



On the circuits of splitting matroids representable over $GF(p)$

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ABSTRACT

We apply the splitting operation defined on binary matroids (Raghunathan et al., 1998) to p -matroids, where p -matroids refer to matroids representable over $GF(p)$. We also characterize circuits, bases, and independent sets of the resulting matroid. Sufficient conditions to yield Eulerian p -matroids from Eulerian and non-Eulerian p -matroids by applying the splitting operation are obtained. A class of connected p -matroids that gives connected p -matroids under the splitting operation is characterized. In Application, we characterize a class of paving p -matroids, which produces paving matroids after the splitting operation.

Keywords: connected matroid, splitting operation, Eulerian matroid, circuits, paving matroids

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1. Introduction

Matroid theory is a framework for studying abstract independence that generalizes concepts from linear algebra and graph theory. A matroid M is an ordered pair (E, \mathcal{I}) consisting of a finite set E and a collection \mathcal{I} of subsets of E satisfying the following three conditions:

- (i) $\emptyset \in \mathcal{I}$;
- (ii) If $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$; and
- (iii) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

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The members of \mathcal{I} are the independent sets of M , and E is the ground set of M . A minimal dependent set in M is called circuit of M , and a maximal independent set is called a basis of M . We shall denote set of circuits of M by $\mathcal{C}(M)$, set of independent sets by $\mathcal{I}(M)$, and set of bases by $\mathcal{B}(M)$. For a matroid $M = (E, \mathcal{I})$ the dual matroid $M^* = (E, \mathcal{I}^*)$ has the bases that are exact complement of bases of M . Observe that $(M^*)^* = M$. The collection $\mathcal{C}(M^*)$ of circuits of dual matroid M^* is also denoted by $\mathcal{C}^*(M)$. The circuits of M^* are called cocircuits of M . Rank of a matroid M is the cardinality of its basis. A matroid M is said to be Eulerian if its ground set is disjoint union of circuits of M . For undefined and standard terminologies in matroid, refer to [9, 14].

The circuits (minimal dependent sets) of a matroid have useful applications in AI and data science. Feature selection in Machine Learning is to choose a minimal subset of features that retains predictive power. Matroids naturally model this problem, where independent sets correspond to feature subsets that avoid redundancy, and circuits indicate minimal redundant feature groups. Greedy algorithms using matroids can efficiently find near-optimal feature subsets. Many AI systems use graph structures, where circuits in graphic matroids correspond to cycles in graphs. Understanding circuit structures helps in optimizing graph traversal, dependency analysis, and causal inference. Neural networks often contain redundant neurons or connections. Matroid circuits help identify minimal sets of neurons that, if removed, would cause dependency issues, leading to efficient pruning strategies.

Fleischer [2] characterized Eulerian graphs using the notion of splitting away a pair of edges from a vertex of degree at least three as shown in Figure 1.

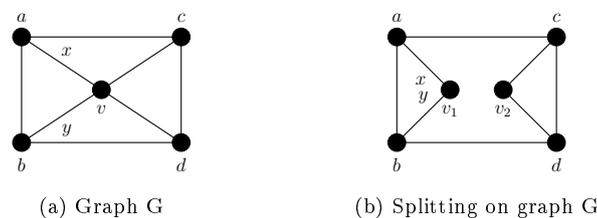


Fig. 1. Splitting operation on graph

Raghunathan et al. [10] extended the splitting operation from graphs to binary matroids, and characterization of Eulerian binary matroids was obtained using this operation as follows:

Theorem 1.1. *A binary matroid M on a set E is Eulerian if and only if M can be transformed by repeated applications of the splitting operation into a matroid in which E is a circuit.*

A sufficient condition was provided by Shikare [11] to obtain connected binary matroids from a 4-connected binary matroids using the splitting operation:

Theorem 1.2. *If M is a 4-connected binary matroid on at least 9 elements and a, b are distinct elements of M , then $M_{a,b}$ is a connected binary matroid.*

It is known that an $m \times n$ matrix A over a field F gives a vector matroid on the ground set $\{1, 2, \dots, n\}$. We may assume that $\text{rank}(A) = m$. From this matrix we can obtain a linear subspace (row space of A) of F^n of dimension m . Such a subspace is called linear code (See [4]). The correspondence of binary matroid and binary linear codes is thoroughly studied by Kashyap in [3]. For finite field F , linear codes have numerous applications in information transmission.

The splitting operation on binary matroids and its various properties are discussed in [5, 6, 7, 12, 13, 16, 18]. In this paper, we intend to study the effect of splitting operation on simple and coloopless matroids representable over $GF(p)$. A matroid M is called a p -matroid if $M \cong M[A]$, where $M[A]$ is the vector matroid of matrix A of size $m \times n$ over the field $F = GF(p)$ for some prime p .

We apply the splitting operation on p -matroids and give a characterization of the circuits of the resulting p -matroid. We give a sufficient condition for the p -matroid to be connected after the splitting operation. A sufficient condition to yield Eulerian p -matroid from Eulerian p -matroid after the splitting operation is also obtained. A class of paving p -matroid which produces paving matroid after splitting operation is characterized.

2. Splitting operation on p -matroids

Let A be a matrix, and $M[A]$ be the corresponding vector matroid. In the following proposition we show that the linear dependence of columns of the matrix A does not change if we multiply the columns of A by non-zero scalars.

