ARS COMBINATORIA 163 (2025) 51–67

Ars Combinatoria
www.combinatorialpress.com/ars

Combinatorial Press

Bounds on inverse sum indeg index of graph operations

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ABSTRACT

Let $G = (V(G), E(G))$ be a simple connected graph. The inverse sum indeg index of G, denoted by ISI(G), is defined as the sum of the weights $\frac{d(u)d(v)}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex in G. In this paper, we first present some lower and upper bound for ISI index in terms of graph parameters such as maximum degree, minimum degree and clique number. Moreover, we compute ISI index of several graph operations like join, cartesian product, composition, corona and strong product of graphs.

Keywords: ISI index, diaz-Metcalf inequality, graph operation

2010 Mathematics Subject Classification: 05C07, 05C15, 05C90.

1. Introduction

All graphs considered in this paper are undirected, simple, connected graphs. Let $G = (V(G), E(G))$ be a graph of order $n = |V(G)|$ and size $m = |E(G)|$. The degree of a vertex u in G, which is written as $d_G(u)$ (or simply d_u), is the number of edges incident to u and the set of neighborhoods of u, denoted by $N_G(u)$, is the set of vertices adjacent to u. The maximum and minimum vertex degree in G are denoted by Δ and δ, respectively.

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Received 15 May 2024; Accepted 13 April 2025; Published Online 28 June 2025.

DOI: 10.61091/ars163-04

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The inverse sum indeg index (ISI index, for short) of graph G is defined as

$$ISI(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}.$$

This significant topological index can predict well the total surface area of octane isomers [36], and has attracted more and more attentions from chemists and mathematicians. The extremal ISI index has been determined by Sedlar et al. [34] for connected graphs, chemical trees, chemical graphs, graphs with given maximum degree, minimum degree, or number of pendent vertices, and trees with k leaves. An and Xiong [3] later obtained the extremal ISI index among graphs with prescribed matching number, vertex connectivity, or independence number. Similar results were also derived by Chen and Deng [5] for graphs with prescribed connectivity, chromatic number, clique number, independence number, covering number, or vertex bipartiteness. Falahati-Nezhad et al. [8] established sharp bounds of ISI index using graph invariants, including the order, radius, size, and number of pendent vertices. Gutman et al. [13] presented several inequalities for ISI index and characterized the extremal graphs. Divya and Manimaran [6] derived Neighborhood inverse sum index for two-dimensional lattice structure of certain Nanotube and Nanotorus. Several lower bounds were obtained in [12]. For more results concerning ISI index, we refer to [1, 17, 30, 32, 10, 37].

It is commonly known that a variety of graph operations can transform simpler graphs into many graphs of general, and particularly chemical, interest. It is crucial to compute invariants of graph operations in order to comprehend the relationship between specific invariants of such composite graphs and the corresponding invariants of their constituent parts.

The topic of calculating topological indices of product graphs was initially examined by Graovac and Pisanski [11], who also provided an exact formula for the Wiener index of the Cartesian product of graphs. The Szeged index of Cartesian product graphs was calculated by Klavžar et al. [24], and the PI index of the Cartesian product of graphs is calculated in [23]. Maji and Ghorai [26] investigated the third leap Zagreb index under a number of graph operations, including the disjunction of two graphs, lexicographic product, strong product, tensor product, Cartesian product, Corona product, and neighborhood Corona product. For the kth generalized transformation graphs, Maji and Ghorai [27] obtained accurate formulas of the F-index and its co-index. The first and second Zagreb index of graph operations, which include the Cartesian product, composition, join, disjunction, and symmetric difference of graphs, were given explicit formulations in [21]. Some inequalities for the atom-bond connectivity index of graph operations were presented by Fath-Tabar et al. [9]. Exact formulae for the values of the eccentric distance sum for the Cartesian product were obtained by Ilić et al. [16] and applied to certain graphs of chemical relevance, such as nanotubes and nanotori. A number of invariants were then calculated by the number of authors for different types of graph operations, such as the symmetric difference, join, disjunction, composition, and Cartesian product of two graphs. For further information, see [2, 4, 7, 14, 21, 18, 19, 20, 22, 25, 28, 29, 38, 39, 40]. Continuing this development, we now examine the ISI index of a few graph operations.

A few graph operations that will be utilized in the study are reviewed.

The graph union $G_1 \cup G_2$, combined with all the edges joining V_1 and V_2, is the join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2.

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and $(a, x)(b, y)$ is an edge of $G_1 \square G_2$ if $a = b$ and $xy \in E(G_2)$, or $ab \in E(G_1)$ and $x = y$.

The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V(G_1[G_2]) = V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2).

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined to be the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. If G_1 is a (p_1, q_1) graph and G_2 is a (p_2, q_2) graph, then it follows from the definition of the corona that $G_1 \circ G_2$ has $p_1(1 + p_2)$ vertices and $q_1 + p_1 q_2 + p_1 p_2$ edges. It is clear that if G_1 is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$.

The strong product $G_1 \nabla G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ and two vertices $(x_1, x_2), (y_1, y_2)$ of $G_1 \nabla G_2$ are adjacent if either (i) $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$, or (ii) $x_1 = y_1$ and $x_2 y_2 \in E(G)$, or (iii) $x_1 y_1 \in E(G_1)$ and $x_2 y_2 \in E(G_2)$. It can be easily checked that $G_1 \nabla G_2$ is a connected graph if and only if both G_1 and G_2 are connected. It is also clear that $G_1 \nabla G_2$ is a complete graph if and only if both factors are complete.

