

On error-detecting open-locating-dominating sets

Devin C. Jean and Suk J. Seo*

ABSTRACT

An open-dominating set S for a graph G is a subset of vertices where every vertex has a neighbor in S . An open-locating-dominating set S for a graph G is an open-dominating set such that each pair of distinct vertices in G have distinct set of open-neighbors in S . We consider a type of a fault-tolerant open-locating dominating set called error-detecting open-locating-dominating sets. We present more results on the topic including its NP-completeness proof, extremal graphs, and a characterization of cubic graphs that permit an error-detecting open-locating-dominating set.

Keywords: domination, fault tolerant detection system, open-locating-dominating sets, cubic graphs, extremal graphs

2020 Mathematics Subject Classification: 05C69.

1. Introduction

An open-locating-dominating set can model a type of detection system which determines the location of a possible “intruder” for a facility or a possible faulty processor in a network of processors [13]. A detection system is an extensively studied graphical concept which is also known as a watching system or discriminating codes [1, 2]. Various detection systems have been defined based on the functionality of each detector in the system. Other well-known and much studied detection systems include identifying codes [9], locating-dominating sets [15] (See Lobstein’s Bibliography [7] for a list of the articles in this field.). In this paper, we consider a fault-tolerant variant of an open-locating dominating set called error-detecting open-locating-dominating sets. We present more results on the topic including its NP-completeness proof, extremal graphs, and a characterization in cubic graphs.

* Corresponding author.

Received 20 Apr 2025; Revised 01 Sep 2025; Accepted 18 Sep 2025; Published Online 25 Dec 2025.

DOI: [10.61091/ars165-05](https://doi.org/10.61091/ars165-05)

© 2025 The Author(s). Published by Combinatorial Press. This is an open access article under the CC BY license (<https://creativecommons.org/licenses/by/4.0/>).

1.1. Notations and definitions

Let G be a graph with vertices $V(G)$ and edges $E(G)$. The *open-neighborhood* of a vertex $v \in V(G)$, denoted $N(v)$, is the set of all vertices adjacent to v : $\{w \in V(G) : vw \in E(G)\}$. The *closed-neighborhood* of a vertex $v \in V(G)$, denoted $N[v]$, is the set of all vertices adjacent to v , as well as v itself: $N(v) \cup \{v\}$. An *open-dominating set* (also called total-dominating) S of a graph G is a subset of vertices where every vertex has a neighbor in S . An *open-locating-dominating set* (OLD set) S of a graph G is an open-dominating set such that each pair of distinct vertices in G have distinct open-neighborhoods within S . For an OLD set $S \subseteq V(G)$ and $u \in V(G)$, we let $N_S(u) = N(u) \cap S$ denote the *dominators* of u and $dom(u) = |N_S(u)|$ denote the (open) *domination number* of u . A vertex $v \in V(G)$ is *k-open-dominated* by an open-dominating set S if $|N_S(v)| = k$. If S is an open-dominating set and $u, v \in V(G)$, u and v are *k-distinguished* if $|N_S(u) \Delta N_S(v)| \geq k$, where Δ denotes the symmetric difference. If S is an open-dominating set and $u, v \in V(G)$, u and v are *k[#]-distinguished* if $|N_S(u) - N_S(v)| \geq k$ or $|N_S(v) - N_S(u)| \geq k$. We will also use terms such as “at least k -dominated” to denote j -dominated for some $j \geq k$.

There are several fault-tolerant variants of OLD sets. For example, a redundant open-locating-dominating set is resilient to a detector being destroyed or going offline [14]. Thus, an open-dominating set $S \subseteq V(G)$ is called a *redundant open-locating-dominating (RED:OLD)* set if $\forall v \in S, S - \{v\}$ is an OLD set. The focus of this paper is another variant of an OLD set called an *error-detecting open-locating-dominating (DET:OLD)* set, which is capable of correctly identifying an intruder even when at most one sensor or detector incorrectly reports that there is no intruder. Hence, DET:OLD sets allow for uniquely locating an intruder in a way which is resilient to up to one false negative. The following Theorem characterizes OLD, RED:OLD, and DET:OLD sets and they are useful in constructing those sets or verifying whether a given set meets their requirements.

Theorem 1.1. *An open-dominating set is*

- i. *an OLD set if and only if every pair of vertices is 1-distinguished [8].*
- ii. *a RED:OLD set if and only if all vertices are at least 2-dominated and all pairs are 2-distinguished [14].*
- iii. *a DET:OLD set if and only if all vertices are at least 2-dominated and all pairs are 2[#]-distinguished [14].*

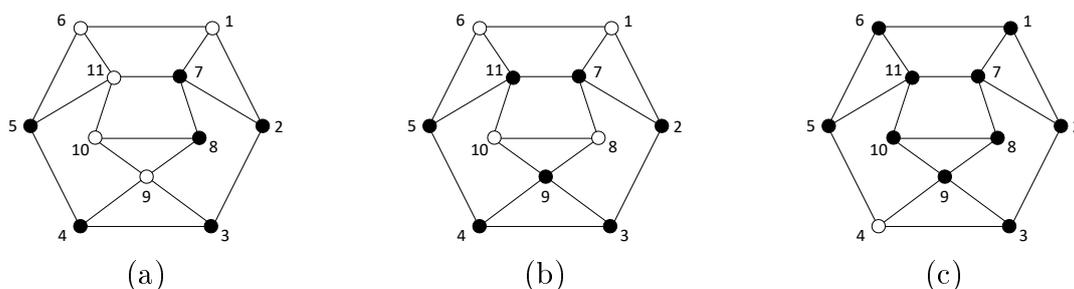


Fig. 1. Optimal OLD (a), RED:OLD (b), and DET:OLD (c) sets. Shaded vertices represent detectors