Proposition 2.1. *Let $M \cong M[A]$ be a p -matroid on ground set $E, \{a, b\} \subset E$ and α_1, α_2 be non-zero elements of $GF(p)$. Obtain the matrix B by multiplying α_1, α_2 to the columns corresponding to the elements a, b of A , respectively. Then $M[A] \cong M[B]$.*

Proof. It is enough to show that the circuit collections of the matroids $M[A]$ and $M[B]$ are same. For instance, let $C = \{v_1, v_2, \dots, v_k\}$ be a circuit of $M[A]$. Suppose, without loss of generality, $v_2 = a, v_3 = b$ and $c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_kv_k = 0, c_i \neq 0 \in GF(p), i \in \{1, 2, \dots, k\}$. Take $c'_2 = \frac{c_2}{\alpha_1}$ and $c'_3 = \frac{c_3}{\alpha_2}$ then $c_1v_1 + c'_2v_2 + c'_3v_3 + \dots + c_kv_k = 0$. This proves that C is a circuit of $M[B]$. Using similar arguments, it can be proved that circuits of $M[B]$ are same as the circuits of $M[A]$. Hence, $M[A] \cong M[B]$. \square

Let M be a p -matroid on the ground set $E, \{a, b\} \subset E$. In the light of above proposition the splitting operation on binary matroids can also be applied to a p -matroids as follows.

Definition 2.2. Let $M \cong M[A]$ be a p -matroid on the ground set $E, \{a, b\} \subset E$. The matrix $A_{a,b}$ is constructed from A by appending an extra row to A which has coordinates equal to 1 in the columns corresponding to the elements a, b respectively and zero elsewhere. Define the splitting matroid $M_{a,b}$ to be the vector matroid $M[A_{a,b}]$. The transformation of M to $M_{a,b}$ is called the splitting operation.

Remark 2.3. a) In the above definition, let B be the matrix obtained by replacing 1s

in the appending extra row by two non zero elements α_1, α_2 in $GF(p)$. Further D be the matrix obtained by multiplying the columns corresponding to elements a, b of B by $\alpha_1^{-1}, \alpha_2^{-1}$. Then $M[B] \cong M[D]$. But in general $M[A_{a,b}] \not\cong M[D]$.

b) In particular, if $\alpha_1 = \alpha_2$, then using similar arguments in the proof of Proposition 2.1 we can prove that $M[A_{a,b}] \cong M[D]$.

Example 2.4. Consider the vector matroid $M[A]$ of a matrix A over a field $GF(3)$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{A}_{3,5} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

For $a = 3, b = 5$ and $\alpha = 2$, the matrix $A_{3,5}$, represents the splitting matroid $M_{3,5}$.

The collection of circuits of M and $M_{3,5}$ is given in the following table.

Circuits of M	Circuits of $M_{3,5}$
$\{1, 2, 4, 8\}, \{1, 2, 6, 7\}, \{3, 5, 6, 8\}, \{4, 6, 7, 8\}$	$\{1, 2, 4, 8\}, \{1, 2, 6, 7\}, \{3, 5, 6, 8\}, \{4, 6, 7, 8\}$
$\{1, 2, 4, 5, 7\}, \{3, 4, 5, 7\}, \{3, 4, 8\}$	$\{1, 2, 3, 4, 5, 7\}, \{3, 4, 5, 6, 7\}, \{3, 4, 5, 7, 8\}$
$\{1, 2, 3\}, \{4, 5, 6\}, \{5, 7, 8\}, \{3, 6, 7\}$	$\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 5, 7, 8\}$
$\{1, 2, 5, 6, 8\}$	—

It is interesting to observe that unlike the splitting in binary matroids, circuits $\{1, 2, 3, 4, 5, 7\}, \{3, 4, 5, 6, 7\}, \{3, 4, 5, 7, 8\}$ of $M_{3,5}$ are neither circuits of M nor disjoint union of circuits of M .

Remark 2.5. Let $\{a, b\}$ be a cocircuit of M . Then $M_{a,b} \cong M$.

Remark 2.6. Observe that $\text{rank}(A) \leq \text{rank}(A_{a,b}) \leq \text{rank}(A) + 1$, which implies that $\text{rank}(M) \leq \text{rank}(M_{a,b}) \leq \text{rank}(M) + 1$.

Remark 2.7. When $p = 2$, the splitting operation coincides with the splitting operation for binary matroids introduced by Raghunathan et al. [10].

In the following discussion, we describe circuits of $M_{a,b}$ with respect to circuits of M .