In this paper, we explore further mathematical properties of inverse sum indeg index of graphs. The content is organized as follows. In Section 2, we present a lower bound for ISI-index by means of some established inequalities. In Section 3, we give sharp bounds for the ISI-index of some graph operations.

2. Lower bound on ISI- index

In this section, we present a lower bound on ISI- index by means of established inequality. Our starting point is the following Diaz-Metcalf inequality which was introduced in [31].

Lemma 2.1. [31] *Let a_i and b_i, $i = 1, 2, \cdots, n$, be real numbers such that $Aa_i \le b_i \le Ba_i$ for each $i = 1, 2, \cdots, n$. Then*

$$(A + B) \sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} b_i^2 + AB \sum_{i=1}^{n} a_i^2,$$

with equality if and only if either $b_i = Aa_i$ or $b_i = Ba_i$ for each $i = 1, 2, \cdots, n$.

The general Randić index of a graph G is defined as

$$R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha,$$

$R_{-1/2}$ is well known Randić index [33].

The general sum connectivity index of a graph G is defined as

$$\chi_\alpha(G) = \sum_{uv\in E(G)} (d_u + d_v)^\alpha,$$

where d_u denotes the degree of a vertex u in G, and α is a real number.

Recently, Hua, Das and Wang [15] presented a sharp lower bound, in terms of some graph invariants, for ABC-index of a nontrivial graph using Diaz-Metcalf inequality. Motivated from [15], we first give a sharp lower bound for ISI-index in terms of some graph invariants using Diaz-Metcalf inequality.

Theorem 2.2. *Let G be a connected graph with m edges, maximum degree Δ, and minimum degree δ. Then*

$$ISI(G) \geq \frac{2\delta^3\Delta^3}{\delta^3+\Delta^3}\left[\frac{1}{4\delta^3\Delta^3}R_2(G) + \chi_{-2}(G)\right], \tag{1}$$

with equality if and only if G is a regular graph.

Proof. Note that the ISI-index can be rewritten as

$$ISI(G) = \sum_{uv\in E(G)} \frac{1}{d_u+d_v} \times d_u d_v.$$

For each edge $uv \in E(G)$, the following holds:

$$\frac{d_u d_v}{\Delta^2(2\Delta)} \leq \frac{1}{d_u+d_v} \leq \frac{d_u d_v}{\delta^2(2\delta)},$$

where the left-hand side equality is attained if and only if $d_u = d_v = \Delta$ for $uv \in E(G)$, and the right-hand side equality is attained if and only if $d_u = d_v = \delta$ for $uv \in E(G)$.

Setting $A = \frac{1}{2\Delta^3}$ and $B = \frac{1}{2\delta^3}$ in Lemma 2.1, we have

$$\begin{aligned} ISI(G) &= \sum_{uv\in E(G)} \frac{1}{d_u+d_v} \times d_u d_v \\ &= \frac{4\delta^3\Delta^3}{2\delta^3+2\Delta^3}\left[\frac{1}{2\Delta^3}+\frac{1}{2\delta^3}\right] \times \sum_{uv\in E(G)} \frac{1}{d_u+d_v} \times d_u d_v \\ &\geq \frac{2\delta^3\Delta^3}{\delta^3+\Delta^3} \times \left[\frac{1}{4\delta^3\Delta^3}\sum_{uv\in E(G)} (d_u d_v)^2 + \sum_{uv\in E(G)}\left(\frac{1}{d_u+d_v}\right)^2\right] \\ &= \frac{2\delta^3\Delta^3}{\delta^3+\Delta^3}\left[\frac{1}{4\delta^3\Delta^3}R_2(G) + \chi_{-2}(G)\right], \end{aligned} \tag{2}$$

which gives the required result in (1). The first part of the proof is done.

Now, the equality holds in (1) if and only if the equality holds in (2), if and only if either $\frac{1}{d_u+d_v} = \frac{1}{2\Delta^3} \times d_u d_v$ or $\frac{1}{d_u+d_v} = \frac{1}{2\delta^3} \times d_u d_v$ for each edge $uv \in E(G)$ by Lemma 2.1, if and only if either $d_u = d_v = \Delta$ or $d_u = d_v = \delta$ for each edge $uv \in E(G)$, i.e., G is regular as G is connected. $\square$

For any nontrivial connected graph G, it is obvious that $R_2(G) = \sum_{uv\in E(G)} (d_u d_v)^2 \geq m\delta^4$ and $\chi_{-2}(G) = \sum_{uv\in E(G)} \frac{1}{(d_u+d_v)^2} \geq \frac{m}{4\Delta^2}$ with either equality if and only if G is regular. The following consequence can be derived immediately from Theorem 2.2.

Corollary 2.3. *Let G be a connected graph with n vertices, m edges, maximum degree Δ, and minimum degree δ. Then*

$$ISI(G) \geq \frac{m\delta^3}{2(\delta^3+\Delta^3)}(\delta+\Delta),$$

with equality holding if and only if G is a regular graph.

