v = x_j$ for some $j \neq 1$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d-1, H-e)$ of H. If e is $x_i y_1^3$ for some $i \neq 1, j$ then, in the da-ecard $(2d-1, H-e)$, at least one d-vertex has a $(d-1)$-neighbour and thus $H-e \not\cong G-g$. Otherwise, e is $x_i y_j$ (if any). Now in the da-ecard $(2d-1, H-e)$, k $(< s)$ d-vertices together form $K_{1,k-1}$ or at least one d-vertex has $(d-1)$-neighbour. Therefore $H-e \not\cong G-g$. Thus no da-ecard of H would be isomorphic to $(2d-1, G-g)$ and hence $dern(G) \leq 2$.

Case 2. There is a $(d+1)$-neighbour, say z of x_1 other than y_1^3, having a d-neighbour other than x_1.

Case 2.1. $s \geq 4$.

For this case, we use the da-ecards $(2d-1, G-f)$ and $(2d-1, G-g)$, where $f = x_1 y_1^3$ and $g = x_1 z$. In $(2d-1, G-g)$, each d-vertex is nonadjacent to the $(d-1)$-vertex and to only one pair of adjacent d-vertices. In $(2d-1, G-f)$, the only $(d-1)$-vertex is x_1 and give the $s+2$ $(s \geq 4)$ d-vertices which together form $K_{1,s-1} \cup K_2$. So, to get an extension H of the da-ecard $(2d-1, G-f)$, we have to join the $(d-1)$-vertex x_1 and any one vertex, say v among the d-vertices in $(2d-1, G-f)$ by an edge. If $v = y_1^3$, then clearly, $H \cong G$. Otherwise, v is equal to x_i for some $i \neq 1$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d-1, H-e)$ of H. If e is $x_i y_1^3$ for some $i \neq 1$, then, in the da-ecard $(2d-1, H-e)$, at least one d-vertex has a $(d-1)$-neighbour and so $H-e \not\cong G-g$. Otherwise, e is $x_i y_j$ (if any). Now in the da-ecard $(2d-1, H-e)$, $k+2$ $(< s+2$ $(s \geq 4))$ d-vertices together form $K_{1,k-1} \cup K_2$ or at least one d-vertex has a $(d-1)$-neighbour. Hence $H-e \not\cong G-g$. Thus no da-ecard of H would be isomorphic to $(2d-1, G-g)$ and hence $dern(G) \leq 2$.

Case 2.2. $s = 3$.

Case 2.2.1. Vertex y_1^3 is adjacent to a neighbour of x_i for some i, $1 \leq i \leq 3$.

Now if $|N(x_1) \cap N(x_2)| = d$, then we consider the two da-ecards $(2d-1, G-f)$ and $(2d, G-g)$, where $f = y_1^3 x_3$ and $g = y_1^3 z$. In $G-g$, exactly one d-vertex (say w_1) has three d-neighbours of which exactly two neighbours (say a_1 and a_2) have a common d-neighbour (other than w_1) and $|N(w_1) \cap N(a_1) \cap N(a_2)| = m$ (say). Hence every extension $H(G-f)$ is either isomorphic to G or in any $(2d, H-e)$ da-ecard,

(i) each d-vertex has at most two d-neighbours; or

(ii) a d-vertex (say w) has three d-neighbours of which exactly two neighbours (say b_1 and b_2) have a common d-neighbour such that $|N(w) \cap N(b_1) \cap N(b_2)| \geq m+1$.

Hence no da-ecard of H is isomorphic to $G-g$. Otherwise, that is, $|N(x_1) \cap N(x_2)| \leq d-1$. Now if a $(d+1)$-vertex z is nonadjacent to x_i $(i = 2, 3)$ and it is adjacent to y_1^3, then choose the two da-ecards $(2d, G-f)$ and $(2d-1, G-g)$, where $f = y_1^3 z$ and $g = x_1 z$. Clearly every extension $H(G-f)$ is either