Lemma 2.8. *If $C \in \mathcal{C}(M)$ and $C \cap \{a, b\} = \emptyset$, then C is a circuit of $M_{a,b}$.*

Proof. Observe that C is a dependent set of $M_{a,b}$. If C is not a circuit of $M_{a,b}$, then C contains a circuit C' of $M_{a,b}$. Note that C' is a dependent set of M and $C' \subset C$ which is not possible. Therefore C is a circuit of $M_{a,b}$. \square

Let $C_k = \{u_1, u_2, \dots, u_l : u_i \in E, i = 1, 2, \dots, l\}$ be a circuit of M for some positive integers k, l and $|C_k \cap \{a, b\}| = 2$. Suppose, without loss of generality, $u_1 = a, u_2 = b$. If

there are non-zero constants $a_1^k, a_2^k, \dots, a_l^k$ in $GF(p)$ such that $\sum_{i=1}^{i=l} a_i^k u_i \equiv 0(\text{mod } p)$ and $a_1^k + a_2^k \equiv 0(\text{mod } p)$ then we call C_k a p -circuit of M . However, if $|C_k \cap \{a, b\}| = 1$ or $|C_k \cap \{a, b\}| = 2$ and $a_1^k + a_2^k \not\equiv 0(\text{mod } p)$, then we call C_k an np -circuit of M .

We denote $\mathcal{C}_0 = \{C \in \mathcal{C}(M) : C \text{ is a } p\text{-circuit or } C \cap \{a, b\} = \phi\}$.

Remark 2.9. Let C and $\{a, b\}$ be a circuit and a cocircuit of a p -matroid M , respectively. If $p = 2$, then every circuit of M is a circuit of $M_{a,b}$. But for $p > 2$ every circuit of M may not be a circuit of $M_{a,b}$. Note that $|C \cap \{a, b\}| \neq 1$, therefore $|C \cap \{a, b\}|$ is even. If $a, b \in C$ and C is an np -circuit of M , then $C \in \mathcal{I}(M_{a,b})$.

Lemma 2.10. Let M be a p -matroid on ground set E and $\{a, b\} \subset E$. If C_1 and C_2 are disjoint np -circuits of M such that $a \in C_1, b \in C_2$, then $C_1 \cup C_2$ is a dependent set of $M_{a,b}$.

Proof. Let $C_1 = \{u_1, u_2, \dots, u_l\}$ and $C_2 = \{v_1, v_2, \dots, v_k\}$. Without loss of generality, assume $u_1 = a$ and $v_1 = b$. Since C_1 and C_2 are circuits, there exist non-zero constants $\alpha_1, \alpha_2, \dots, \alpha_l$ and $\beta_1, \beta_2, \dots, \beta_k$ such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_l u_l \equiv 0(\text{mod } p), \quad (1)$$

$$\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_k v_k \equiv 0(\text{mod } p). \quad (2)$$

Multiplying Eq. (1) by α_1^{-1} and Eq. (2) by $-\beta_1^{-1}$ we get

$$u_1 + \alpha'_2 u_2 + \dots + \alpha'_l u_l \equiv 0(\text{mod } p),$$

$$-v_1 + \beta'_2 v_2 + \dots + \beta'_k v_k \equiv 0(\text{mod } p).$$

Thus we get non-zero scalars $1, \alpha'_i, i = 2, 3, \dots, l$ and $-1, \beta'_j, j = 2, 3, \dots, k$ such that

$$u_1 + \alpha'_2 u_2 + \dots + \alpha'_l u_l + (-1)v_1 + \beta'_2 v_2 + \dots + \beta'_k v_k \equiv 0,$$

and

$$\text{coeff.}(u_1) + \text{coeff.}(v_1) \equiv 0(\text{mod } p).$$

Therefore $C_1 \cup C_2$ is a dependent set of $M_{a,b}$. \square

Remark 2.11. Let C_1, C_2 be disjoint np -circuits of M such that $a \in C_1, b \in C_2$ and $I \neq \phi$ be an independent set of M . Then $C_1 \cup C_2 \cup I$ can not be a circuit of $M_{a,b}$ as it contains the dependent set $C_1 \cup C_2$ of $M_{a,b}$.

Consider subsets of E of the type $C \cup I$ where $C = \{u_1, u_2, \dots, u_l\}$ is an np -circuit of M which is disjoint from an independent set $I = \{v_1, v_2, \dots, v_k\}$ and $\{a, b\} \subset (C \cup I)$. We say $C \cup I$ is p -dependent if it contains no member of \mathcal{C}_0 and there are non-zero constants $\alpha_1, \alpha_2, \dots, \alpha_l$ and $\beta_1, \beta_2, \dots, \beta_k$ such that $\sum_{i=0}^l \alpha_i u_i + \sum_{j=0}^k \beta_j v_j = 0(\text{mod } p)$ and $\text{coeff.}(a) + \text{coeff.}(b) = 0(\text{mod } p)$.

Theorem 2.12 characterizes the circuits of the splitting matroid $M_{a,b}$ with respect to circuits of M .

Theorem 2.12. *Let M be a p -matroid on ground set E and $\{a, b\} \subset E(M)$. Then $\mathcal{C}(M_{a,b}) = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ where*

$$\mathcal{C}_0 = \{C \in \mathcal{C}(M) : C \text{ is a } p\text{-circuit or } C \cap \{a, b\} = \emptyset\};$$

$\mathcal{C}_1 =$ *Set of minimal members of $\{C \cup I : (C \cup I) \text{ is } p\text{-dependent and there are no disjoint } np\text{-circuits } C'_1, C'_2 \text{ such that } C'_1 \cup C'_2 \subset (C \cup I)\}$; and*

$\mathcal{C}_2 =$ *Set of minimal members of $\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}(M), a \in C_1, b \in C_2, C_1 \cap C_2 = \emptyset$ and there is no $C \in \mathcal{C}_0 \cup \mathcal{C}_1$ such that $C \subset C_1 \cup C_2\}$.*

Proof. By the Definition 2.2 and Lemma 2.10, it is clear that every element of $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ is a circuit of $M_{a,b}$. Conversely, let $C \in \mathcal{C}(M_{a,b})$, then C is a dependent set of M .