3. Upper bound on ISI- index

In [15], the authors presented an upper bound for ABC-index in terms of clique number. Motivated from [15], in this section, we present a upper bound for ISI-index in terms of clique number. First, we recall Turán Theorem, which is stated as follows.

Lemma 3.1. [35] *Let G be a connected K_{q+1}-free graph of order n and size m. Then $m \leq \left(1-\frac{1}{q}\right)\frac{n^2}{2}$, with equality if and only if G is a complete q-partite graph in which all classes are of equal cardinality.*

We now present an upper bound on ISI-index of graphs.

Theorem 3.2. *Let G be a connected graph with n vertices, m edges and clique number ω. Then*

$$ISI(G) \leq \frac{\Delta^2}{2}\left[1+\frac{1}{\delta}\left(\frac{n^2(\omega-1)}{2\omega}-\Delta\right)\right], \tag{3}$$

where Δ and $\delta \geq 2$ are the maximum degree and the minimum degree of graph G, respectively. Moreover, equality is achieved if and only if G is a complete ω-partite graph in which all classes are of equal cardinality.

Proof. Let u be the maximum degree vertex in G, i.e., $d_u = \Delta$. Then

$$\sum_{v\in N_G(u)} \frac{d_u d_v}{d_u+d_v} = \sum_{v\in N_G(u)} \frac{1}{\frac{1}{d_u}+\frac{1}{d_v}} = \sum_{v\in N_G(u)} \frac{1}{\frac{1}{\Delta}+\frac{1}{d_v}} \leq \Delta\left(\frac{1}{\frac{1}{\Delta}+\frac{1}{\Delta}}\right) = \frac{\Delta^2}{2}, \tag{4}$$

with equality holding if and only if $d_v = \delta$ for $v \in N_G(u)$.

Using the arithmetic-geometric mean inequality, we obtain

$$\frac{d_v d_w}{d_v+d_w} \leq \frac{\Delta^2}{2\sqrt{d_v d_w}} \leq \frac{\Delta^2}{2\delta},$$

for any edge $v_v v_w \in E(G)$. It is easy to see that the equality in the above is attained if and only if $d_v = d_w = \delta = \Delta$.

Thus, we have

$$\sum_{\substack{vw \in E(G) \\ v,w \neq u}} \frac{d_v d_w}{d_v + d_w} \leq (m - \Delta)\frac{\Delta^2}{2\delta}, \tag{5}$$

with equality holding if and only if $d_v = d_w = \delta = \Delta$ for any edge $vw \in E(G)$.

Since G has clique number ω, G is $K_{\omega+1}$-free graph. By Lemma 3.1,

$$m \leq \frac{n^2(\omega - 1)}{2\omega}, \tag{6}$$

with equality if and only if G is a complete ω-partite graph in which all classes are of equal cardinality.

Using the above results in (4), (5) and (6), we obtain

$$\begin{aligned} ISI(G) &= \sum_{v \in N_G(u)} \frac{d_u d_v}{d_u + d_v} + \sum_{\substack{vw \in E(G) \\ v,w \neq u}} \frac{d_v d_w}{d_v + d_w} \\ &\leq \frac{\Delta^2}{2} + (m - \Delta)\frac{\Delta^2}{2\delta} \\ &\leq \frac{\Delta^2}{2} + \left(\frac{n^2(\omega - 1)}{2\omega} - \Delta\right)\frac{\Delta^2}{2\delta} \\ &= \frac{\Delta^2}{2}\left[1 + \frac{1}{\delta}\left(\frac{n^2(\omega - 1)}{2\omega} - \Delta\right)\right], \end{aligned}$$

with equality if and only if $\Delta = \delta$ and G is a complete ω-partite graph in which all classes are of equal cardinality, i.e., G is a complete ω-partite graph in which all classes are of equal cardinality.

This completes the proof of the theorem. □

4. ISI-index of graph operations

In this section, some bounds for ISI-index of the join, Cartesian product, composition, corona and strong product of graphs are presented. We also prove our bounds are tight.

Theorem 4.1. *Let G and H be arbitrary graphs. Then*

$$\begin{aligned} ISI(G + H) \leq & \frac{\Delta_G}{\delta_G + |H|} ISI(G) + \frac{|E(G)||H|[2\Delta_G + |H|]}{2\,(\delta_G + |H|)} \\ & + \frac{\Delta_H}{\delta_H + |G|} ISI(H) + \frac{|E(H)||G|[2\Delta_H + |G|]}{2\,(\delta_H + |G|)} \\ & + |G||H|\left[\frac{(\Delta_G + |H|)\,(\Delta_H + |G|)}{\delta_G + \delta_H + |G| + |H|}\right], \end{aligned}$$

with equality if and only if G and H are regular, where $|G|$ and $|E(G)|$ denote the number of vertices and edges in G respectively.