Figure 1 shows OLD, RED:OLD, and DET:OLD sets on the given graph G ; we can

verify exhaustively that these sets of detectors meet the requirements of Theorem 1.1. To understand why the conditions of DET:OLD established in Theorem 1.1 are necessary, consider the following example scenarios. Suppose we use the RED:OLD detector set from Figure 1 (b) and there is an intruder at v_9 ; then precisely two vertices, v_3 and v_4 , will detect the intruder. However, if v_3 gives a false negative, we will have only v_4 detecting an intruder. This detection pattern is identical to an alternative scenario where the intruder is placed at v_5 and v_{11} gives a false negative response. Because there are two scenarios in which we observe the same detection pattern for two different intruder locations (v_5 and v_9), we fail to locate the intruder; in particular, we find that v_5 and v_9 are 2-distinguished but not $2^\#$ -distinguished. However, if we reattempt the same scenario using instead the detector set from Figure 1 (c), we see that the intruder at v_9 causes detections at v_3 , v_8 , and v_{10} . Suppose one of these three detectors reports a false negative; for the sake of example let v_{10} give a false negative. We still have two positive responses from v_3 and v_8 , so the intruder must be within $N(v_3) \cap N(v_8)$, which contains only v_9 ; thus, we have successfully located the intruder despite the false negative. From Theorem 1.1, we see that the only difference between RED:OLD and DET:OLD is the switch from 2-distinguishing in RED:OLD to the stronger $2^\#$ -distinguishing in DET:OLD. It is ultimately this additional requirement of $2^\#$ -distinguishing that allows DET:OLD to overcome the possibility of false negatives and still locate intruders; the full proof of this fact is available in [14].

Naturally, our goal is to install a minimum number of detectors in any detection system. For finite graphs, the notations $\text{OLD}(G)$, $\text{RED:OLD}(G)$, and $\text{DET:OLD}(G)$ represent the cardinality of the smallest possible OLD, RED:OLD, and DET:OLD sets on graph G , respectively [8, 12, 14]. We can verify exhaustively that the sets presented in Figure 1 are optimal, as there are no smaller sets meeting their respective requirements on this graph. Therefore, we have $\text{OLD}(G) = 6$, $\text{RED:OLD}(G) = 7$, and $\text{DET:OLD}(G) = 10$.

When measuring detector sets on infinite graphs, instead of cardinality, we use the *density* of the subset, which is conceptually the ratio of the number of detectors to the total number of vertices. Formally, the density of $S \subseteq V(G)$ is defined as $\limsup_{r \rightarrow \infty} \frac{|B_r(x) \cap S|}{|B_r(x)|}$, where $B_r(v) = \{u \in V(G) : d(u, v) \leq r\}$ is the ball of radius r about v and x is a particular choice of center point for the expansion. Notably, this definition of density is a function of the center vertex, x ; however, it has been proven that graphs exhibiting the “slow-growth” property of $\lim_{r \rightarrow \infty} \frac{|B_{r+1}(x)|}{|B_r(x)|} = 1$ for some choice of $x \in V(G)$ have the additional property that the density is invariant of the chosen center point [10]. In this paper, we will only consider graphs which exhibit this slow-growth property. With the formal definition of density established for both finite and infinite graphs, the notations $\text{OLD}\%(G)$, $\text{RED:OLD}\%(G)$, and $\text{DET:OLD}\%(G)$ represent the minimum density of such a set on G .

In Section 2, we present extremal graphs with the highest value of $\text{DET:OLD}\%(G)$. In Section 3, we provide a proof that the problem of determining $\text{DET:OLD}(G)$ for an arbitrary graph is NP-complete. Sections 4 and 5 discuss DET:OLD sets on several infinite grids and cubic graphs, respectively.

2. Extremal graphs on DET:OLD sets

In this section, we consider extremal graphs with $\text{DET:OLD}(G) = n$. Let $S \subseteq V(G)$ be a DET:OLD set for G ; because non-detectors do not aid in locating intruders, it must be that S is a DET:OLD set for $G - (V(G) - S)$, implying the smallest graph with DET:OLD will have $\text{DET:OLD}(G) = n$. The following theorem shows that the smallest graphs with DET:OLD have $n = 7$.

Theorem 2.1. *If G has a DET:OLD, then $n \geq 7$.*

Proof. Assume to the contrary, $n \leq 6$. Clearly G has a cycle because DET:OLD requires $\delta(G) \geq 2$. Firstly, we consider the cases when the smallest cycle in the graph is C_n , with $n = 6, 5, 4$. If the smallest cycle is a C_6 subgraph $abcdef$, then a and c cannot be distinguished without having $n \geq 7$, a contradiction. Suppose the smallest cycle is a C_5 subgraph $abcde$. To distinguish a and c , by symmetry we can assume that $\exists u \in N(a) - N(c) - \{a, b, c, d, e\}$. Similarly, to distinguish b and e we can assume by symmetry that $\exists v \in N(b) - N(e) - \{a, b, c, d, e\}$. If $u = v$, there would be a smaller cycle, implying $n \geq 7$, a contradiction; thus, we can assume $u \neq v$. Suppose the smallest cycle is a C_4 subgraph $abcd$. To distinguish a and c , we can assume $p, q \in N(a) - N(c) - \{a, b, c, d\}$. Similarly, to distinguish b and d we can assume by symmetry that $\exists x, y \in N(b) - N(d) - \{a, b, c, d\}$. We know that $\{p, q\} \cap \{x, y\} = \emptyset$ because otherwise we create a smaller cycle; thus, $n \geq 8$, a contradiction. Otherwise, we can assume that G has a triangle.

Next, we will show that if G contains a K_4 subgraph, then $n \geq 7$; let $abcd$ be the vertices of a K_4 subgraph in G . To distinguish pairs of vertices in $abcd$, without loss of generality we can assume that $\exists x \in N(a)$, $\exists y \in N(b)$, and $\exists z \in N(c)$. Clearly $\{x, y, z\} \cap \{a, b, c, d\} = \emptyset$ because x, y , and z are used to distinguish the vertices a, b, c , and d ; however, we do not yet know if x, y , and z are distinct. If $x = y = z$, then distinguishing a, b , and c requires at least another two vertices, so we would have at least 7 vertices and would be done; otherwise, without loss of generality we can assume $x \neq z$. Suppose $x = y$; we see that a and b are not distinguished and $n = 6$, so without loss of generality let $bz \in E(G)$ to distinguish a and b . If $\{dx, dz\} \cap E(G) = \emptyset$, then (d, x) and (d, z) cannot be distinguished; otherwise without loss of generality assume $dx \in E(G)$. We now see that a and d are not distinguished, but they are symmetric, so without loss of generality let $dz \in E(G)$ to distinguish a and d . We see that d and b become closed twins and cannot be distinguished. Otherwise, we can assume $x \neq y$ and by symmetry $y \neq z$. Thus, $n \geq 7$, and we would be done. Thus, if G has a K_4 subgraph, then we would be done.