Case 1. If C is a minimally dependent set of M , then either $C \cap \{a, b\} = \emptyset$ or C is a p -circuit of M . That is, $C \in \mathcal{C}_0$.

Case 2. If C is not a minimally dependent set of M , then it contains a minimally dependent set, say C_1 . Note that $C_1 \notin \mathcal{C}_0$. Thus C contains no member of \mathcal{C}_0 .

If $a, b \in C_1$, then $C \setminus C_1$ is an independent set of M ; otherwise $C \setminus C_1$ contains a circuit, say C' , such that $C' \in \mathcal{C}_0$ which is not possible. Thus $C = C_1 \cup I$ where C_1 is an np -circuit, $I = C \setminus C_1 \in \mathcal{I}(M)$ and it contains no member of \mathcal{C}_0 . Therefore C is a minimal p -dependent set of M and it does not contain a disjoint union of two np -circuits. Thus $C \in \mathcal{C}_1$.

If $a \in C_1, b \in C \setminus C_1$ and $C \setminus C_1$ is an independent set of M , denoted by I , then $C = C_1 \cup I$. Note that $C_1 \cup I$ is a minimal p -dependent set of M and it does not contain a disjoint union of two np -circuits. Thus $C \in \mathcal{C}_1$.

If $a \in C_1, b \in C \setminus C_1$ and $C \setminus C_1$ is dependent set of M , then there is a circuit, say C_2 , contained in $C \setminus C_1$ and $b \in C_2$. Therefore $C = C_1 \cup C_2 \cup I$ where both C_1 and C_2 are np -circuits and $I = C \setminus (C_1 \cup C_2)$ is an independent set of M . By Remark 2.11, $I = \emptyset$. Further, $C = C_1 \cup C_2$ does not contain p -dependent set of M and a disjoint union of two np -circuits. Therefore $C \in \mathcal{C}_2$. \square

In the Example 2.4, $\mathcal{C}(M_{3,5}) = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ where

$$\mathcal{C}_0 = \{\{1, 2, 4, 8\}, \{1, 2, 6, 7\}, \{3, 5, 6, 8\}, \{4, 6, 7, 8\}\};$$

$$\mathcal{C}_1 = \{\{1, 2, 3, 4, 5, 7\}, \{3, 4, 5, 6, 7\}, \{3, 4, 5, 7, 8\}\};$$

$$\mathcal{C}_2 = \{\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 5, 7, 8\}\}.$$

2.1. Independent sets, bases and rank function of $M_{a,b}$

Now we describe independent sets, bases and the rank function of $M_{a,b}$ in terms of independent sets, bases and the rank function of M , respectively.

Let $\mathcal{I}_0 = \mathcal{I}(M)$ and $\mathcal{I}_1 = \{C \cup I : (C \cup I) \text{ is not } p\text{-dependent and it contains no union of two disjoint } np\text{-circuits}\}$.

Proposition 2.13. $\mathcal{I}(M_{a,b}) = \mathcal{I}_0 \cup \mathcal{I}_1$.

Proof. Clearly $\mathcal{I}_0 \cup \mathcal{I}_1 \subseteq \mathcal{I}(M_{a,b})$. Conversely, assume $S \in \mathcal{I}(M_{a,b})$. If S is an independent

set of M , then $S \in \mathcal{I}_0$. If S is dependent set of M , then it contains an np -circuit of M . In the light of Lemma 2.10, S does not contain union of two disjoint np -circuits. Therefore $S = C \cup I$ for some $C \in \mathcal{C}(M)$, $I \in \mathcal{I}(M)$ and $C \cup I$ is not p -dependent. Hence $S \in \mathcal{I}_1$. \square

We use r and r' to denote the rank functions of matroid M and $M_{a,b}$ respectively.

Theorem 2.14. *Let M be a p -matroid and $\{a, b\} \subset E$. If M contains an np -circuit, then $\mathcal{B}(M_{a,b}) = \{B \cup e : B \in \mathcal{B}(M), e \notin B \text{ and } B \cup e \text{ contains neither } p\text{-circuit nor } p\text{-dependent set}\}$.*

Proof. Let $B \in \mathcal{B}(M)$ and $e \notin B$. If $B \cup e$ contains no p -circuit and no p -dependent set, then $B \cup e$ is an independent set of $M_{a,b}$. Moreover, $B \cup e$ is a maximal independent set of $M_{a,b}$ because $r'(M_{a,b}) \leq r(M) + 1$.

Conversely, let B' be a basis of $M_{a,b}$. Note that B' is an independent subset of M if and only if M contains no np -circuit. Since M contains an np -circuit, B' is a dependent set of M . If B' contains a p -circuit, then B' becomes dependent set of $M_{a,b}$, a contradiction. Therefore B' contains an np -circuit, say C . Thus $B' = C \cup I$ where $I = B' \setminus C$. Note that I is an independent set of M , otherwise there is a np -circuit, say C_1 , of M contained in I and $C \cup C_1$ gives a dependent subset of B' in $M_{a,b}$, a contradiction.