Proof. By the definition of ISI-index,

$$ISI\left(G+H\right)=\sum_{uv\in E(G+H)}\frac{d_u d_v}{d_u+d_v}$$
$$=\sum_{uv\in E(G)}\frac{d_u d_v}{d_u+d_v}+\sum_{uv\in E(H)}\frac{d_u d_v}{d_u+d_v}+\sum_{u\in V(G),v\in V(H)}\frac{d_u d_v}{d_u+d_v}.$$

It is clear that if $u\in V(G)$, then $d_u=d_G(u)+|H|$ and

$$\sum_{uv\in E(G)}\frac{d_u d_v}{d_u+d_v}=\sum_{uv\in E(G)}\frac{\left(d_G(u)+|H|\right)\left(d_G(v)+|H|\right)}{d_G(u)+d_G(v)+2\left|H\right|}.$$

On the other hand,

$$\frac{\left(d_G(u)+|H|\right)\left(d_G(v)+|H|\right)}{d_G(u)+d_G(v)+2\left|H\right|}$$
$$=\frac{d_G(u)d_G(v)+|H|\left[d_G(u)+d_G(v)\right]+|H|^2}{d_G(u)+d_G(v)+2\left|H\right|}$$
$$=\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)+2\left|H\right|}+\frac{|H|\left[d_G(u)+d_G(v)\right]}{d_G(u)+d_G(v)+2\left|H\right|}+\frac{|H|^2}{d_G(u)+d_G(v)+2\left|H\right|}$$
$$=\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}\times\frac{d_G(u)+d_G(v)}{d_G(u)+d_G(v)+2\left|H\right|}+\frac{|H|\left[d_G(u)+d_G(v)\right]}{d_G(u)+d_G(v)+2\left|H\right|}$$
$$+\frac{|H|^2}{d_G(u)+d_G(v)+2\left|H\right|}$$
$$\leq\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}\times\frac{2\Delta_G}{2\delta_G+2\left|H\right|}+\frac{|H|\left[2\Delta_G\right]}{2\delta_G+2\left|H\right|}+\frac{|H|^2}{2\delta_G+2\left|H\right|}$$
$$=\frac{\Delta_G}{\delta_G+|H|}\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}+|H|\right]+\frac{|H|^2}{2\delta_G+2\left|H\right|},\tag{7}$$

with equality if and only if $\Delta_G=d_G(u)=d_G(v)=\delta_G$.

In a similar way, we have

$$\frac{\left(d_H(u)+|G|\right)\left(d_H(v)+|G|\right)}{d_H(u)+d_H(v)+2\left|G\right|}\leq\frac{\Delta_H}{\delta_H+|G|}\left[\frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)}+|G|\right]+\frac{|G|^2}{2\delta_H+2\left|G\right|},$$

with equality if and only if $\Delta_H=d_H(u)=d_H(v)=\delta_H$.

For $u\in V(G)$ and $v\in V(H)$,

$$d_u+d_v=d_G(u)+|H|+d_H(v)+|G|\geq\delta_G+\delta_H+|G|+|H|\,,$$

and $d_u d_v=\left(d_G(u)+|H|\right)\left(d_H(v)+|G|\right)\leq\left(\Delta_G+|H|\right)\left(\Delta_H+|G|\right)$ and

$$\frac{d_u d_v}{d_u+d_v}\leq\frac{\left(\Delta_G+|H|\right)\left(\Delta_H+|G|\right)}{\delta_G+\delta_H+|G|+|H|},$$

with equality if and only if $\Delta_G=d_G(u)=\delta_G$ and $\Delta_H=d_H(v)=\delta_H$.

So,

$$\begin{aligned}
ISI\left(G+H\right)=&\sum_{uv\in E(G+H)}\frac{d_ud_v}{d_u+d_v}\\
\leq&\sum_{uv\in E(G)}\left[\frac{\Delta_G}{\delta_G+|H|}\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}+|H|\right)+\frac{|H|^2}{2\left(\delta_G+|H|\right)}\right]\\
&+\sum_{uv\in E(H)}\left[\frac{\Delta_H}{\delta_H+|G|}\left(\frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)}+|G|\right)+\frac{|G|^2}{2\left(\delta_H+|G|\right)}\right]\\
&+\sum_{u\in V(G),v\in V(H)}\left[\frac{\left(\Delta_G+|H|\right)\left(\Delta_H+|G|\right)}{\delta_G+\delta_H+|G|+|H|}\right]\\
=&\frac{\Delta_G}{\delta_G+|H|}\left(ISI(G)+|E(G)||H|\right)+\frac{|E(G)||H|^2}{2\left(\delta_G+|H|\right)}\\
&+\frac{\Delta_H}{\delta_H+|G|}\left(ISI(H)+|E(H)||G|\right)+\frac{|E(H)||G|^2}{2\left(\delta_H+|G|\right)}\\
&+|G|\,|H|\left[\frac{\left(\Delta_G+|H|\right)\left(\Delta_H+|G|\right)}{\delta_G+\delta_H+|G|+|H|}\right]\\
=&\frac{\Delta_G}{\delta_G+|H|}ISI(G)+\frac{|E(G)||H|\left[2\Delta_G+|H|\right]}{2\left(\delta_G+|H|\right)}\\
&+\frac{\Delta_H}{\delta_H+|G|}ISI(H)+\frac{|E(H)||G|\left[2\Delta_H+|G|\right]}{2\left(\delta_H+|G|\right)}\\
&+|G|\,|H|\left[\frac{\left(\Delta_G+|H|\right)\left(\Delta_H+|G|\right)}{\delta_G+\delta_H+|G|+|H|}\right],
\end{aligned}$$

with equality if and only if G and H are regular. □

Theorem 4.2. *Let G and H be arbitrary graphs. Then*

$$\begin{aligned}
ISI\left(G+H\right)\geq&\left|\frac{\delta_G}{\Delta_G+|H|}ISI(G)-\frac{|E(G)||H|\left[2\delta_G+|H|\right]}{2\left(\Delta_G+|H|\right)}\right|\\
&+\left|\frac{\delta_H}{\Delta_H+|G|}ISI(H)-\frac{|E(H)||G|\left[2\delta_H+|G|\right]}{2\left(\Delta_H+|G|\right)}\right|\\
&+|G|\,|H|\,\frac{\left(\delta_G+|H|\right)\left(\delta_H+|G|\right)}{\Delta_G+\Delta_H+|G|+|H|},
\end{aligned}$$

with equality if and only if G and H are empty graphs.

