ARS COMBINATORIA 163 (2025) 113–121

Ars Combinatoria
www.combinatorialpress.com/ars

Combinatorial Press

Type I and type II codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$

Ankur Singh[1,2,✉], R. Kumar[3]

[1] *Department of Mathematics, VIT-AP University Amaravati, A.P. India*
[2] *Department of Mathematics (SoT) PDEU Gandhinagar, Gujarat, India*
[3] *Department of Mathematics, Amity University, Gwalior, Madhya Pradesh, India*

ABSTRACT

We consider a ring $R_{u^3} = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2, u^4 = 0$. We discuss the structure of self-dual codes, Type I codes and Type II codes over the ring R_{u^3} in terms of the structure of their Torsion and Residue codes. We construct Type I and Type II optimal codes over the ring R_{u^3} for different lengths.

Keywords: self-dual codes, gray map, residue code, torsion code

2020 Mathematics Subject Classification: 94B05.

1. Introduction

There are several researchers working on self-dual codes from the long period of time. Mallows and Sloane [12] discuss the bound for self-dual codes over the field which we can use to differentiate the Type of the code. Conway et al. [3] discuss the binary self-dual code for different length using the bound to talk about the optimality of self-dual codes upto length 32. We can further look at [15] for the study of self-dual codes. Latter on scope of self-dual codes have been extended to finite rings. We can refer to series of papers [1, 2, 5, 8, 9, 11, 16, 17] for more detailed description of self-dual codes.

Construction of self-dual codes have been discussed in the following series of papers [7, 10, 13, 18, 21, 20]. Gaborit [6] consider the ring $\mathbb{Z}_4$ and $\mathbb{F}_q + u\mathbb{F}_q$ and discuss the mass formula for self-dual codes over these rings. He also discuss the Residue and Torsion code

✉ Corresponding author.
E-mail address: ankur786ankur@gmail.com (A. Singh).

Accepted 11 June 2020; Published Online 28 June 2025.

DOI: 10.61091/ars163-08

and give a mass formula for quaternary Type II codes, doubly even binary codes and proved the equivalence of self-dual codes with the characterization of self-dual codes over these rings. Dougherty et al. [4] consider the commutative Frobenius ring $R_u = \mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$ and discuss some theoretical concept on self-dual codes over the ring R_u and give the classification of Type I and Type II codes for different lengths.

We consider a Frobenius ring $R_{u^3} = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2, u^4 = 0$ and use some theoretical concept from [19] to discuss the self-dual codes of different length and give some optimal codes over the ring. Our main focus in this paper is to discuss the self-dual codes over R_{u^3}. We also characterize the structure of self-dual, Type I and Type II codes over the ring R_{u^3} with generator matrix in terms of the structure of their Torsion and Residue codes.

2. Preliminaries

We define an ordinary inner product $\langle\cdot,\cdot\rangle$ on $R_{u^3}^n$ by

$$\langle x, y\rangle = \sum_{i=0}^{n} x_i y_i,$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $R_{u^3}^n$. The dual code $C^\perp$ of C is the code

$$C^\perp = \{x \in R_{u^3}^n \mid \langle x, y\rangle = 0, \forall y \in C\}.$$

We say that C is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. Note that the length of binary self-dual codes is always even, but self-dual codes over R_{u^3} can have odd length.

Definition 2.1. A linear code C over the ring R_{u^3} of length n is an R_{u^3}-submodule of $R_{u^3}^n$.

Definition 2.2. Let $\phi : (\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2)^n \to \mathbb{F}_2^{4n}$ be the map given by

$$\phi(a_1 + ub_1 + u^2c_1 + u^3d_1) = (a_1 + b_1 + c_1 + d_1, c_1 + d_1, b_1 + d_1, d_1).$$

It is easy to see that ϕ is a linear map and takes length n binary linear code over $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$ to a binary linear code of length $4n$.