Let $e \in C$ and $B = B' \setminus e = (C \setminus e) \cup I$. Now if B is a dependent set of M , then $r[(C \setminus e) \cup I] < r(M)$. It implies $r[(C \cup I)] < r(M)$ and $r'(C \cup I) \leq r(M)$. That is, $r'(B') \leq r(M)$, a contradiction. Therefore $B = B' \setminus e$ is an independent set of M . Moreover, $|B| = r(M)$, which implies that B is a basis of M . Thus $B' = B \cup e$. \square

In the following corollary, we provide the rank function of $M_{a,b}$ with respect to the rank function of M .

Corollary 2.15. *Suppose $S \subseteq E(M_{a,b})$. Then*

$$r'(S) = \begin{cases} r(S), & \text{if } S \text{ contains no } np\text{-circuit of } M; \\ r(S) + 1, & \text{if } S \text{ contains an } np\text{-circuit of } M. \end{cases} \quad (3)$$

Proof. Let B and B' be the bases of $M|_S$ and $M_{a,b}|_S$, respectively, then $r(S) = |B|$ and $r'(S) = |B'|$. If S contains no np -circuit, then $|B'| = |B|$. Therefore $r(S) = r'(S)$. If S contains an np -circuit, then by the Theorem 2.14, $B' = B \cup e$ for some $B \in \mathcal{B}(M|_S)$, $e \in (S \setminus B)$ and $B \cup e$ contains a unique np -circuit which is the fundamental circuit of e with respect to B . Therefore $r'(S) = r'(M_{a,b}|_S) = |B'| = |B \cup e| = |B| + 1 = r(S) + 1$. \square

Corollary 2.16. *Let M be a p -matroid and $\{a, b\} \subset E$, then*

- a) $r'(M_{a,b}) = r(M)$ if and only if every circuit $C \in \mathcal{C}(M)$ is a p -circuit. Moreover, $M \cong M_{a,b}$.
- b) If M contains an np -circuit, then $r'(M_{a,b}) = r(M) + 1$.

Remark 2.17. Let M be a p -matroid and $\{a, b\} \subset E$. If M contains an np -circuit, then $\{a, b\}$ is a cocircuit of $M_{a,b}$. Assume the contrary, that is, $\{a, b\}$ is not a cocircuit of $M_{a,b}$. Then there is a basis B' of $M_{a,b}$ such that $B' \cap \{a, b\} = \emptyset$. By Theorem 2.14, $B' = B \cup e$ for some $B \in \mathcal{B}(M)$ and $e \in E \setminus B$. Now $B' = B \cup e$ contains a unique np -circuit of M which is a contradiction to the assumption that there is a basis B' of $M_{a,b}$ such that $B' \cap \{a, b\} = \emptyset$. Therefore, $\{a, b\}$ is a cocircuit of $M_{a,b}$.

3. Connectivity of splitting matroids

Let M be a matroid having ground set E . The 1-separation of the matroid M is a partition (S, T) of E such that, $|S|, |T| \geq 1$ and $r(S) + r(T) - r(M) < 1$. We say M is connected if M has no 1-separation. In depth study of matroid connectivity is given in [15].

Observe that in Example 2.4, the matroid M as well as the splitting matroid $M_{3,5}$ are connected. In general, the connectivity of a p -matroid is not closed under the splitting operation, which is observed in the following example.

Example 3.1. Consider the vector matroid $M \cong M[A]$ represented by the matrix A over the field $GF(3)$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_{1,4} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the matroid M is connected. But the splitting matroid $M_{1,4} \cong M[A_{1,4}]$ is not connected.

Remark 3.2. Let M be a p -matroid on ground set E and $\{a, b\} \subset E$. If M contains no np -circuit, then $M_{a,b} \cong M$. In this case, M is connected if and only if $M_{a,b}$ is connected.

In the next result, a sufficient condition is provided for the matroid $M_{a,b}$ to be connected.

Lemma 3.3. *Let M be a connected p -matroid and $\{a, b\} \subset E(M)$. If for every proper subset S of $E(M)$ with $|S| \geq 1$, either S or $T = E(M) \setminus S$ contains an np -circuit of M , then $M_{a,b}$ is connected.*

Proof. Note that, M is a connected p -matroid, which implies that M has no 1-separation. On the contrary, assume $M_{a,b}$ is not connected. That is, $M_{a,b}$ has 1-separation, say (S, T) . Therefore

$$r'(S) + r'(T) - r'(M_{a,b}) < 1.$$

If S and T both contain np -circuits then, by lemma 2.15, we have

$$r(S) + 1 + r(T) + 1 - r(M) - 1 = r(S) + r(T) - r(M) + 1 < 1.$$

Thus we get a contradiction to the fact $r(S) + r(T) - r(M) \geq 0$. Further, if only one of S or T , say S , contains an np -circuit, then we have

$$r'(S) + r'(T) - r'(M_{a,b}) = r(S) + 1 + r(T) - r(M) - 1 < 1.$$

That is

$$r(S) + r(T) - r(M) < 1.$$

Thus (S, T) gives a 1-separation of M which is not possible. \square

For $n \geq 2$, a matroid M is said to be vertically n -connected if for any positive integer $k < n$, there does not exist a partition (S, T) of $E(M)$, such that $r(S) + r(T) - r(M) < k$, and $r(S), r(T) \geq k$.