Proof. For $u,v\in V(G)$, we have

$$\begin{aligned}
\frac{\left(d_G(u)+|H|\right)\left(d_G(v)+|H|\right)}{d_G(u)+d_G(v)+2\,|H|}=&\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}\times\frac{d_G(u)+d_G(v)}{d_G(u)+d_G(v)+2\,|H|}\\
&+\frac{|H|\left[d_G(u)+d_G(v)\right]}{d_G(u)+d_G(v)+2\,|H|}+\frac{|H|^2}{d_G(u)+d_G(v)+2\,|H|}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} \times \frac{2\delta_G}{2\Delta_G + 2\,|H|} \\
&\quad + \frac{|H|\,2\delta_G}{2\Delta_G + 2\,|H|} + \frac{|H|^2}{2\Delta_G + 2\,|H|} \\
&= \frac{\delta_G}{\Delta_G + |H|} \times \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} + \frac{|H|\,[2\delta_G + |H|]}{2\,(\Delta_G + |H|)},
\end{aligned}
$$

with equality if and only if $\Delta_G = d_G(u) = d_G(v) = \delta_G$.

Clearly, $a + b \geq |a - b|$ with equality if and only if $a = 0$ or $b = 0$, where a and b are non-negative. Thus,

$$
\frac{(d_G(u) + |H|)\,(d_G(v) + |H|)}{d_G(u) + d_G(v) + 2\,|H|} \geq \left| \frac{\delta_G}{\Delta_G + |H|} \times \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} - \frac{|H|\,[2\delta_G + |H|]}{2\,(\Delta_G + |H|)} \right|,
$$

with equality if and only if $\delta_G = 0$.

For $u, v \in V(H)$, by a similar argument we can get

$$
\frac{(d_H(u) + |G|)\,(d_H(v) + |G|)}{d_H(u) + d_H(v) + 2\,|G|} \geq \left| \frac{\delta_H}{\Delta_H + |G|} \times \frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)} - \frac{|G|\,[2\delta_H + |G|]}{2\,(\Delta_H + |G|)} \right|,
$$

with equality if and only if $\Delta_H = d_H(u) = d_H(v) = \delta_H = 0$.

Let $u \in V(G)$ and $v \in V(H)$. Then

$$
d_u + d_v = d_G(u) + d_H(v) + |G| + |H| \leq \Delta_G + \Delta_H + |G| + |H|\,,
$$

and

$$
d_u d_v = (d_G(u) + |H|)\,(d_H(v) + |G|) \geq (\delta_G + |H|)\,(\delta_H + |G|)\,.
$$

Thus

$$
\frac{d_u d_v}{d_u + d_v} \geq \frac{(\delta_G + |H|)\,(\delta_H + |G|)}{\Delta_G + \Delta_H + |G| + |H|},
$$

with equality if and only if $\Delta_G = d_G(u) = \delta_G$ and $\Delta_H = d_H(v) = \delta_H$.

So,

$$
\begin{aligned}
ISI(G+H) &= \sum_{uv \in E(G+H)} \frac{d_u d_v}{d_u + d_v} \\
&\geq \sum_{uv \in E(G)} \left| \frac{\delta_G}{\Delta_G + |H|} \times \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} - \frac{|H|\,[2\delta_G + |H|]}{2\,(\Delta_G + |H|)} \right| \\
&\quad + \sum_{uv \in E(H)} \left| \frac{\delta_H}{\Delta_H + |G|} \times \frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)} - \frac{|G|\,[2\delta_H + |G|]}{2\,(\Delta_H + |G|)} \right| \\
&\quad + \sum_{u \in V(G), v \in V(H)} \frac{(\delta_G + |H|)\,(\delta_H + |G|)}{\Delta_G + \Delta_H + |G| + |H|} \\
&\geq \left| \sum_{uv \in E(G)} \left(\frac{\delta_G}{\Delta_G + |H|} \times \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} - \frac{|H|\,[2\delta_G + |H|]}{2\,(\Delta_G + |H|)} \right) \right|
\end{aligned}
$$

$$+\left|\sum_{uv\in E(H)}\left(\frac{\delta_H}{\Delta_H+|G|}\times\frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)}-\frac{|G|\left[2\delta_H+|G|\right]}{2\left(\Delta_H+|G|\right)}\right)\right|$$
$$+|G||H|\frac{\left(\delta_G+|H|\right)\left(\delta_H+|G|\right)}{\Delta_G+\Delta_H+|G|+|H|}$$
$$=\left|\frac{\delta_G}{\Delta_G+|H|}ISI(G)-\frac{|E(G)||H|\left[2\delta_G+|H|\right]}{2\left(\Delta_G+|H|\right)}\right|$$
$$+\left|\frac{\delta_H}{\Delta_H+|G|}ISI(H)-\frac{|E(H)||G|\left[2\delta_H+|G|\right]}{2\left(\Delta_H+|G|\right)}\right|$$
$$+|G||H|\frac{\left(\delta_G+|H|\right)\left(\delta_H+|G|\right)}{\Delta_G+\Delta_H+|G|+|H|}, \qquad (8)$$