Next, we will show that the existence of a ‘‘diamond’’ subgraph, which is an (almost) K_4 subgraph minus one edge, implies that $n \geq 7$. Let $abcd$ be a C_4 subgraph and assume $ac \in E(G)$ but $bd \notin E(G)$, which forms said diamond subgraph. To distinguish b and d , we can assume by symmetry that $\exists u, v \in N(b) - N(d) - \{a, b, c, d\}$. Similarly, to distinguish a and c , we can assume that $\exists w \in N(a) - N(c) - \{a, b, c, d\}$. We know that $u \neq v$ by assumption, so $n \leq 6$ requires by symmetry that $w = u$. We can assume that

$uc \notin E(G)$ because otherwise this would create a K_4 subgraph and fall into a previous case. We see that $cv \in E(G)$ is required to distinguish u and c . To distinguish u and v , by symmetry we can assume $uv \in E(G)$. Now, we see that a and v cannot be distinguished without creating a K_4 subgraph, so we are done with the diamond case.

For the final case, we know there must be a triangle, abc , but there cannot be any K_4 or diamond subgraphs. To distinguish vertices in abc , we can assume by symmetry that $\exists u \in N(a) - \{a, b, c\}$ and $\exists v \in N(b) - \{a, b, c\}$; further, we know that $u \neq v$ because otherwise this would create a diamond subgraph. We know that $\{av, bu, cu, cv\} \cap E(G) = \emptyset$ because any of these edges would create a diamond subgraph. Suppose $uv \in E(G)$; then distinguishing a and v within the bounds of $n \leq 6$ requires $\exists w \in N(a) - N(v) - \{a, b, c, u, v\}$. Similarly, distinguishing u and b requires $w \in N(b)$; however, this creates a diamond subgraph, so we would be done. Otherwise, we can assume that $uv \notin E(G)$. To 2-dominate u and v , we require $\exists p \in N(u) - \{a, b, c, u, v\}$ and $\exists q \in N(v) - \{a, b, c, u, v\}$. However, $n \leq 6$ requires that $p = q$. We see that u and v cannot be distinguished without creating a diamond subgraph or having $n \geq 7$, completing the proof. \square

Let $G_{n,m}$ have n vertices and m edges. Then $G_{7,11}$, as shown in Figure 2, is the first graph that permits a DET:OLD set in the lexicographic ordering of (n, m) tuples; i.e., the graph with the smallest number of edges given the smallest number of vertices.

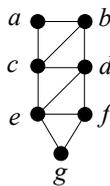


Fig. 2. $G_{7,11}$ with $\text{DET:OLD}(G_{7,11}) = 7$

Next we consider extremal graphs with $\text{DET:OLD}(G) = n$ with the fewest number of edges.

Observation 2.2. *If S is a DET:OLD set on G and $v \in V(G)$, then v has at most one degree 2 neighbor.*

Theorem 2.3. *If $G_{n,m}$ has DET:OLD then $m \geq \left\lceil \frac{3n - \lfloor \frac{n}{2} \rfloor}{2} \right\rceil$.*

Proof. Because G has DET:OLD, we know $\delta(G) \geq 2$; let p be the number of degree 2 vertices in G . By Observation 2.2, every degree 2 vertex, v , must have at least one neighbor, u , of at least degree 3, and said neighbor u is not adjacent to any degree 2 vertices other than v . Thus, we can pair each of the p degree 2 vertices with a unique degree 3 or higher vertex. From this we know that $p \leq \lfloor \frac{n}{2} \rfloor$, and the $n - 2p$ vertices that are not pairs are all at least degree 3. Thus, $\sum_{v \in V(G)} \text{deg}(v) \geq (2 + 3)p + 3(n - 2p) = 3n - p \geq 3n - \lfloor \frac{n}{2} \rfloor$. However, we also know that the degree sum of any graph must be

even, so we can strengthen this to $\sum_{v \in V(G)} \deg(v) \geq 2 \left\lceil \frac{3n - \lfloor \frac{n}{2} \rfloor}{2} \right\rceil$. Dividing the degree sum by 2 completes the proof. \square

The lower bound given in Theorem 2.3 on the minimum number of edges in a graph with DET:OLD is sharp for all $n \geq 9$ and Figure 3 shows a construction for an infinite family of graphs achieving the extremal value for $9 \leq n \leq 20$.

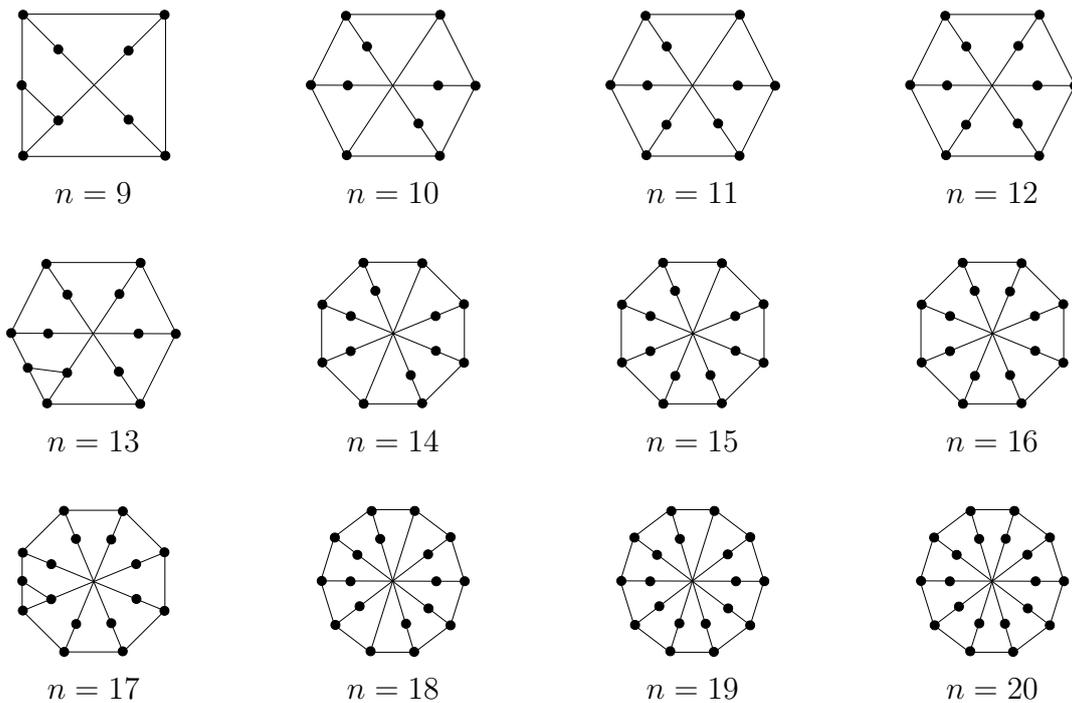


Fig. 3. Extremal family of graphs with $\text{DET:OLD}(G) = n$ with the smallest number of edges