Definition 2.3. For any element $a_1 + ub_1 + u^2c_1 + u^3d_1 \in R_{u^3}$, we define the Lee weight-w_L as $w_L(a_1 + ub_1 + u^2c_1 + u^3d_1) = w_H(a_1 + b_1 + c_1 + d_1, c_1 + d_1, b_1 + d_1, d_1)$, where w_H denotes the Hamming weight for binary codes.

Proposition 2.4. *If C is a linear code over R_{u^3} of length n, size 2^k and minimum Lee distance d, then $\phi(C)$ is a binary $[4n, k, d]-$linear code.*

Proposition 2.5. *If C is a self-dual linear code over R_{u^3} of length n, then $\phi(C)$ is a self-dual binary linear code of length $4n$.*

In the ring R_{u^3}, we have four elements $(1, 1+u, 1+u^2, 1+u+u^2+u^3)$ of Lee weight 1, six elements $(u, u^2, u+u^2, u+u^3, u^2+u^3, u+u^2+u^3)$ of Lee weight 2, four elements $(1+u^3, 1+u+u^3, 1+u^2+u^3, 1+u+u^2)$ of Lee weight 3, and one element u^3 of weight 4.

The paper is organized as follows. In Section 2, we discuss self-dual codes over R_{u^3} and give the form of the generator matrix with which we discuss Torsion and Residue of the code and use them to discuss few results on the self-dual codes over R_{u^3}. In Section **??**, we construct self-dual codes for different lengths over R_{u^3} and characterize Type I and Type II codes.

3. Self-dual codes over R_{u^3}

Definition 3.1. Let $R_{u^3} := \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$ and $R_u = \mathbb{F}_2 + u\mathbb{F}_2$. A self-dual code over R_{u^3} is said to be Type II if the Lee weight of every codeword is a multiple of 4 and Type I otherwise.

Definition 3.2. Let C be a linear code over R_{u^3}. We define the Residue code $C_{(1)}$ and the Torsion code $C_{(2)}$ of C as follows:

$$C_{(1)} = \{x \in (R_u)^n \mid \exists y \in (R_u)^n \mid x + u^2 y \in C\},$$

and

$$C_{(2)} = \{x \in (R_u)^n \mid u^2 x \in C\}.$$

We denote by $(C^{\perp})_1$ and $(C^{\perp})_2$ the Residue and Torsion of the dual code $C^{\perp}$ respectively. It is easy to see that $|C| = |C_{(1)}||C_{(2)}|$.

Definition 3.3. We denote the residue of $C_{(1)}$ as $Res(C_{(1)})$ and the torsion of $C_{(2)}$ as $Tor(C_{(2)})$ and define as:

$$Res(C_{(1)}) = \{x \in \mathbb{F}_2^n \mid \exists y \in \mathbb{F}_2^n \mid x + uy \in C_{(1)}\},$$

and

$$Tor(C_{(2)}) = \{x \in \mathbb{F}_2^n \mid ux \in C_{(2)}\}.$$

Lemma 3.4. *If C is a linear code over R_{u^3}, then*
(1) $(C^{\perp})_1 \subseteq C_{(2)}^{\perp}$ *and*
(2) $(C^{\perp})_2 = C_{(1)}^{\perp}$.

Proof. (1) Let $x \in (C^{\perp})_1$, there exists $y \in R_u^n$ such that $x + u^2 y \in C^{\perp}$. For $z \in C_{(2)}, u^2 z \in C$. Then $u^2 z \cdot (x + u^2 y) = 0 \Rightarrow u^2(z \cdot x) = 0$. Thus $x \in C_{(2)}^{\perp}$. Hence $(C^{\perp})_1 \subseteq C_{(2)}^{\perp}$.