with equality if and only if G and H are empty. □

Theorem 4.3. *Let G and H be arbitrary graphs. Then*

$$ISI(G\Box H)\leq\frac{\Delta_G}{\delta_G+\delta_H}|H|\,ISI(G)+\frac{\Delta_H}{\delta_G+\delta_H}|G|\,ISI(H)$$
$$+\left(|H|\,|E(G)|+|G|\,|E(H)|\right)\frac{\Delta_G\Delta_H}{\delta_G+\delta_H},$$

with equality if and only if G and H are regular.

Proof. Let $uv\in E(G\Box H)$, where $u=(a,b)$ and $v=(c,d)$ are vertices of $G\Box H$. Then $d_u=d_G(a)+d_H(b)$, $d_v=d_G(c)+d_H(d)$ and

$$\frac{d_ud_v}{d_u+d_v}=\frac{\left(d_G(a)+d_H(b)\right)\left(d_G(c)+d_H(d)\right)}{\left(d_G(a)+d_H(b)\right)+\left(d_G(c)+d_H(d)\right)}$$
$$=\frac{d_G(a)d_G(c)+d_G(a)d_H(d)+d_H(b)d_G(c)+d_H(b)d_H(d)}{d_G(a)+d_H(b)+d_G(c)+d_H(d)}$$
$$=\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\times\frac{d_G(a)+d_G(c)}{d_G(a)+d_H(b)+d_G(c)+d_H(d)}$$
$$+\frac{d_G(a)d_H(d)}{d_G(a)+d_H(b)+d_G(c)+d_H(d)}$$
$$+\frac{d_H(b)d_G(c)}{d_G(a)+d_H(b)+d_G(c)+d_H(d)}+\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}$$
$$\times\frac{d_H(b)+d_H(d)}{d_G(a)+d_H(b)+d_G(c)+d_H(d)}$$
$$\leq\left(\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\right)\left(\frac{2\Delta_G}{2\delta_G+2\delta_H}\right)+\frac{d_G(a)d_H(d)+d_H(b)d_G(c)}{d_G(a)+d_H(b)+d_G(c)+d_H(d)}$$
$$+\left(\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}\right)\left(\frac{2\Delta_H}{2\delta_G+2\delta_H}\right)$$
$$\leq\left(\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\right)\left(\frac{\Delta_G}{\delta_G+\delta_H}\right)+\left(\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}\right)\left(\frac{\Delta_H}{\delta_G+\delta_H}\right)$$
$$+\frac{\Delta_G\Delta_H+\Delta_H\Delta_G}{2\delta_G+2\delta_H}$$

$$= \left(\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\right)\left(\frac{\Delta_G}{\delta_G+\delta_H}\right)+\left(\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}\right)\left(\frac{\Delta_H}{\delta_G+\delta_H}\right)+\frac{\Delta_G\Delta_H}{\delta_G+\delta_H},$$

i.e.,

$$\frac{d_ud_v}{d_u+d_v} \le \left(\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\right)\left(\frac{\Delta_G}{\delta_G+\delta_H}\right)+\left(\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}\right)\left(\frac{\Delta_H}{\delta_G+\delta_H}\right)+\frac{\Delta_G\Delta_H}{\delta_G+\delta_H},$$

with equality if and only if $\Delta_G = d_G(a) = d_G(c) = \delta_G$ and $\Delta_H = d_H(b) = d_H(d) = \delta_H$.

So,

$$\begin{aligned}
ISI(G\Box H) = \sum_{\substack{uv\in E(G\Box H)\\ u=(a,b),v=(c,d)}} \frac{d_ud_v}{d_u+d_v} &\le \frac{\Delta_G}{\delta_G+\delta_H}\sum_{\substack{uv\in E(G\Box H)\\ u=(a,b),v=(c,d)}}\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\\
&+\frac{\Delta_H}{\delta_G+\delta_H}\sum_{\substack{uv\in E(G\Box H)\\ u=(a,b),v=(c,d)}}\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}+\sum_{\substack{uv\in E(G\Box H)\\ u=(a,b),v=(c,d)}}\frac{\Delta_G\Delta_H}{\delta_G+\delta_H}\\
&=\frac{\Delta_G}{\delta_G+\delta_H}|H|\sum_{ac\in E(G)}\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}+\frac{\Delta_H}{\delta_G+\delta_H}|G|\sum_{bd\in E(H)}\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}\\
&+(|H|\,|E(G)|+|G|\,|E(H)|)\frac{\Delta_G\Delta_H}{\delta_G+\delta_H}\\
&=\frac{\Delta_G}{\delta_G+\delta_H}|H|\,ISI(G)+\frac{\Delta_H}{\delta_G+\delta_H}|G|\,ISI(H)\\
&+(|H|\,|E(G)|+|G|\,|E(H)|)\frac{\Delta_G\Delta_H}{\delta_G+\delta_H},
\end{aligned}$$

with equality if and only if G and H are regular. □

Theorem 4.4. *Let G and H are arbitrary graphs. Then*

$$\begin{aligned}
ISI(G[H]) \le &\frac{1}{|H|\,\delta_G+\delta_H}\Bigg[\left(|H|^2\,\Delta_G\right)\left(ISI(G)+|G|\,|E(H)|\,\frac{\Delta_G^2}{2\delta_G}\right)\\
&+\Delta_H\left(|E(G)|\,\frac{\Delta_H^2}{2\delta_H}+|G|\,ISI(H)\right)+|E\left(G\left[H\right]\right)|\,|H|\,\Delta_G\Delta_H\Bigg],
\end{aligned}$$

with equality if and only if G and H are regular.