isomorphic to G or in any da-ecard $(2d - 1, H - e)$, the neighbour of the unique $(d-1)$-vertex has three d-neighbours of which at least two are adjacent to a d-vertex. Hence $H - e$ is not isomorphic to $G - g$ and thus $dern(G) \leq 2$. Otherwise, choose the two da-ecards $(2d-1, G - f)$ and $(2d, G - g)$ where $f = y_1^3 x_3$ and $g = y_1^3 z$. In $G - g$, exactly one d-vertex (say w_1) is adjacent to three d-neighbours of which exactly two (say a_1 and a_2) have a common d-neighbour, $|N(a_1) \cap N(a_2)| = m_1$ (say) and $|N(w_1) \cap N(a_1) \cap N(a_2)| = m_2$ (say), where $m_1 > m_2$. Clearly, every extension $H(G - f)$ is either isomorphic to G or in any da-ecard $(2d, H - e)$, each d-vertex has at most two d-neighbours or the unique d-vertex (say w) has three d-neighbours of which exactly two (say b_1 and b_2) have a common d-neighbour such that $|N(w) \cap N(b_1) \cap N(b_2)| \geq m_2 + 1$ or $|N(b_1) \cap N(b_2)| \leq m_1 - 1$. Hence no da-ecard of H is isomorphic to $G - g$ and thus $dern(G) \leq 2$.

Case 2.2.2. Vertex y_1^3 is not adjacent to any neighbour of x_i for $1 \leq i \leq 3$.

Let the neighbours of y_1^3 be $w_1, w_2, ..., w_{d-2}$. Then choose two da-ecards as below.

Choose $A_1 = (2d, G - f)$ and $B_1 = (2d - 1, G - g)$, where $f = y_1^3 w_1$ and $g = x_1 z$, when w_1 has two d-neighbours; choose $A_2 = (2d - 1, G - f)$ and $B_2 = (2d - 1, G - g)$ when w_1 has exactly one d-neighbour (say a_1), where $f = x_1 y_1^3$ and $g = w_1 a_1$; choose A_2 and $B_3 = (2d, G - g)$ when w_1 is adjacent to a vertex w_2, where $g = w_1 w_2$; choose A_2 and $B_4 = (2d, G - g)$ when no two w_i's are adjacent and a neighbour (say w) of w_1 has no d-neighbours, where $g = w w_1$; choose A_2 and $B_5 = (2d - 1, G - g)$ when no two w_i's are adjacent and a neighbour (say w) of w_1 has exactly one d-neighbour (say a_2), where $g = w a_2$; or choose finally A_2 and $B_6 = (2d, G - g)$, where $g = y_1^3 w_1$, when no two w_i's are adjacent and a neighbour of w_1 is adjacent to two d-vertices.

Among the d-vertices in B_1, exactly one pair of vertices is adjacent. Hence every extension $H(A_1)$ is either isomorphic to G or in any da-ecard $(2d - 1, H(A_1) - e)$, at least two pairs of d-vertices are adjacent or at least one d-vertex has at least two d-neighbours. In B_i ($i \geq 2$), all the d-vertices are mutually nonadjacent. Hence every extension $H(A_2)$ is either isomorphic to G or in any da-ecard $(k, H(A_2) - e)$, where $k = 2d$ or $2d - 1$, at least two d-vertices are adjacent and hence no da-ecard of $H(A_i)$ ($i = 1,\ 2$) is isomorphic to B_i. Thus, in all the cases, we have shown that G has at most one da-ecard in common with H, where $H \not\cong G$ and hence $dern(G) \leq 2$. $\square$

Theorem 2.12. *If at least two $(d+1)$-vertices of a bidegreed graph G has at least three d-neighbours, then $dern(G) = 1$ or 2.*

Proof. We assume that G does not satisfy the hypotheses of Theorems 2.1, 2.2 and 2.3 and hence $dern(G) \geq 2$ and at least two $(d + 1)$-vertices in G are adjacent. Since G contains at least two y_i^3s, we proceed by two cases depending on their adjacencies.

Case 1. At least two $y_i^{m_1}$'s are adjacent, where m_1 (≥ 3) is minimum.

Let A be the collection of all y_i^3's in G. Then $|A| \geq 2$. Let B be the set of all vertices in A that are adjacent to the minimum number of (or m_1) d-vertices. Let C be the set of all vertices in B that are adjacent to a vertex in B. Then the set C is non-empty by Case 1. Choose a vertex, say $y_p^{m_1}$ in C that is adjacent to the minimum number (say m_2) of $(d + 1)$-vertices in B.