(2) Let $z \in C_{(1)}^{\perp}$ and $x + u^2 y \in C$. Then $x \in C_{(1)}$ and $x \cdot z = 0$. Thus $u^2 z \cdot (x + u^2 y) = u^2(z \cdot x) = 0$. Therefore $u^2 z \in C^{\perp}$ and $z \in (C^{\perp})_2$. So $C_{(1)}^{\perp} \subseteq (C^{\perp})_2$. Now let $z \in (C^{\perp})_2$, then $u^2 z \in C^{\perp}$. For each $x + u^2 y \in C$, $x \in C_{(1)}$ and $u^2 z \cdot (x + u^2 y) = 0 \Rightarrow u^2(z \cdot x) = 0 \Rightarrow z \in C_{(1)}^{\perp}$. Thus $(C^{\perp})_2 \subseteq C_{(1)}^{\perp}$. Hence $(C^{\perp})_2 = C_{(1)}^{\perp}$. □

Lemma 3.5. *If C is a self-orthogonal linear code over R_{u^3} and $C_{(1)}^{\perp} = C_{(2)}$, then C is a self-dual code.*

Proof. Since C is a self-orthogonal code, $C \subseteq C^{\perp}$. By Lemma 3.4, we have $|C^{\perp}| = |(C^{\perp})_1||(C^{\perp})_2| \leq |C_{(2)}^{\perp}||C_{(1)}^{\perp}|$. Therefore, $|C^{\perp}| \leq |C_{(2)}^{\perp}||C_{(1)}^{\perp}| = |C_{(1)}||C_{(2)}| = |C|$. Thus $C = C^{\perp}$. □

Theorem 3.6. *Let C be a linear code over R_{u^3}, then C is self-dual if and only if*

(1) *$C_{(1)}$ is self-orthogonal even code,*
(2) *$x \cdot t + y \cdot s = 0$ for $x + u^2 y, s + u^2 t \in C$, $x, t \in R_u^n$ and*
(3) *$C_{(2)} = C_{(1)}^{\perp}$.*

Proof. Let C be a self-dual code over R_{u^3}.

(1) Let $x, s \in C_{(1)}$. Then, there exist y and t in $(R_u)^n$ such that $x + u^2 y, s + u^2 t \in C$. Since C is self-dual, $(x + u^2 y) \cdot (s + u^2 t) = 0$. This implies that $x \cdot s = 0$. Hence $C_{(1)} \subseteq C_{(1)}^{\perp}$. As the residue code $C_{(1)}$ is self-orthogonal, it contains all even weight vectors. Hence, $C_{(1)}$ is even.

(2) Let $x + u^2 y, s + u^2 t \in C$. Since C is self-dual, $(x + u^2 y) \cdot (s + u^2 t) = 0$. This gives $x \cdot t + y \cdot s = 0$.

(3) Let $x \in C_{(2)}$, then $u^2 x \in C$. For $s \in C_{(1)}$, there exists $t \in R_u^n$ such that $s + u^2 t \in C$. Since C is self-dual, $u^2 x \cdot (s + u^2 t) = 0 \Rightarrow u^2 (x \cdot s) = 0 \Rightarrow x \in C_{(1)}^{\perp}$. Thus $C_{(2)} \subseteq C_{(1)}^{\perp}$. Let $x \in C_{(1)}^{\perp}$ and $s + u^2 t \in C$. Then $s \in C_{(1)}$ and $u^2 x \cdot (s + u^2 t) = u^2 (x \cdot s) = 0 \Rightarrow u^2 x \in C^{\perp} = C \Rightarrow x \in C_{(2)}$. Thus $C_{(1)}^{\perp} \subseteq C_{(2)}$. Hence $C_{(1)}^{\perp} = C_{(2)}$.

Conversely, suppose Conditions 1-3 hold. Let $x + u^2 y, s + u^2 t \in C$. From Conditions (1) and (2), $(x + u^2 y) \cdot (s + u^2 t) = 0$. Hence C is self-orthogonal. From Lemma 3.5, C is self-dual. □

The following corollaries follow from the above theorem.