Theorem 3.4. *Let M be a connected, vertically 3-connected, simple p -matroid and $\{a, b\} \subset E$. Then $M_{a,b}$ is connected p -matroid if and only if for every $e \in E$ there is an np -circuit of M not containing e .*

Proof. On the contrary, suppose that $M_{a,b}$ is not connected. And (S, T) is a 1-separation of $E(M_{a,b})$. Consequently, $|S|, |T| \geq 1$ and $r'(S) + r'(T) - r'(M_{a,b}) \leq 0$. Following are the possible cases:

Case 1. $|S| = 1$.

Suppose $S = \{x\}$ where $x \in E$. As M contains an np -circuit C such that $\{x\} \cap C = \emptyset$, which gives $C \subseteq T$. By Corollary 2.15, $r'(T) = r(T) + 1$. We get $r(S) + r(T) + 1 - r(M) - 1 \leq 0$. Consequently, $r(S) + r(T) - r(M) \leq 0$ and $|S|, |T| \geq 1$, which forms a 1-separation of M , a contradiction.

Case 2. $|S|, |T| > 1$.

If either S or T contains an np -circuit then using Lemma 3.3 we conclude that $M_{a,b}$ is connected.

Suppose both S and T do not contain an np -circuit. Then $r'(S) + r'(T) - r'(M_{a,b}) \leq 0$. Using Corollary 2.15 we get, $r(S) + r(T) - r(M) - 1 \leq 0$. That is $r(S) + r(T) - r(M) \leq 1$. Since M is simple, $r(S), r(T) \geq 2$. It implies (S, T) is a vertical 2-separation of M , a contradiction.

If each of S and T contains an np -circuit, then $r'(S) + r'(T) - r'(M_{a,b}) \leq 0$. Again by Corollary 2.15 we get $r(S) + 1 + r(T) + 1 - r(M) - 1 \leq 0$. That is $r(S) + r(T) - r(M) \leq -1$. Consequently, (S, T) is a 1 separation of M , which is a contradiction.

Therefore $M_{a,b}$ has no 1-separation. We conclude that $M_{a,b}$ is a connected p -matroid.

To check the necessity of the condition, suppose that $M_{a,b}$ is connected. On the contrary, assume that there is an element $e \in E$ which is contained in every np -circuit of M . Let $S = \{e\}$ and $T = E \setminus S$, then T contains no np -circuit of M . Thus $r'(S) = 1$, $r'(T) = r(T) = r(M)$ and $|S|, |T| \geq 1$. Further, $r'(S) + r'(T) - r'(M_{a,b}) = 1 + r(T) - r(M) - 1 = 0$. Thus (S, T) forms a 1-separation of $M_{a,b}$, a contradiction. Therefore we conclude that for every $e \in E(M)$ there is an np -circuit of M not containing e . \square

A matroid M is connected if and only if, for every pair of distinct elements of E , there is a circuit containing both. This result is used to prove the following proposition.

Proposition 3.5. *Let M be a non-connected p -matroid on ground set E with exactly two connected components M_1, M_2 , and $\{a, b\} \subset E$. Let $a \in M_1, b \in M_2$. If C and C' are np -circuits of M contained in M_1 and M_2 , respectively, then $C \cup C'$ is a circuit of $M_{a,b}$.*

Proof. By Lemma 2.10, $C \cup C'$ is a dependent set of $M_{a,b}$. Therefore there is a circuit C_1 of $M_{a,b}$ such that $C_1 \subseteq C \cup C'$. Note that C_1 is a dependent set of M .

Suppose $C_1 \subset C \cup C'$. Then $C_1 \not\subseteq C$ and $C_1 \not\subseteq C'$ because C and C' are minimal dependent sets of M . Let $C_1 \cap C \neq \phi$ and $C_1 \cap C' \neq \phi$. Then we have the following two cases.

Case 1. Let $C_1 \cap \{a, b\} = \phi$.

Then C_1 is a p -circuit containing elements of M_1 and M_2 which imply that M is a connected matroid, a contradiction.

Case 2. Let $C_1 \cap \{a, b\} \neq \phi$.

Then $a, b \in C_1$ because $\{a, b\}$ is a co-circuit of $M_{a,b}$. Since M is not connected, C_1 can not be a circuit of M . Thus there is a circuit, say C_2 , of M such that $C_2 \subset C_1$. Now by similar argument as above, $C_2 \not\subseteq C$ and $C_2 \not\subseteq C'$. Consequently, $C_2 \cap C \neq \phi$ and $C_2 \cap C' \neq \phi$. Thus C_2 is a circuit of M containing elements of M_1 and M_2 . Therefore M is connected which is a contradiction.

Hence, in either case, we get a contradiction. Thus $C_1 = C \cup C'$. □

Corollary 3.6. *Let M be a non-connected p -matroid on ground set E with exactly two connected components M_1, M_2 and $\{a, b\} \subset E$. Let $a \in M_1, b \in M_2$. Then $M_{a,b}$ is a connected p -matroid.*

Proof. It is enough to show that for every pair $x, y \in E$, there is a circuit of $M_{a,b}$ containing x and y .