Consider the two da-ecards $(2d - 1, G - x_p y_p^{m_1})$ and $(2d, G - y_p^{m_1} y_q^{m_1})$, where $\{x_p, y_q^{m_1}\} \in N(y_p^{m_1})$. In $G - y_p^{m_1} y_q^{m_1}$, one of the two nonadjacent d-vertices is adjacent to m_1 (respectively $m_2 - 1$) vertices of degree d (respectively $d + 1$) and each of these $(d + 1)$-vertices has at least $m_1 + 1$ d-neighbours. In $(2d - 1, G - x_p y_p^{m_1})$, the only $(d - 1)$-vertex is x_p and the m_1 (≥ 3) d-vertices together form $K_{1, m_1 - 1}$ such that $y_p^{m_1}$ has m_2 $(d + 1)$-neighbours in B. So, to get an extension H of the da-ecard

$(2d - 1, G - x_p y_p^{m_1})$, we have to join the $(d-1)$-vertex x_p and any one vertex, say v among the d-vertices in $(2d - 1, G - x_p y_p^{m_1})$ by an edge. If $v = y_p^{m_1}$, then clearly, $H \cong G$. Otherwise, v is equal to x_i for some $i \neq p$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d, H - e)$ of H. If e is $w_i w_j$, where w_i or $w_j \in N(y_p^{m_1})$ then, in the da-ecard $(2d, H - e)$, one of the two nonadjacent d-vertices is adjacent to at least m_1 (respectively at most $m_2 - 1$) vertices of degree d (respectively $d + 1$) and each of these $(d + 1)$-vertex has at least $m_1 + 1$ d-neighbours and thus $H - e \not\cong G - y_p^{m_1} y_q^{m_1}$. Otherwise, e is $w_i w_j$, where $w_i,\ w_j \notin N(y_p^{m_1})$. Now, in the da-ecard $(2d - 1, H - e)$, one of the two nonadjacent d-vertices is adjacent to $m_1 - 1$ (respectively m_2) vertices of degree d (respectively $d + 1$) and each of these $(d + 1)$-vertices has at least $m_1 + 1$ d-neighbours and thus $H - e \not\cong G - y_p^{m_1} y_q^{m_1}$. Thus no da-ecard of H would be isomorphic to $(2d, G - y_p^{m_1} y_q^{m_1})$ and hence $dern(G) \leq 2$.

Case 2. Any two $y_i^{m_1}$'s are nonadjacent, where $m_1\ (\geq 3)$ is minimum.

Let $n_0, n_1, n_2, \cdots, n_s$ be the number of $(d+1)$-vertices adjacent to exactly $0, 1, 2, \cdots, s$ vertices of degree d, respectively. Let y_p^i and w be adjacent, respectively, to exactly i and j vertices of degree d.

Case 2.1. $|i - j| \geq 2,\ 0 \leq j < i \leq s,\ i \geq 3$.

Case 2.1.1. $j \geq 1$.

Consider the two da-ecards $(2d - 1, G - x_p y_p^i)$ and $(2d - 1, G - x_r w)$, where $x_p \in N(y_p^i)$ and $x_r \in N(w)$. In $G - x_r w$, there is a unique d-vertex having exactly $j - 1$ d-neighbours and each d-vertex is not adjacent to the $(d - 1)$-vertex. In $(2d - 1, G - x_p y_p^i)$, the only $(d - 1)$-vertex is x_p and the above $i\ (\geq 3)$ d-vertices together form $K_{1,i-1}$. So, to get an extension H of the da-ecard $(2d-1, G-x_p y_p^i)$, we have to join the $(d-1)$-vertex x_p and any one vertex, say v among the d-vertices in $(2d - 1, G - x_p y_p^i)$ by an edge. If $v = y_p^i$, then clearly, $H \cong G$. Otherwise, v is equal to x_k for some $k \neq p$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d-1, H - e)$ of H. If e is $x_k z$, then the da-ecard $(2d - 1, H - e)$ has a unique d-vertex adjacent to at least j d-vertices or at least one d-vertex is adjacent to the $(d - 1)$-vertex. Hence $H - e \not\cong G - x_r w$ and no da-ecard of H would be isomorphic to $(2d - 1, G - x_r w)$.

Case 2.1.2. $j = 0$.