Corollary 3.7. *If C is a self-orthogonal code over R_u, then $C + u^2 C^{\perp}$ is a self-dual code over R_{u^3}.*

Corollary 3.8. *If C is a self-orthogonal linear code over R_{u^3}, then $Res(C_{(1)})$ is a self-orthogonal even binary code.*

Proof. If C is self-orthogonal then $C_{(1)}$ is self-orthogonal even code over R_u. Hence, $\text{Res}(C_{(1)})$ is a self-orthogonal even binary code. □

Let C be a linear code which has a generator matrix G [14] after a suitable permutation of coordinates can be written in the form:

$$G = \begin{pmatrix} I_{k_1} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & uI_{k_2} & uA_{23} & uA_{24} & uA_{25} \\ 0 & 0 & u^2 I_{k_3} & u^2 A_{34} & u^2 A_{35} \\ 0 & 0 & 0 & u^3 I_{k_4} & u^3 A_{45} \end{pmatrix}, \tag{1}$$

where A_{ij}, $1 \leq i \leq 4$ and $2 \leq j \leq 5$ are matrices over $\mathbb{R}_{u^3}$. We use a representation $A_{1j} = A_{1j}^{(1)} + uA_{1j}^{(2)} + u^2A_{1j}^{(3)} + u^3A_{1j}^{(4)}$ and $uA_{2j} = u(A_{2j}^{(1)} + uA_{2j}^{(2)} + u^2A_{2j}^{(3)} + u^3A_{2j}^{(4)})$, where $A_{1j}^{(i)}$ and $A_{2j}^{(i)}$, for $i = 1, 2, 3, 4$ are matrices over $\mathbb{F}_2$. A generator matrix of $C_{(1)}$ is given by

$$G_1 = \begin{pmatrix} I_{k_1} & A_{12}^{(1)} + uA_{12}^{(2)} & A_{13}^{(1)} + uA_{13}^{(2)} & A_{14}^{(1)} + uA_{14}^{(2)} & A_{15}^{(1)} + uA_{15}^{(2)} \\ 0 & uI_{k_2} & uA_{23}^{(1)} & uA_{24}^{(1)} & uA_{25}^{(1)} \end{pmatrix}, \tag{2}$$

and a generator matrix of $C_{(2)}$ is given by

$$G_2 = \begin{pmatrix} I_{k_1} & A_{12}^{(1)} + uA_{12}^{(2)} & A_{13}^{(1)} + uA_{13}^{(2)} & A_{14}^{(1)} + uA_{14}^{(2)} & A_{15}^{(1)} + uA_{15}^{(2)} \\ 0 & uI_{k_2} & uA_{23}^{(1)} & uA_{24}^{(1)} & uA_{25}^{(1)} \\ 0 & 0 & I_{k_3} & A_{34}^{(1)} + uA_{34}^{(2)} & A_{35}^{(1)} + uA_{35}^{(2)} \\ 0 & 0 & 0 & uI_{k_4} & uA_{45}^{(1)} \end{pmatrix}. \tag{3}$$

Theorem 3.9. *Let C be a linear code over R_{u^3} which is permutation-equivalent to a code with a generator matrix G of the form given in Eq. (1). Then C is self-dual iff*
(1) *$C_{(1)}$ is self-orthogonal even code,*
(2) *$C_{(2)} = C_{(1)}^{\perp}$.*

Proof. From Theorem 3.6, if C is a self-dual code, then $C_{(1)}$ is self-orthogonal. Also Res $(C_{(1)})$ is self-orthogonal. Thus,

$$\begin{aligned} &\begin{pmatrix} I_{k_1} & A_{12}^{(1)} + uA_{12}^{(2)} & A_{13}^{(1)} + uA_{13}^{(2)} & A_{14}^{(1)} + uA_{14}^{(2)} & A_{15}^{(1)} + uA_{15}^{(2)} \end{pmatrix} \\ &\times \begin{pmatrix} I_{k_1} & A_{12}^{(1)} + uA_{12}^{(2)} & A_{13}^{(1)} + uA_{13}^{(2)} & A_{14}^{(1)} + uA_{14}^{(2)} & A_{15}^{(1)} + uA_{15}^{(2)} \end{pmatrix}^T = 0. \end{aligned}$$

Also for the generator matrix G, and for $x, t \in R_u^n$ $(x + u^2y) \cdot (s + u^2t) = x \cdot s + u^2(x \cdot t + y \cdot s) \Rightarrow x \cdot t + y \cdot s = 0$. Second part of the theorem follows from Lemma 3.4 and Theorem 3.6. □

Lemma 3.10. *If C is a Type II code over R_{u^3}, then the residue code $C_{(1)}$ contains the all u-vector* $\mathbf{u}$.