3. NP-completeness of error-detecting OLD

It has been shown that finding the cardinality of the smallest OLD set, when phrased as a decision problem, is NP-complete [13]. This NP-completeness has also been demonstrated for many other forms of detection systems explored in other works, such as the Locating-Dominating set [5, 3] and the Identifying Code [3, 4]. We will now prove that the problem of determining the smallest DET:OLD set is also NP-complete. For additional information about NP-completeness, see Garey and Johnson [6].

Question 3.1 (3-SAT). *Let X be a set of N variables. Let ψ be a conjunction of M clauses, where each clause is a disjunction of three literals from distinct variables of X . Is there is an assignment of values to X such that ψ is true?*

Question 3.2 (Error-Detecting Open-locating dominating set (DET-OLD)). *A graph G and integer K with $7 \leq K \leq |V(G)|$. Is there a DET:OLD set S with $|S| \leq K$? Or equivalently, is $\text{DET:OLD}(G) \leq K$?*

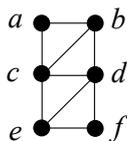


Fig. 4. Subgraph G_6

Lemma 3.3. *In the G_6 subgraph given in Figure 4, all six vertices internal to G_6 , as well as at least one external vertex adjacent to b must be detectors in order for DET:OLD to exist.*

Proof. Note that vertices a , b , and d are permitted to have any number of edges going to vertices outside of the G_6 subgraph under consideration, but no other edges incident with these 6 vertices are allowed. Suppose S is a DET:OLD for G . Firstly, to 2-dominate f , we require $\{e, d\} \subseteq S$. To distinguish e and f , we require $\{c, f\} \subseteq S$. To distinguish c and f , we require $\{a, b\} \subseteq S$. Finally, e and b are not distinguished unless b is adjacent to at least one detector external to this G_6 subgraph, completing the proof. \square

Theorem 3.4. *The DET-OLD problem is NP-complete.*

Proof. Clearly, DET-OLD is NP, as every possible candidate solution can be generated nondeterministically in polynomial time (specifically, $O(n)$ time), and each candidate can be verified in polynomial time using Theorem 1.1. To complete the proof, we will now show a reduction from 3-SAT to DET-OLD.

Let ψ be an instance of the 3-SAT problem with M clauses on N variables. We will construct a graph, G , as follows. For each variable x_i , create an instance of the F_i graph (Figure 5); this includes a vertex for x_i and its negation \bar{x}_i . For each $i \in \{1, \dots, N\}$, let $\{\bar{x}_i x_k, \bar{x}_i \bar{x}_k\} \subseteq E(G)$ for $k = (i \bmod N) + 1$. For each clause c_j of ψ , create a new instance of the H_j graph (Figure 5). For each clause $c_j = \alpha \vee \beta \vee \gamma$, create an edge from the y_j vertex to α , β , and γ from the variable graphs, each of which is either some x_i or \bar{x}_i ; for an example, see Figure 6. The resulting graph has precisely $8N + 6M$ vertices and $16N + 12M$ edges, and can be constructed in polynomial time. To complete the problem instance, we define $K = 7N + 6M$.

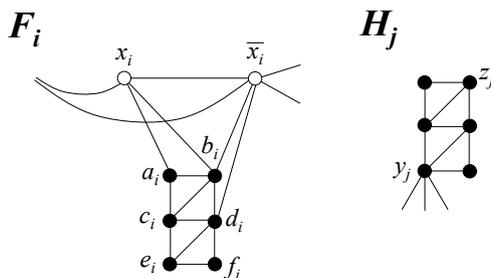


Fig. 5. Variable and clause graphs

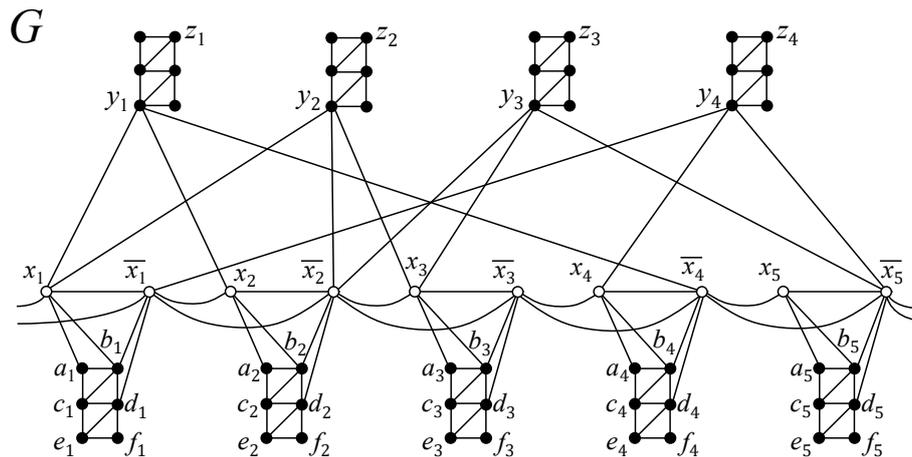


Fig. 6. Construction of G from $(x_1 \vee x_2 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_5) \wedge (\bar{x}_1 \vee x_4 \vee \bar{x}_5)$ with $N = 5$, $M = 4$, $K = 59$

Suppose $S \subseteq V(G)$ is a DET:OLD on G with $|S| \leq K$. By Lemma 3.3, we require at least the $6N + 6M$ detectors shown by the shaded vertices in Figure 5. Additionally, Lemma 3.3 gives us the additional requirement that each b_i vertex must be dominated by at least one additional detector outside of its G_6 subgraph; thus, for each i , $\{x_i, \bar{x}_i\} \cap S \neq \emptyset$, giving us at least N additional detectors. Thus, we find that $|S| \geq 7N + 6M = K$, implying that $|S| = K$, so $|\{x_i, \bar{x}_i\} \cap S| = 1$ for each $i \in \{1, \dots, N\}$. Applying Lemma 3.3 again to the clause graphs yields that each y_j vertex must be dominated by at least one additional detector outside of its G_6 subgraph. As no more detectors may be added, it must be that each y_j is now dominated by one of its three neighbors in the F_i graphs; therefore, ψ is satisfiable.