Case 1. Let $x, y \in M_1$.

Then there is a circuit, say C , of M containing x, y . If C is a p -circuit, then it is the desired circuit of $M_{a,b}$. If C is an np -circuit of M , then $a \in C$. Let C' be an np -circuit of M contained in M_2 . Then by Proposition 3.5, $C \cup C'$ is a circuit of $M_{a,b}$ containing x and y .

Case 2. Let $x \in M_1$ and $y \in M_2$.

As M_1, M_2 are connected there are circuits, say C_1, C_2 , containing x, a and y, b , respectively, in M . Note that C_1, C_2 are np -circuits. By Proposition 3.5, $C_1 \cup C_2$ is a circuit of $M_{a,b}$ containing x and y . □

4. Applications

Let M be Eulerian matroid on ground set E . Then there are disjoint circuits C_1, C_2, \dots, C_k of M such that $E = C_1 \cup C_2 \cup \dots \cup C_k$. In binary matroids the properties of being Euler and bipartite are dual concepts are proved by Welsh [17].

Eulerian p -matroids are not closed under the splitting operation as observed in the

following example.

Example 4.1. $M = M[A]$ is Eulerian p -matroid over $GF(3)$ but the splitting matroid $M_{1,2} = M[A_{1,2}]$ is not Eulerian p -matroid.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}, \quad A_{1,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\{a, b\} \subset E$ and $M_{a,b}$ be the splitting matroid. We say that the collection $\tilde{C} = \{C_1, C_2, \dots, C_k\}$ is a p -decomposition of M if $a, b \in C_i$, for some $i \in \{1, 2, \dots, k\}$, and C_i is a p -circuit or $a \in C_i, b \in C_j$ for some $i \neq j$ in $\{1, 2, \dots, k\}$ and $(C_i \cup C_j) \in \mathcal{C}_2$.

In the following result we give a sufficient condition to yield Eulerian p -matroids from Eulerian p -matroids after the splitting operation.

Proposition 4.2. *Let M be a p -matroid on ground set E and $\{a, b\} \subset E$. If M is Eulerian matroid having a p -decomposition, then $M_{a,b}$ is Eulerian.*

Proof. Let $\tilde{C} = \{C_1, C_2, \dots, C_k\}$ be a p -decomposition of M . Then

$$E = C_1 \cup C_2 \cup \dots \cup C_k.$$

Case 1. If $a, b \in C_i$ some $i \in \{1, 2, \dots, k\}$ and C_i is a p -circuit.

Then the same collection \tilde{C} is a disjoint circuit decomposition of $E(M_{a,b})$. Therefore $M_{a,b}$ is Eulerian matroid.

Case 2. Assume, without loss of generality, $a \in C_1, b \in C_2$.

By hypothesis, \tilde{C} is a p -decomposition, therefore $(C_1 \cup C_2)$ is a circuit of $M_{a,b}$. Denote $(C_1 \cup C_2)$ by C . Then the collection $\{C, C_3, \dots, C_k\}$ is a disjoint circuit decomposition of $M_{a,b}$. Hence $M_{a,b}$ is Eulerian matroid. \square

Proposition 4.3. *If $M_{a,b}$ is Eulerian p -matroid having a disjoint circuit decomposition which contains no member of \mathcal{C}_1 , then M is Eulerian.*

Proof. Let $E(M_{a,b}) = C_1 \cup C_2 \cup \dots \cup C_k$ be a disjoint circuit decomposition which contains no member of \mathcal{C}_1 . If $C_i \in \mathcal{C}_0, C_j \in \mathcal{C}_2$ for some $i, j \in \{1, 2, \dots, k\}$, then $C_i \in \mathcal{C}(M)$ and $C_j = C_j^1 \cup C_j^2$ where $C_j^1, C_j^2 \in \mathcal{C}(M)$ and $C_j^1 \cap C_j^2 = \phi$. Therefore

$$E(M) = C_1 \cup C_2 \cup \dots \cup C_i \cup \dots \cup C_j^1 \cup C_j^2 \cup \dots \cup C_i,$$

is a disjoint circuit decomposition of $E(M)$. Therefore M is Eulerian. \square

Applying Proposition 4.2 and 4.3 for $p = 2$, we obtain the following result of Raghunthan et al. [10] for binary matroids.

Corollary 4.4. *Let M be a binary matroid and $a, b \in E$. Then M is Eulerian if and only if $M_{a,b}$ is Eulerian.*

Note that, the splitting operation on non-Eulerian p -matroids may yield Eulerian p -matroids.

Example 4.5. Consider the vector matroid $M \cong M[A]$ represented by the matrix A over field $GF(5)$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{A}_{3,5} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The circuits of M are $\{1, 2, 3, 4\}, \{1, 2, 5\}, \{3, 4, 5\}$. Note that M is non-Eulerian matroid. For $a = 3, b = 5$, $A_{3,5}$ represents the splitting matroid $M_{3,5}$. Here the matroid $M_{3,5}$ has only one circuit $\{1, 2, 3, 4, 5\}$. Therefore it is Eulerian matroid over $GF(5)$.

In the following result, we provide a sufficient condition for the splitting operation on non-Eulerian p -matroids to yield Eulerian p -matroids.