Proof. Let $x \in C_{(2)}$, then $u^2x \in C$. Since C is a Type II code, x contains even number of units (1 and $1+u$). Therefore, the all u-vector $\mathbf{u}$ is orthogonal to x, i.e., $x \in C_{(2)}^{\perp} = C_{(1)}$. Hence, $C_{(1)}$ contains the all u-vector $\mathbf{u}$. □

If C is a code of length n over R_{u^3} and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two vectors of C, we denote by $n_{x,y}(a, b)$ the number of occurrences of the couples (a, b) and (b, a) in $((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$, where a and b are elements of R_{u^3}. By abuse of notation, we simply denote $n_{x,y}(a, b)$ by $n(a, b)$.

Lemma 3.11. *If C is a self-dual code over R_{u^3} and if $x, y \in C$ such that $4 \mid w_L(x)$ and $4 \mid w_L(y)$ then $4 \mid w_L(x + y)$ and $4 \mid w_L(\alpha x)$,where $\alpha \in R_{u^3}$.*

Proof. Let $x = x_1 + u^2x_2$ and $y = y_1 + u^2y_2$ be codewords of C, where $x_i, y_i \in (R_u)^n$. We follow the idea for proving these results from [1] and [4] and write a 16 by 16 table (which we are not writing here, as it is very long table) to visualize the structure of $w_L(x+y)$ and $w_L(\alpha x)$. Now from the definition of Lee weight we have

$w_L(x+y) = w_L(x) + w_L(y) - 2A \pmod 4$ and $w_L(\alpha x) = w_L(\alpha)\cdot w_L(x) \equiv 0 \pmod 4$, where

$$\begin{aligned} A =& n(u^3,1) + n(u^3,1+u) + n(u^3,1+u^2) + n(u^3,1+u^3) + n(u^3,1+u+u^2) \\ &+ n(u^3,1+u+u^3) + n(u^3,1+u^2+u^3) + n(u^3,1+u+u^2+u^3) + n(1,1) \\ &+ n(1,u)x + n(1,u^2) + n(1,u+u^2+u^3) + n(1,1+u+u^2) + n(1,1+u+u^3) \\ &+ n(1,1+u^2+u^3) + n(u,u^2) + n(u,1+u) + n(u,u+u^2) + n(u,u^2+u^3) \\ &+ n(u,u+u^2+u^3) + n(u,1+u^3) + n(u,1+u+u^3) + n(u^2,1+u^2) \\ &+ n(u^2,u+u^2) + n(u^2,u+u^3) + n(u^2,u+u^2+u^3) + n(u^2,1+u^3) \\ &+ n(u^2,1+u^2+u^3) + n(1+u,1+u) + n(1+u,u+u^2) + n(1+u,u^2+u^3) \\ &+ n(1+u,1+u^3) + n(1+u,1+u+^3) + n(1+u,1+u^2+u^3) + n(1+u^2,1+u^2) \\ &+ n(1+u^2,u+u^2) + n(1+u^2,u+u^3) + n(1+u^2,1+u^3) + n(1+u^2,1+u+u^2) \\ &+ n(1+u^2,1+u+u^3) + n(u+u^2,u+u^3) + n(u+u^2,u^2+u^3) \\ &+ n(u+u^2,1+u+u^3) + n(u+u^2,1+u^2+u^3) + n(u+u^3,u^2+u^3) \\ &+ n(u+u^3,u+u^2+u^3) + n(u+u^3,1+u+u^2) + n(u+u^3,1+u^2+u^3) \\ &+ n(u+u^3,1+u+u^2+u^3) + n(u^2+u^3,u+u^2+u^3) + n(u^2+u^3,1+u+u^2) \\ &+ n(u^2+u^3,1+u+u^3) + n(u^2+u^3,1+u+u^2+u^3) + n(u+u^2+u^3,1+u^3) \\ &+ n(u+u^2+u^3,1+u+u^2) + n(u+u^2+u^3,1+u+u^2+u^3) + n(1+u^3,1+u^3) \\ &+ n(1+u^3,1+u+u^2+u^3) + n(1+u+u^2,1+u+u^2) + n(1+u+u^3,1+u+u^3) \\ &+ n(1+u+u^3,1+u+u^2+u^3) + n(1+u^2+u^3,1+u^2+u^3) \\ &+ n(1+u^2+u^3,1+u+u^2+u^3) + n(1+u+u^2+u^3,1+u+u^2+u^3). \end{aligned}$$