For the converse, suppose we have a solution to the 3-SAT problem ψ ; we will show that there is a DET:OLD, S , on G with $|S| \leq K$. We construct S by first including all of the $6N + 6M$ vertices as shown in Figure 5. Next, for each variable, x_i , if x_i is true then we let the vertex $x_i \in S$; otherwise, we let $\bar{x}_i \in S$. Thus, the fully-constructed S has $|S| = 7N + 6M = K$. Each b_i has its required external dominator due to having $x_i \in S$ or $\bar{x}_i \in S$. Additionally, because S was constructed from a satisfying truth assignment for the 3-SAT problem, by hypothesis each y_j vertex must also be dominated by one of its (external) term vertices in the F_i subgraphs. Therefore, each G_6 subgraph in G satisfies Lemma 3.3, and so are internally sufficiently dominated and distinguished. Since all G_6 subgraphs are sufficiently far apart, it is also the case that all vertex pairs in distinct G_6 subgraphs are distinguished. Indeed, it can be shown that all vertices are 2-dominated and $2^\#$ -distinguished, so S is a DET:OLD set for G with $|S| \leq K$, completing the proof. \square

4. DET:OLD sets for infinite grids

In this section we consider minimum-sized DET:OLD sets on the infinite hexagonal grid (HEX), the infinite square grid (SQR), the infinite triangular grid (TRI), and the infinite king grid (KNG). Figure 7 presents the best constructions that have been discovered so far.

Solutions (b), (c), and (f) are new solutions, while the others were first presented in other papers [8, 14]. The solutions for HEX, SQR, and TRI are tight bounds, while the exact value for KNG is currently unknown. Subfigure (c) gives the best (i.e., lowest-density) solution we have constructed for KNG.

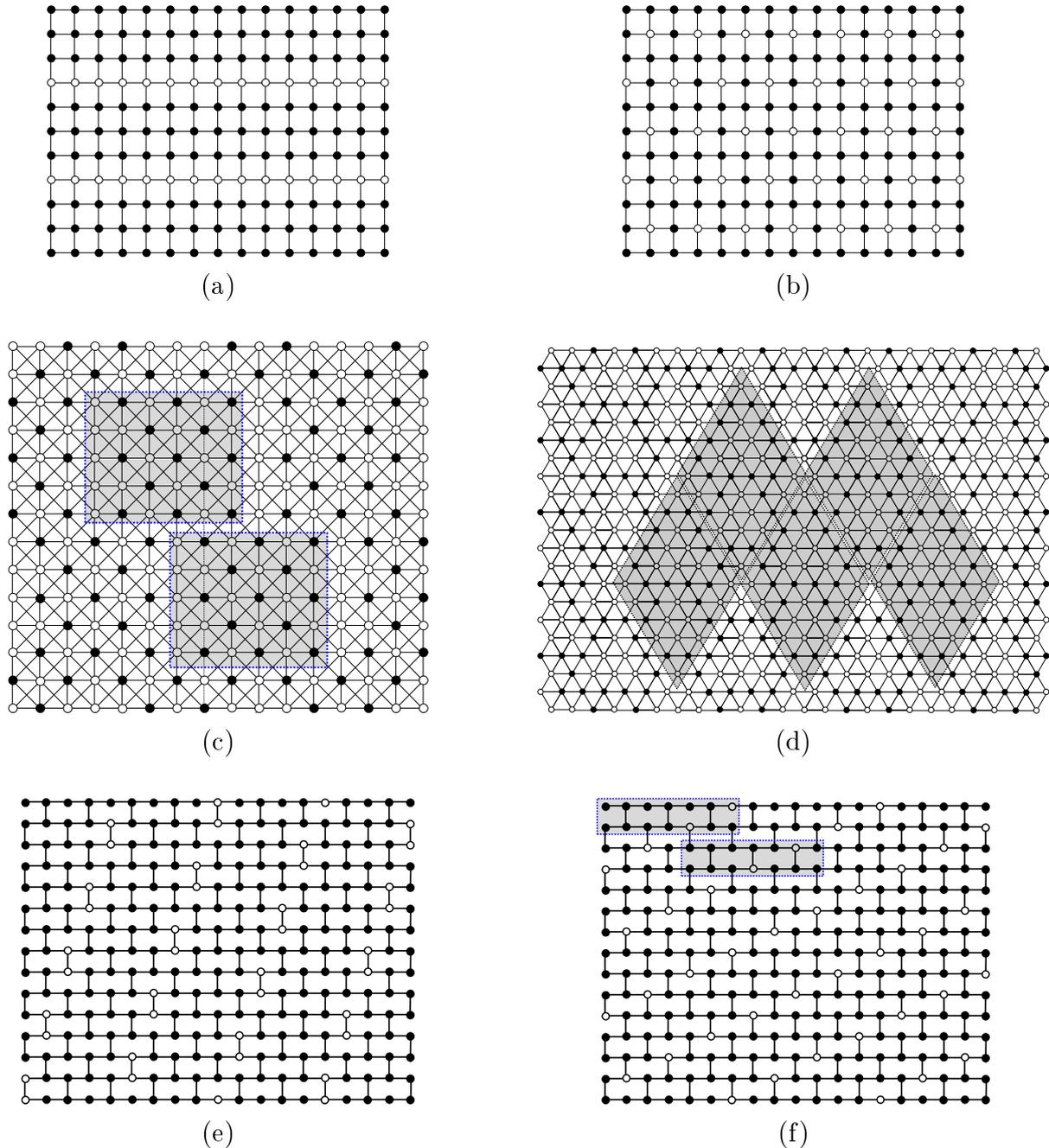


Fig. 7. Our best constructions of DET:OLD on SQR (a) and (b), KNG (c), TRI (d), and HEX (e) and (f). Shaded vertices denote detectors

To establish lower bounds for the minimum densities of domination-based parameters in infinite graphs, we employ a concept known as a *share argument*, originally introduced by Slater [16] and extensively utilized by other literature [8, 11]. Essentially, this technique inverts the problem: instead of directly seeking a lower bound for the density of a

DET:OLD set, we determine an upper bound on the average *share* of a detector vertex $v \in S$, denoted $sh(v) = \sum_{u \in N(v)} \frac{1}{dom(u)}$, representing its total contribution to the domination of its neighbors. Because S is an open-dominating set, every vertex is at least 1-open-dominated, and each vertex with $dom(u) = k$ contributes precisely $\frac{1}{k}$ to the share of each of its k dominators, for a total of 1 per vertex in $V(G)$. Therefore, an upper bound for the average share of all detectors can be inverted to yield a lower bound for the density of S in $V(G)$.