Proof. Let $uv \in E(G[H])$, where $u = (a, b)$ and $v = (c, d)$ are vertices of $G[H]$. Then $d_u = d_{G[H]}(u) = |H|d_G(a) + d_H(b)$, $d_v = d_{G[H]}(v) = |H|d_G(c) + d_H(d)$ and

$$\begin{aligned}
\frac{d_ud_v}{d_u+d_v} &= \frac{(|H|\,d_G(a)+d_H(b))\,(|H|\,d_G(c)+d_H(d))}{|H|\,d_G(a)+d_H(b)+|H|\,d_G(c)+d_H(d)}\\
&=\frac{(|H|^2\,d_G(a)d_G(c)+|H|\,d_G(a)d_H(d)+|H|\,d_H(b)d_G(c)+d_H(b)d_H(d)}{|H|\,d_G(a)+d_H(b)+|H|\,d_G(c)+d_H(d)}\\
&\le |H|^2\left(\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\right)\left(\frac{d_G(a)+d_G(c)}{|H|\,d_G(a)+d_H(b)+|H|\,d_G(c)+d_H(d)}\right)
\end{aligned}$$

$$+\frac{|H|\Delta_G\Delta_H+|H|\Delta_H\Delta_G}{|H|\delta_G+\delta_H+|H|\delta_G+\delta_H}+\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}$$
$$\times\frac{d_H(b)+d_H(d)}{|H|d_G(a)+d_H(b)+|H|d_G(c)+d_H(d)}$$
$$\leq |H|^2\left(\frac{2\Delta_G}{|H|\delta_G+\delta_H+|H|\delta_G+\delta_H}\right)\left(\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}\right)+\frac{2|H|\Delta_G\Delta_H}{2(|H|\delta_G+\delta_H)}$$
$$+\left(\frac{2\Delta_H}{|H|\delta_G+\delta_H+|H|\delta_G+\delta_H}\right)\left(\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}\right),$$

with equality if and only if $\Delta_G=d_G(a)=d_G(c)=\delta_G$ and $\Delta_H=d_H(b)=d_H(d)=\delta_H$.

Therefore,

$$ISI(G[H])=\sum_{uv\in E(G[H])}\frac{d_ud_v}{d_u+d_v}$$
$$\leq |H|^2\left(\frac{\Delta_G}{|H|\delta_G+\delta_H}\right)\sum_{\substack{ac\in E(G)\ or\\ a=c,bd\in E(H)}}\frac{d_G(a)d_G(c)}{d_G(a)+d_G(c)}$$
$$+\left(\frac{\Delta_H}{|H|\delta_G+\delta_H}\right)\sum_{\substack{ac\in E(G)\ or\\ a=c,bd\in E(H)}}\frac{d_H(b)d_H(d)}{d_H(b)+d_H(d)}+|E(G[H])|\left(\frac{|H|\Delta_G\Delta_H}{|H|\delta_G+\delta_H}\right)$$
$$\leq\frac{1}{|H|\delta_G+\delta_H}\Bigg[\left(|H|^2\Delta_G\right)\left(ISI(G)+|G||E(H)|\frac{\Delta_G^2}{2\delta_G}\right)$$
$$+\Delta_H\left(|E(G)|\frac{\Delta_H^2}{2\delta_H}+|G|ISI(H)\right)+|E(G[H])||H|\Delta_G\Delta_H\Bigg],$$

with equality if and only if G and H are regular. □

Theorem 4.5. *Let G and H be arbitrary graphs. Then*

$$ISI(G\circ H)\leq\frac{\Delta_G}{\delta_G+|H|}\left(ISI(G)+|E(G)||H|\right)+\frac{|H|^2}{2(\delta_G+|H|)}|E(G)|$$
$$+|G|\left[\frac{\Delta_H}{\delta_H+1}ISI(H)+\frac{2\Delta_H+1}{2(\delta_H+1)}|E(H)|\right]$$
$$+|G||H|\left[\frac{(\Delta_G+|H|)(\Delta_H+1)}{\delta_G+|H|+\delta_H+1}\right],$$

with equality if and only if G and H are regular.