If C is self-dual, we can see that $A \equiv 0 \pmod 2$. Thus, $w_L(x+y) \equiv w_L(x) + w_L(y) \equiv 0 \pmod 4$. Hence, $4 \mid w_L(x+y)$ and $4 \mid w_L(\alpha x)$. □

Theorem 3.12. *Let C be a linear code of even length over R_{u^3} which is permutation-equivalent to a code with a generator matrix G of the form given in Eq. (1). Then C is Type II if and only if*

(1) *The Residue code $C_{(1)}$ is even and contains the all u-vector* $\mathbf{u}$.
(2) $C_{(2)} = C_{(1)}^{\perp}$.
(3) $w_L \equiv 0$ *(mod 4) for the first $k_1 + k_2$ rows of G.*

Proof. If C is Type II, then the Lee weights of every codeword is a multiple of 4. So $w_L \equiv 0 \pmod 4$ for the first $k_1 + k_2$ rows of G. Conditions 1,2 and 3 follow from Theorem 3.9 and Lemma 3.10. Conversely, suppose the conditions 1-3 hold, then the code C is self-dual code (from Theorem 3.9). Since $\mathbf{u} \in \mathbf{C_{(1)}} = \mathbf{C_{(2)}^{\perp}}$, a vector $x \in C_{(2)}$ contains even number of units (1 and $1+u$). Thus, $w_L(u^2x) \equiv 0 \pmod 4$ and $w_L \equiv 0 \pmod 4$ for the generator matrix G. Hence, from Lemma 3.11, the code C is a Type II code. □

Proposition 3.13. *Let C be a self-orthogonal code of even length over R_u. If C contains the all u-vector and weights of all codewords of C are multiple of 4, then $C + u^2C^{\perp}$ is a Type II code over R_{u^3}.*

Proof. From Corollary 3.7, $C + u^2C^{\perp}$ is self-dual. Since C contains the all u-vector, a vector $y \in C^{\perp}$ contains even number of units (1 and $1+u$). Thus, $w_L(u^2y) \equiv 0 \pmod 4$. Also, for $x + u^2y \in C + u^2C^{\perp}$, $w_L(x + u^2y) \equiv w_L(x) + w_L(u^2y) \equiv 0 \pmod 4$. Hence, $C + u^2C^{\perp}$ is a Type II code. □

Corollary 3.14. *If C is a Type II code over R_u, then $C + u^2C$ is a Type II code over R_{u^3}.*

4. Examples

We discuss few examples of self-dual codes that are Type I and Type II for lengths 1 to 6. We use results discussed in section 2 to construct self-dual codes of different length.

Type I Code of length 1

Let $C_{1,1} = \langle (u^2 + u^3) \rangle$. $\phi(C_{1,1})$ is a Type I code over the ring R_{u,u^2}, and its binary parameters is $[4, 2, 2]^*$.

4.1. *Self-dual codes of length* 2

Let $C_{2,1} = \langle (1, 1) \rangle$. $\phi(C_{2,1})$ is a Type I code over the ring R_{u,u^2} with the binary parameters $[8, 4, 2]^*$.