As an example of how share could be used to produce a lower bound for $DET:OLD\%(G)$, suppose G is k -regular and let $v \in S$ be an arbitrary detector vertex in some DET:OLD set $S \subseteq V(G)$. We know that v has k neighbors and each neighbor must be at least 2-dominated due to S being a DET:OLD set. Therefore, $sh(v) \leq \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} + \frac{1}{2} = \frac{k}{2}$. Because v was selected arbitrarily, this value of $\frac{k}{2}$ serves as an upper bound for the average share of all detectors, and so its inverse, $\frac{2}{k}$, represents a lower bound for the minimum density of a DET:OLD set in G . Because KNG is 8-regular, we get a trivial lower bound of $DET:OLD\%(KNG) \geq \frac{1}{4}$; next, we will present a proof for a better lower bound. In the proof, as a shorthand, we will let σ_A denote $\sum_{k \in A} \frac{1}{k}$ for some sequence of single-character symbols, A . For example, $\sigma_a = \frac{1}{a}$, $\sigma_{ab} = \frac{1}{a} + \frac{1}{b}$, and so on.

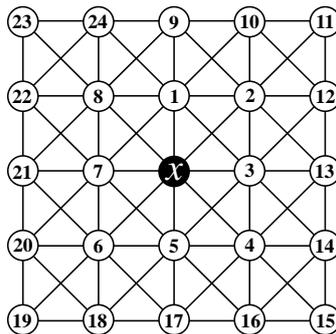


Fig. 8. KNG labeling scheme. Vertices v_1, v_2, \dots, v_{24} are labeled as $1, 2, \dots, 24$

Theorem 4.1. *The infinite king grid, KNG, has $DET:OLD\%(KNG) \geq \frac{3}{8}$.*

Proof. We will use the labeling shown in Figure 8. Let $S \subseteq V(G)$ be a DET:OLD set and $x \in S$. We will show that $sh(x) \leq \frac{8}{3}$, which would complete the proof. Firstly, we know that each vertex must be at least 2-dominated. If there are two distinct vertices $p, q \in N(x)$ with $dom(p) = dom(q) = 2$, then p and q cannot be distinguished; therefore, we may assume that there is at most one 2-dominated vertex in $N(x)$. Suppose that $dom(v_1) \leq 3$ and $dom(v_2) \leq 3$ and $dom(v_3) \leq 3$. By inspecting v_1 and v_2 , we see that if $(N_S(v_1) \cap N_S(v_2)) - \{x\} \neq \emptyset$, then v_1 and v_2 cannot be distinguished; thus, we can assume that $\{v_3, v_9, v_{10}\} \cap S = \emptyset$. By applying this same logic to v_2 and v_3 , we arrive at $\{v_1, v_{12}, v_{13}\} \cap S = \emptyset$. And by applying this logic a final time with v_1 and v_3 , we have $v_2 \notin S$. We see that vertex v_2 can only be 2-dominated, so it must be the (only) 2-dominated vertex in $N(x)$. Now suppose that $dom(v_4) = dom(v_5) = 3$. By applying similar pairwise reasoning to vertices in $\{v_3, v_4, v_5\}$, we find that $dom(v_4) =$

2, a contradiction. Otherwise, we know that $\text{dom}(v_4) \geq 4$ or $\text{dom}(v_5) \geq 4$, and by symmetry we also have that $\text{dom}(v_7) \geq 4$ or $\text{dom}(v_8) \geq 4$. In any case, we see that $sh(x) \leq \sigma_{24433333} = \frac{1}{2} + \frac{2}{4} + \frac{5}{3} = \frac{8}{3}$, and we would be done. Otherwise, we know that vertices $\{v_1, v_2, v_3\}$ must contain at least one vertex which is at least 4-dominated, and by symmetry for any $A \in \{\{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_1, v_7, v_8\}\}$, A must contain at least one vertex which is at least 4-dominated. These four ‘‘corner’’ regions may be satisfied by as few as two 4-dominated vertices, resulting in $sh(x) \leq \sigma_{24433333} = \frac{8}{3}$ and completing the proof. \square

The following theorem summarizes upper and lower bounds for DET:OLD on the four infinite grids we have explored.

Theorem 4.2. *The upper and lower bounds on DET:OLD:*

- i. *For the infinite hexagonal grid HEX, $\text{DET:OLD}\%(HEX) = \frac{6}{7}$. (Seo and Slater [14])*
- ii. *For the infinite square grid SQR, $\text{DET:OLD}\%(SQR) = \frac{3}{4}$. (Seo and Slater [14])*
- iii. *For the infinite triangular grid TRI, $\text{DET:OLD}\%(TRI) = \frac{1}{2}$. (Jean and Seo [8])*
- iv. *For the infinite king grid KNG, $\frac{3}{8} \leq \text{DET:OLD}\%(KNG) \leq \frac{13}{30}$. (Theorem 4.1 and Figure 7 (c))*

5. Extremal cubic graphs

In this section, we characterize cubic graphs that permit a DET:OLD set. We also consider extremal cubic graphs on DET:OLD(G).

Observation 5.1. *A cubic graph G has DET:OLD if and only if G is C_4 -free.*

Proof. Let $abcd$ be a 4-cycle in G . We see that a and c cannot be distinguished, a contradiction. For the converse, we will show that $S = V(G)$ is a DET:OLD for a C_4 -free cubic graph. Let $u, v \in V(G)$; we know that $|N(u) \cap N(v)| \leq 1$ because G is C_4 -free. Thus, $|N(u) - N(v)| \geq 2$, implying u is distinguished from v , completing the proof. \square

Definition 5.2. For $v \in V(G)$ and $r \in \mathbb{N}_0$, let $T_r(v)$ denote the set of terminal vertices among all trails (i.e., walks which forbid repeated edges) of length r starting from v .