Proof. By the definitions of ISI-index and $G\circ H$, we have:

$$ISI(G\circ H)=\sum_{\substack{uv\in E(G\circ H)\ and\\ u,v\in V(G)}}\frac{d_ud_v}{d_u+d_v}+|G|\sum_{\substack{uv\in E(G\circ H)\ and\\ u,v\in V(H)}}\frac{d_ud_v}{d_u+d_v}+\sum_{\substack{uv\in E(G\circ H)\ and\\ u\in V(G),v\in V(H)}}\frac{d_ud_v}{d_u+d_v}.$$

(i) For $uv\in E(G\circ H)$ and $u,v\in V(G)$, we have

$$\frac{d_ud_v}{d_u+d_v}=\frac{(d_G(u)+|H|)(d_G(v)+|H|)}{d_G(u)+d_G(v)+2|H|}$$

$$
\begin{aligned}
&= \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} \times \frac{d_G(u)+d_G(v)}{d_G(u)+d_G(v)+2|H|} + \frac{|H|(d_G(u)+d_G(v))}{d_G(u)+d_G(v)+2|H|} \\
&\quad + \frac{|H|^2}{d_G(u)+d_G(v)+2|H|} \\
&\le \frac{\Delta_G}{\delta_G+|H|}\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} + |H|\right] + \frac{|H|^2}{2\left(\delta_G+|H|\right)},
\end{aligned}
$$

with equality if and only if $\Delta_G = d_G(u) = d_G(v) = \delta_G$.

(ii) For $uv \in E(G \circ H)$ and $u, v \in V(H)$, we have

$$
\begin{aligned}
\frac{d_u d_v}{d_u+d_v} &= \frac{(d_H(u)+1)\,(d_H(v)+1)}{d_H(u)+1+d_H(v)+1} \\
&= \frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)+2} + \frac{d_H(u)+d_H(v)}{d_H(u)+d_H(v)+2} + \frac{1}{d_H(u)+d_H(v)+2} \\
&= \frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)} \times \frac{d_H(u)+d_H(v)}{d_H(u)+d_H(v)+2} + \frac{d_H(u)+d_H(v)+1}{d_H(u)+d_H(v)+2} \\
&\le \left(\frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)}\right)\left(\frac{2\Delta_H}{2\delta_H+2}\right) + \frac{2\Delta_H+1}{2\delta_H+2},
\end{aligned}
$$

with equality if and only if $\Delta_H = d_H(u) = d_H(v) = \delta_H$.

(iii) For $uv \in E(G \circ H)$, $u \in V(G)$ and $v \in V(H)$, we have

$$
\frac{d_u d_v}{d_u+d_v} = \frac{(d_G(u)+|H|)\,(d_H(v)+1)}{d_G(u)+|H|+d_H(v)+1} \le \frac{(\Delta_G+|H|)\,(\Delta_H+1)}{\delta_G+|H|+\delta_H+1},
$$

with equality if and only if $\Delta_G = d_G(u) = d_G(v) = \delta_G$ and $\Delta_H = d_H(u) = d_H(v) = \delta_H$.

Therefore,

$$
\begin{aligned}
ISI(G \circ H) \le &\sum_{uv \in E(G)} \left[\frac{\Delta_G}{\delta_G+|H|}\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} + |H|\right) + \frac{|H|^2}{2\left(\delta_G+|H|\right)}\right] \\
&+ |G|\left[\sum_{uv \in E(H)} \left(\frac{\Delta_H}{\delta_H+1}\right)\left(\frac{d_H(u)d_H(v)}{d_H(u)+d_H(v)}\right) + \frac{2\Delta_H+1}{2\delta_H+2}\right] \\
&+ |G|\,|H|\left[\frac{(\Delta_G+|H|)\,(\Delta_H+1)}{\delta_G+|H|+\delta_H+1}\right] \\
= &\frac{\Delta_G}{\delta_G+|H|}\left(ISI(G) + |E(G)|\,|H|\right) + \frac{|H|^2}{2\left(\delta_G+|H|\right)}\,|E(G)| \\
&+ |G|\left[\frac{\Delta_H}{\delta_H+1} ISI(H) + \frac{2\Delta_H+1}{2\left(\delta_H+1\right)}\,|E(H)|\right] \\
&+ |G|\,|H|\left[\frac{(\Delta_G+|H|)\,(\Delta_H+1)}{\delta_G+|H|+\delta_H+1}\right],
\end{aligned}
$$

with equality if and only if G and H are regular. □

Using a method similar to Theorems 4.1- 4.5, we can also obtain the following result.

Theorem 4.6. *Let G and H be arbitrary graphs. Then*

$$\begin{aligned}ISI(G\nabla H) \leq & \frac{1}{\delta_G+\delta_H+\delta_G\delta_H}\Bigg[\Delta_G\left(|H|ISI(G)+|G||E(H)|\frac{\Delta_G^2}{2\delta_G}+|E(H)|ISI(G)\right)\\ & +\Delta_H\left(|H||E(G)|\frac{\Delta_H^2}{2\delta_H}+|G|ISI(H)+|E(G)|ISI(H)\right)\\ & +|E(G\nabla H)|\left(\frac{2\Delta_G\Delta_H+2\Delta_G^2\Delta_H+2\Delta_G\Delta_H^2+\Delta_G^2\Delta_H^2}{2\left(\delta_G+\delta_H+\delta_G\delta_H\right)}\right)\Bigg],\end{aligned}$$

with equality if and only if G and H are regular.

5. Conclusion

In this paper, the sharp bounds for ISI-index by means of some known inequalities have been presented. Also, the sharp bounds for the ISI-index of some graph operations have also been discussed.

Conflicts of Interest

On behalf of all authors, the corresponding author declares that there is no conflict of interest.

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