We take $C_{2,2} = \langle (1, 1 + u^3) \rangle$. $\phi(C_{2,2})$ is a Type II code over the ring R_{u,u^2} with its binary parameters $[8, 4, 4]^*$.

We take $C_{2,3} = \langle (1 + u^2 + u^3, 1 + u^2) \rangle$. $\phi(C_{2,3})$ is a Type II code over the ring R_{u,u^2} with its binary parameters $[8, 4, 4]^*$.

4.2. *Type I code of length* 3

We take $C_{3,1} = \langle (0, u^3, u^2), (1, 1 + u + u^3, u) \rangle$. $\phi(C_{3,1})$ is a Type I code over the ring R_{u,u^2} and its binary parameters is $[12, 6, 4]^*$.

4.3. *Self-dual codes of length* 4

We take $C_{4,1} = \langle (1 + u + u^3, u^3, u, 1 + u^3), (1 + u, 1 + u^2 + u^3, 1, 1 + u + u^2 + u^3) \rangle$. $\phi(C_{4,1})$ is a Type I code over the ring R_{u,u^2} with its binary parameters $[16, 8, 4]^*$.

Let $C_{4,2} = \langle (1 + u, 1 + u + u^3, 0, 0), (1 + u, 1 + u + u^3, 1, 1 + u^3) \rangle$. $\phi(C_{4,2})$ is a Type II code over the ring R_{u,u^2} and its binary parameters is $[16, 8, 4]^*$.

4.4. Self-dual codes of length 5

We take $C_{5,1} = \langle(1, 1+u^2+u^3, 0, 0, 0), (u^2+u^3, u^2+u^3, u^3, u^2, u^3), (1+u^2, 1+u^3, u^2, u^3, u^2+u^3), (1+u^3, 1+u^2, 0, u^2, u^2)\rangle$. $\phi(C_{5,1})$ is a Type I code over the ring R_{u,u^2} with its binary parameters $[20, 10, 4]^*$.

4.5. Self-dual codes of length 6

We take $C_{6,1} = \langle(1, 0, 0, 1+u^3, u^2, u^2), (0, 1, 0, u^2, 1+u+u^3, u+u^3), (0, 0, 1, u^2, u+u^3, 1+u+u^3)\rangle$. $\phi(C_{6,1})$ is a Type II code over the ring R_{u,u^2} with its binary parameters $[24, 12, 4]$.

We take $C_{6,1} = \langle(1, 0, 0, 1+u^2+u^3, u^3, u^2+u^3), (0, 1, 0, u^3, 1+u^2+u^3, u^2), (0, 0, 1, u^2+u^3, u^2, 1+u^3)\rangle$. $\phi(C_{6,2})$ is a Type I code over the ring R_{u,u^2} with its binary parameters $[24, 12, 4]$.

References

[1] E. Bannai, M. Harada, T. Ibukiyama, A. Munemasa, and M. Oura. Type ii codes over $\mathbb{F}_2 + u\mathbb{F}_2$ and applications to hermitian modular forms. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 73, pages 13–42. Springer, 2003. https://doi.org/10.1007/BF02941267.

[2] A. Bonnecaze and P. Udaya. Correspondence-cyclic codes and self-dual codes over $\mathbb{F}_2+u\mathbb{F}_2$. *IEEE Transactions on Information Theory*, 45(4):1250–1254, 1999. https://doi.org/10.1109/18.761278.

[3] J. H. Conway, V. Pless, and N. J. A. Sloane. The binary self-dual codes of length up to 32: a revised enumeration. *Journal of Combinatorial Theory, Series A*, 60(2):183–195, 1992. https://doi.org/10.1016/0097-3165(92)90003-D.

[4] S. T. Dougherty, P. Gaborit, M. Harada, and P. Solé. Type ii codes over $\mathbb{F}_2 + u\mathbb{F}_2$. *IEEE Transactions on Information Theory*, 45(1):32–45, 1999. https://doi.org/10.1109/18.746770.