Observation 5.3. *Let S be a DET:OLD set on cubic graph G . For all vertices $v \in V(G) - S$, we have $T_2(v) \cup T_4(v) \subseteq S$.*

Proof. Assume to the contrary that there exist $u, v \in V(G) - S$ such that $u \in T_2(v) \cup T_4(v)$. We know that $u \neq v$ because G is C_4 -free due to the existence of DET:OLD. We consider the three non-isomorphic ways to form a length 2 or 4 trail from u to v as illustrated in Figure 9. We see that in Figure 9 (a), x cannot be 2-dominated, in (b), x and y cannot be distinguished, and in (c), v and x cannot be distinguished. All three cases contradict the existence of S , completing the proof. \square

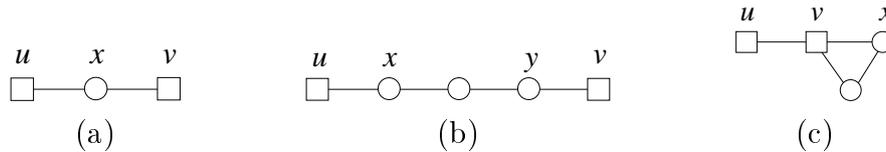


Fig. 9. The three non-isomorphic constructions of length 2 and 4 trails from u to v

Theorem 5.4. *Let G be a C_4 -free cubic graph, and let $\bar{S} \subseteq V(G)$ such that for all distinct $u, v \in \bar{S}$, $u \notin T_2(v) \cup T_4(v)$. Then $S = V(G) - \bar{S}$ is a DET:OLD on G .*

Proof. Let $u \in V(G)$. We see that for any distinct $x, y \in N(u)$, $x \in T_2(y)$. Thus, by assumption it must be that $|N(u) \cap \bar{S}| \leq 1$, implying that $\text{dom}(u) \geq 2$. We now know that all vertices are at least 2-dominated by S . Next, we consider the following three cases depending on the distance between a pair of vertices and show the pair is $2^\#$ -distinguished.

Case 1. Suppose $u, v \in V(G)$ with $d(u, v) = 1$.

Because G is twin-free (due to being a C_4 -free cubic graph) and regular, it must be that $\exists x \in N(u) - N[v]$ and $\exists y \in N(v) - N[u]$. Suppose $\exists w \in N(u) \cap N(v)$. Then we see that $\forall p \in N[u] \cup N[v]$, $(N[u] \cup N[v]) - \{p\} \subseteq T_2(p) \cup T_4(p)$, implying that $|(N[u] \cup N[v]) \cap \bar{S}| \leq 1$, from which it can be seen that u and v must be distinguished. Otherwise, we can assume that $N(u) \cap N(v) = \emptyset$. If $u \in \bar{S}$, then $N(v) - N[u] \subseteq T_2(u) \subseteq S$, so v is distinguished from u and we would be done; otherwise we assume $u \in S$ and by symmetry $v \in S$. We see that $|(N(u) - N[v]) \cap \bar{S}| \leq 1$, so u is distinguished from v .

Case 2. Suppose $u, v \in V(G)$ with $d(u, v) = 2$, and let $N(u) \cap N(v) = \{w\}$, as G is C_4 -free.

Let $N(u) - N[v] = \{u', u''\}$ and let $N(v) - N[u] = \{v', v''\}$. We see that $\forall p \in \{u', u'', v', v''\}$, $\{u', u'', v', v''\} - \{p\} \subseteq T_2(p) \cup T_4(p) \subseteq S$, which implies that $|\{u', u'', v', v''\} \cap \bar{S}| \leq 1$. Therefore, u and v must be distinguished.

Case 3. Suppose $u, v \in V(G)$ with $d(u, v) \geq 3$, and let $N(u) = \{u', u'', u'''\}$ and $N(v) = \{v', v'', v'''\}$.

We see that for any $p \in N(u)$, $\{u', u'', u'''\} - \{p\} \subseteq T_2(p) \subseteq S$. Thus u and v must be distinguished, completing the proof. \square

From Observations 5.1 and 5.3 and Theorem 5.4, we have the following full characterization of DET:OLD in cubic graphs.

Corollary 5.5. *Let G be a cubic graph and $S \subseteq V(G)$. Then S is a DET:OLD if and only if G is C_4 -free and for all distinct $u, v \in V(G) - S$, $u \notin T_2(v) \cup T_4(v)$.*

The upper bound on DET:OLD(G) for cubic graphs is known to be $\frac{45}{46}n$ [12], and we can improve it using Corollary 5.5.

Theorem 5.6. *If G is a cubic graph that permits DET:OLD, then $\text{DET:OLD}\%(G) \leq \frac{30}{31}$.*

Proof. We will show that we can construct a set \bar{S} with the property that $\forall v \in V(G)$, $\exists u \in \bar{S}$ such that $v \in T_0(u) \cup T_2(u) \cup T_4(u)$. Because $|T_0(u)| = 1$, $|T_2(u)| \leq 6$, and $|T_4(u)| \leq$

24, this construction will result in a detector set $S = V(G) - \bar{S}$ with density at most $\frac{6+24}{1+6+24} = \frac{30}{31}$. Assume to the contrary that we have a maximal \bar{S} set, but $\exists x \in V(G)$ such that $x \notin T_0(u) \cup T_2(u) \cup T_4(u)$ for any $u \in \bar{S}$, implying that $x \notin \bar{S}$. Then by Corollary 5.5, we see that $\bar{S} \cup \{x\}$ still satisfies the requirements of our \bar{S} set, contradicting maximality of \bar{S} . Therefore, we have a DET:OLD set S with density at most $\frac{30}{31}$, completing the proof. \square

Corollary 5.7. *If G is a cubic graph that permits DET:OLD, then $\text{DET:OLD}(G) \leq n-1$.*

Figure 10 shows extremal cubic graphs with the highest density on n vertices for $16 \leq n \leq 24$. The $n = 22$ graph shown in Figure 10 has the highest density we have found so far of $\frac{21}{22}$, and we conjecture the density $\frac{21}{22}$ is the tight upper bound for all cubic graphs.