If at least two y_k^0's are adjacent, then choose the two da-ecards $(2d-1, G-x_p y_p^i)$ and $(2d, G-y_p^0 y_q^0)$, where $x_p \in N(y_p^i)$ and $y_q^0 \in N(y_p^0)$. In $G - y_p^0 y_q^0$, all the d-vertices are mutually nonadjacent. In $(2d - 1, G - x_p y_p^i)$, $i\ (\geq 3)$ d-vertices together form $K_{1,i-1}$. So, to get an extension H of the da-ecard $(2d-1, G-x_p y_p^i)$, we have to join the $(d-1)$-vertex x_p and any one vertex, say v among the d-vertices in $(2d - 1, G - x_p y_p^i)$ by an edge. If $v = y_p^i$, then clearly, $H \cong G$. Otherwise, v is equal to x_k for some $k \neq p$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d, H - e)$ of H. If e is $w_i w_j$, then, in the da-ecard $(2d, H - e)$, $i\ (\geq 2)$ d-vertices together form $K_{1,i-1}$ and thus $H - e \not\cong G - y_p^0 y_q^0$. Thus no da-ecard of H would be isomorphic to $(2d, G - y_p^0 y_q^0)$. Otherwise, choose the two da-ecards $(2d - 1, G - x_p y_p^i)$ and $(2d, G - y_p^0 y_p^i)$, where $\{x_p, y_p^0\} \subseteq N(y_p^i)$. In $G - y_p^0 y_p^i$, the unique d-vertex has i d-neighbours. In $(2d-1, G-x_p y_p^i)$, $i\ (\geq 3)$ d-vertices together form $K_{1,i-1}$. So, to get an extension H of the da-ecard $(2d - 1, G - x_p y_p^i)$, we have to join the $(d - 1)$-vertex x_p and any one vertex, say v among the d-vertices in $(2d - 1, G - x_p y_p^i)$ by an edge. If $v = y_p^i$, then clearly, $H \cong G$. Otherwise, v is equal to x_k for some $k \neq p$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d, H - e)$ of H. If e is $w_i w_j$, then, in the da-ecard $(2d, H - e)$, the unique d-vertex has at most $i - 1$ d-neighbours or at least two d-vertices adjacent to other d-vertices and thus $H - e \not\cong G - y_p^0 y_p^i$. Thus no da-ecard of H would be isomorphic to $(2d, G - y_p^0 y_p^i)$ and hence $dern(G) \leq 2$.

Case 2.2. $|i - j| \leq 1$.

The proof follows by Case 1 when $i = j$. For $i \neq j$, consider the da-ecards $(2d - 1, G - x_p y_p^i)$ and $(2d, G - y_p^i y_p^{i-1})$, where $\{x_p, y_p^{i-1}\} \subseteq N(y_p^i)$. In $G - y_p^i y_p^{i-1}$, exactly two nonadjacent d-vertices such that one of them adjacent to i d-vertices and the other adjacent to $i - 1$ d-vertices. In $(2d - 1, G - x_p y_p^i)$, the only $(d - 1)$-vertex is x_p and the above i (≥ 3) d-vertices together form $K_{1,i-1}$. So, to get an extension H of the da-ecard $(2d - 1, G - x_p y_p^i)$, we have to join the $(d - 1)$-vertex x_p and any one vertex, say v among the d-vertices in $(2d - 1, G - x_p y_p^i)$ by an edge. If $v = y_p^i$, then clearly, $H \cong G$. Otherwise, v is equal to x_k for some $k \neq p$ so that it will become a $(d+1)$-vertex in H. Now consider all possible da-ecards $(2d, H - e)$ of H. If e is $w_i w_j$, then, in the da-ecard $(2d, H - e)$, at least three nonadjacent d-vertices adjacent to other d-vertices and thus $H - e \not\cong G - y_p^i y_p^{i-1}$. Thus no da-ecard of H would be isomorphic to $(2d, G - y_p^i y_p^{i-1})$ and hence $dern(G) \leq 2$. $\qquad\square$

Myrvold et al. [18] proved that bidegreed graphs are edge reconstructible. But the edge reconstruction number of bidegreed graphs, except for a few classes, is not known. This paper together with [1] have completely determined the degree associated edge reconstruction number of all bidegreed graphs and it is at most four. The degree associated edge reconstruction parameter might be a strong tool for providing evidence to support or reject the Edge Reconstruction Conjecture that remains open.

Acknowledgments

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