[5] J. Fields, V. Pless, and J. S. Leon. All self-dual Z_4 codes of length 15 or less are known. In *Proceedings of IEEE International Symposium on Information Theory*, page 202. IEEE, 1997. https://doi.org/10.1109/ISIT.1997.613117.

[6] P. Gaborit. Mass formulas for self-dual codes over $\mathbb{F}_4$ and $\mathbb{F}_q+u\mathbb{F}_q$ rings. *IEEE Transactions on Information Theory*, 42(4):1222–1228, 1996. https://doi.org/10.1109/18.508845.

[7] W. Huffman. Automorphisms of codes with applications to extremal doubly even codes of length 48. *IEEE Transactions on Information Theory*, 28(3):511–521, 1982. https://doi.org/10.1109/TIT.1982.1056499.

[8] W. C. Huffman. On the decomposition of self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ with an automorphism of odd prime order. *Finite Fields and Their Applications*, 13(3):681–712, 2007. https://doi.org/10.1016/j.ffa.2006.02.003.

[9] W. C. Huffman. Self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ with an automorphism of odd order. *Finite Fields and Their Applications*, 15(3):277–293, 2009. https://doi.org/10.1016/j.ffa.2007.07.003.

[10] P. K. Kewat, B. Ghosh, and S. Pattanayak. Cyclic codes over the ring $\mathbb{Z}_p[u, v]/<u^2, v^2, uv-vu>$. *Finite Fields and Their Applications*, 34:161–175, 2015.

[11] H. J. Kim and Y. Lee. Construction of extremal self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ with an automorphism of odd order. *Finite Fields and Their Applications*, 18(5):971–992, 2012. https://doi.org/10.1016/j.ffa.2012.05.004.

[12] C. L. Mallows and N. J. Sloane. An upper bound for self-dual codes. *Information and Control*, 22(2):188–200, 1973. https://doi.org/10.1016/S0019-9958(73)90273-8.

[13] M. Ozeki. Hadamard matrices and doubly even self-dual error-correcting codes. *Journal of Combinatorial Theory, Series A*, 44(2):274–287, 1987. https://doi.org/10.1016/0097-3165(87)90034-3.

[14] Z. ö. Özger, Ü. Ü. Kara, and B. Yıldız. Linear, cyclic and constacyclic codes over $S_4 = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$. *Filomat*, 28(5):897–906, 2014. https://www.jstor.org/stable/24896855.

[15] V. Pless, R. A. Brualdi, and W. C. Huffman. *Handbook of Coding Theory*. Elsevier Science Inc., 1998.

[16] V. Pless, J. S. Leon, and J. Fields. All z4codes of type ii and length 16 are known. *Journal of Combinatorial Theory, Series A*, 78(1):32–50, 1997. https://doi.org/10.1006/jcta.1996.2750.

[17] A. K. Singh and P. K. Kewat. On cyclic codes over the ring $\mathbb{Z}_p[u]/<u^k>$. *Designs, Codes and Cryptography*, 74:1–13, 2015. https://doi.org/10.1007/s10623-013-9843-2.

[18] V. D. Tonchev. Self-orthogonal designs and extremal doubly even codes. *Journal of Combinatorial Theory, Series A*, 52(2):197–205, 1989. https://doi.org/10.1016/0097-3165(89)90030-7.

[19] B. Yildiz and S. Karadeniz. Self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$. *Journal of the Franklin Institute*, 347(10):1888–1894, 2010. https://doi.org/10.1016/j.jfranklin.2010.10.007.

[20] V. Yorgov. A method for constructing inequivalent self-dual codes with applications to length 56. *IEEE Transactions on Information Theory*, 33(1):77–82, 1987. https://doi.org/10.1109/TIT.1987.1057273.

[21] V. Yorgov. Doubly even extremal codes of length 64. *Problemy Peredachi Informatsii*, 22(4):35–42, 1986.