Proposition 4.6. *Let M be non-Eulerian p -matroid on ground set E . Then E can be written as follows:*

$$E = C_1 \cup C_2 \cup \dots \cup C_k \cup I,$$

where $C_i \in \mathcal{C}(M), I \in \mathcal{I}(M), C_i \cap C_j = \phi$ when $i \neq j \in \{1, 2, \dots, k\}$ and $C_i \cap I = \phi$ for all $i \in \{1, 2, \dots, k\}$. Let $\{a, b\} \subset E$. If there is a circuit C_i such that $C_i \cup I \in \mathcal{C}_1$, then $M_{a,b}$ is Eulerian Matroid.

An Eulerian p -matroid M on E can be transformed by repeated applications of splitting operations into a matroid in which E is a circuit, which is observed in the following example.

Example 4.7. The matrix A gives $GF(3)$ representation of the Eulerian matroid $S(5, 6, 12)$. Let M denote the vector matroid of A . Consider the following sequence of the splitting operations: $M^1 = M_{1,6}, M^2 = M_{1,4}^1, M^3 = M_{1,3}^2, M^4 = M_{3,9}^3$, and $M^5 = M_{3,11}^4$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

The matroid M^5 is Eulerian in which $E(M)$ forms a circuit. We observe that each matroid M and $M^k, k \in \{1, 2, 3, 4\}$, have a p -decomposition. Therefore by Proposition 4.2 the matroids $M^k, k \in \{1, 2, \dots, 5\}$ are Eulerian.

In the light of Example 4.5, the Theorem 1.1 by Raghunathan et al. [10] does not hold, in general, for p -matroids, $p > 2$.

For $p > 2$, it remains open to prove or disprove the existence of a sequence of splitting operations on Eulerian p -matroid M which transforms the ground set $E(M)$ into a circuit. The following algorithm transforms $E(M)$ into a circuit, if a sequence of splitting operations, described earlier, exists; otherwise, $E(M)$ can not be transformed into a circuit.

Algorithm 4.8. Let M be Eulerian p -matroid,

Step 1 : Find all the circuits of M .

Step 2 : List all the circuit decompositions of M as $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\}$.

Step 3 : If there is a pair $a, b \in E(M)$ such that \mathcal{D}_i is a p -decomposition of M , and $|\mathcal{D}_i| > 1$, for some $i \in [k]$, then the splitting matroid $M_{a,b}$ is Eulerian by Proposition 4.2. Replace M by $M_{a,b}$ and go to **Step 1**.

If no such pair $a, b \in E(M)$ exists, then go to **Step 4**.

Step 4 : Then $E(M)$ is the circuit of M or $E(M)$ can not be transformed into a circuit.

Let M be a matroid of rank r . M is called a paving matroid, if every circuit of M is of the size r or $r + 1$. In [1] Ackett characterized all binary paving matroids. Further Oxley [8] provided a characterization of ternary paving matroids. The class of paving p -matroids is not closed under the splitting operation. The following result characterizes a class of paving p -matroids which yield paving matroid under the splitting operation.

Lemma 4.9. *Let M be a paving p -matroid of rank $r, \{a, b\} \subset E(M)$. Then the splitting matroid $M_{a,b}$ is also paving if and only if every circuit $C \in \mathcal{C}(M)$ of size r is an np -circuit.*

Proof. Note that every circuit of M is of the size r or $r + 1$. By the assumption, every circuit of M of the size r is an np -circuit. All such circuits become independent sets of the splitting matroid $M_{a,b}$. Further, $M_{a,b}$ cannot have a circuit of the size less than $r + 1$ that is a member of \mathcal{C}_1 , and \mathcal{C}_2 . Thus the splitting matroid $M_{a,b}$ cannot have a circuit of size less than or equal to r . Therefore $M_{a,b}$ is also a paving matroid of rank $r + 1$.

Let $M_{a,b}$ be a paving p -matroid of rank $r + 1$. Then $M_{a,b}$ has circuits of size $r + 1$ or $r + 2$. By the hypothesis, M is a paving matroid of rank r . Therefore, M has circuits of size r or $r + 1$. Let C be a circuit of M of size r . We claim that C is an np -circuit. If C is a p -circuit, then C is also circuit of $M_{a,b}$ of size r , a contradiction. \square

Example 4.10. The matroid $M(K_4)$ is a paving matroid of rank 3. All the circuits of size 3 of matroid $M(K_4)$ are np -circuits with respect to the splitting of non-adjacent edges

of K_4 . Therefore, the corresponding splitting matroid is also paving.

Corollary 4.11. *Let M be a paving p -matroid of rank r with at least three disjoint circuits of size r . Then there does not exist $\{a, b\} \subset E(M)$ such that $M_{a,b}$ is paving.*

Proof. On the contrary, assume that there exists $\{a, b\} \subset E(M)$ such that the splitting matroid $M_{a,b}$ is a paving matroid of rank $r + 1$. Since M has at least three disjoint circuits at most two of them can be np -circuit. Thus, there exists at least one p -circuit of M of size of r . Such a circuit also remains a circuit of $M_{a,b}$, which gives a contradiction to the fact $M_{a,b}$ is a paving matroid of rank $r + 1$. \square

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