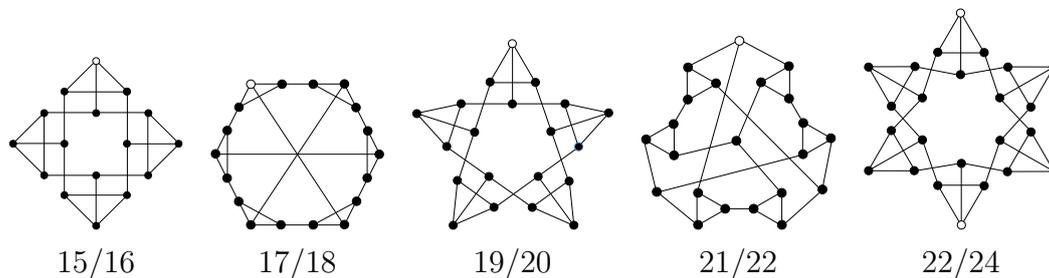


Fig. 10. Extremal cubic graphs on n vertices with the highest density for $16 \leq n \leq 24$

6. Conclusion and future direction

In this paper, we have explored various results for minimum-density error-detecting open-locating-dominating (DET:OLD) sets in several classes of graphs. In Theorem 2.3, we discovered that any graph permitting DET:OLD must have $m \geq \left\lceil \frac{3n - \lfloor \frac{n}{2} \rfloor}{2} \right\rceil$, and we have constructed an infinite family of graphs on $n \geq 9$ vertices (shown in Figure 3) which achieves this extremal minimum number of edges. In Theorem 3.4, we found that the decision problem of determining if a general graph permits a DET:OLD of a certain maximum size is NP-complete. Future work could determine if there are any particular classes of graphs for which this problem can be solved in polynomial time. For infinite grids, the minimum density of DET:OLD on HEX, SQR, and TRI is already known; in this paper, we have improved the bounds for KNG to $\frac{3}{8} \leq \text{DET:OLD}\%(KNG) \leq \frac{13}{30}$. Future research may further reduce this gap or determine the exact value of $\text{DET:OLD}\%(KNG)$.

For cubic graphs, Observation 5.1 revealed that DET:OLD exists if and only if G is C_4 -free. In Theorem 5.6, we demonstrated that any cubic graph permitting DET:OLD must have $\text{DET:OLD}\%(G) \leq \frac{30}{31}$. In Figure 10, we have provided a collection of cubic graphs on $16 \leq n \leq 24$ vertices with the highest minimum density of DET:OLD. Future work could extend this to an infinite family and discover whether the $\frac{30}{31}$ upper bound is tight. However, from computer-assisted exploration, we believe this $\frac{30}{31}$ bound is not tight and instead propose the following conjecture:

Conjecture 6.1. *If G is a cubic graph permitting DET:OLD, then $\text{DET:OLD}\%(G) \leq \frac{20}{21}$.*

References

- [1] D. Auger, I. Charon, O. Hudry, and A. Lobstein. Watching systems in graphs: an extension of identifying codes. *Discrete Applied Mathematics*, 161(12):1674–1685, 2013. <https://doi.org/10.1016/j.dam.2011.04.025>.
- [2] I. Charon, G. Cohen, O. Hudry, and A. Lobstein. Links between discriminating and identifying codes in the binary hamming space. *Lecture Notes in Computer Science*:267–270, 2007.
- [3] I. Charon, O. Hudry, and A. Lobstein. Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard. *Theoretical Computer Science*, 290:2109–2120, 2003. [https://doi.org/10.1016/S0304-3975\(02\)00536-4](https://doi.org/10.1016/S0304-3975(02)00536-4).
- [4] G. Cohen, I. Honkala, A. Lobstein, and G. Zémor. On identifying codes. In *Proceedings of DIMACS Workshop on Codes and Association Schemes '99*, volume 56, pages 97–109, Piscataway, USA, 2001.
- [5] C. Colbourn, P. Slater, and L. Stewart. Locating dominating sets in series parallel networks. *Congressus Numerantium*, 56:135–162, 1987.
- [6] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman, San Francisco, 1979.
- [7] D. Jean and A. Lobstein. Watching systems, identifying, locating-dominating and discriminating codes in graphs. <https://dragazo.github.io/bibdom/main.pdf>, 2025.
- [8] D. Jean and S. Seo. Optimal error-detecting open-locating-dominating set on the infinite triangular grid. *Discussiones Mathematicae Graph Theory*, 43(2):445–455, 2023. <https://doi.org/10.7151/dmgt.2374>.
- [9] M. Karpovsky, K. Chakrabarty, and L. Levitin. On a new class of codes for identifying vertices in graphs. *IEEE Transactions on Information Theory*, IT-44:599–611, 1998. <https://doi.org/10.1109/18.661507>.
- [10] R. Sampaio, G. Sobral, and Y. Wakabayashi. Density of identifying codes of hexagonal grids with finite number of rows. *RAIRO-Operations Research*, 58(2):1633–1651, 2024. <https://doi.org/10.1051/ro/2024046>.
- [11] S. Seo. Open-locating-dominating sets in the infinite king grid. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 104:31–47, 2018.
- [12] S. Seo. Fault-tolerant detectors for distinguishing sets in cubic graphs. *Discrete Applied Mathematics*, 293:25–33, 2021. <https://doi.org/10.1016/j.dam.2021.01.008>.
- [13] S. Seo and P. Slater. Open neighborhood locating-dominating sets. *Australasian Journal of Combinatorics*, 46:109–119, 2010.
- [14] S. Seo and P. Slater. Fault tolerant detectors for distinguishing sets in graphs. *Discussiones Mathematicae Graph Theory*, 35:797–818, 2015. <https://doi.org/10.7151/dmgt.1838>.
- [15] P. Slater. Domination and location in acyclic graphs. *Networks*, 17:55–64, 1987. <https://doi.org/10.1002/net.3230170105>.

-
- [16] P. Slater. Fault-tolerant locating-dominating sets. *Discrete Mathematics*, 249:179–189, 2002.
[https://doi.org/10.1016/S0012-365X\(01\)00244-8](https://doi.org/10.1016/S0012-365X(01)00244-8).

Devin C. Jean

Computer Science Department, Middle Tennessee State University

E-mail devin.jean@mtsu.edu

Suk J. Seo

Computer Science Department, Middle Tennessee State University

E-mail Suk.Seo@mtsu.edu