



# Regular pandiagonal sparse magic squares of order $n \equiv 5 \pmod{6}$ with density 6

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## ABSTRACT

Sparse magic squares are a generalization of magic squares and can be used to the magic labeling of graphs. An  $n \times n$  array based on  $\mathcal{X} = \{0, 1, \dots, nd\}$  is called a *sparse magic square of order  $n$  with density  $d$*  ( $d < n$ ), denoted by  $\text{SMS}(n, d)$ , if each non-zero element of  $\mathcal{X}$  occurs exactly once in the array, and its row-sums, column-sums and two main diagonal sums is the same. An  $\text{SMS}(n, d)$  is called *pandiagonal* (or *perfect*) denoted by  $\text{PSMS}(n, d)$ , if the sum of all elements in each broken diagonal is the same. A  $\text{PSMS}(n, d)$  is called *regular* if there are exactly  $d$  positive entries in each row, each column and each main diagonal. In this paper, some construction of regular pandiagonal sparse magic squares is provided and it is proved that there exists a regular  $\text{PSMS}(n, 6)$  for all positive integer  $n \equiv 5 \pmod{6}$ ,  $n > 6$ .

*Keywords:* magic square, sparse magic square, pandiagonal, regular

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## 1. Introduction

A *magic square of order  $n$*  is an  $n \times n$  array with entries consisting of  $n^2$  consecutive nonnegative integers such that the sum of elements in each row, each column and each main diagonal is the same. The sum is called *the magic number* or *magic sum*. Usually, the main diagonal from upper left to lower right is called *the left diagonal*, another is called *the right diagonal*.

Magic squares and their various generalizations have been objects of interest for many centuries and in many cultures. A lot of work had been done on the constructions of magic squares, for more details, the interested reader may refer to [1, 2, 3, 12] and the

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references therein.

Sparse magic squares are a generalization of magic squares. Let  $n$  and  $d$  be two positive integers with  $d < n$ ,  $A$  be an  $n \times n$  array over  $\mathcal{X} = \{0, 1, \dots, nd\}$ . The array  $A$  is called a *sparse magic square of order  $n$  with density  $d$* , denoted by  $\text{SMS}(n, d)$ , if each non-zero element of  $\mathcal{X}$  occurs exactly once in  $A$ , the sum of elements in each row, each column, and each main diagonal is the same. An  $\text{SMS}(n, d)$  is called *regular* if there are exactly  $d$  non-zero elements in each row, each column and each main diagonal.

A sparse magic square can be used to the magic labeling of graphs, see [4, 5, 6, 7, 8, 9, 10, 11, 14]. The existence of a regular  $\text{SMS}(n, d)$  has been solved completely by Li et al. [13].

**Lemma 1.1.** ([13]) *There exists a regular  $\text{SMS}(n, d)$  if and only if  $d \geq 3$  when  $n \equiv 1 \pmod{2}$ , or  $d \geq 4$  and  $d \equiv 0 \pmod{2}$  when  $n \equiv 0 \pmod{2}$ .*

A sparse magic square  $A = (a_{i,j})$  of order  $n$  with density  $d$  is called *pandiagonal* (or *perfect*), denoted by  $\text{PSMS}(n, d)$  in short, if the sum of all elements in each broken diagonal is the same, i.e., for each  $0 \leq j \leq n-1$ ,  $\sum_{i=0}^{n-1} a_{i,j+i}$  and  $\sum_{i=0}^{n-1} a_{i,j-i}$  is the same, where the additions in the subscripts are all taken modulo  $n$ . As an example, a regular  $\text{PSMS}(11, 6)$  is listed in Example 2.3 in Section 3.

In this paper, we investigate the existence of a regular pandiagonal sparse magic square of order  $n \equiv 5 \pmod{6}$  with density 6. We shall prove the following.

**Theorem 1.2.** *There exists a regular  $\text{PSMS}(n, 6)$  for all positive integer  $n \equiv 5 \pmod{6}$  and  $n > 6$ .*

In Section 2, a construction of regular pandiagonal sparse magic squares will be given. The proof of Theorem 1.2 will be given in Section 3.

## 2. A construction of regular $\text{PSMS}(n, 6)$ with $n \equiv 5 \pmod{6}$

In this section, we shall provide a construction of regular pandiagonal sparse magic squares, in which a row latin matrix with some special properties will be useful. Let  $m, n$  be positive integers. An  $m \times n$  matrix  $A = (a_{i,j})$  based on  $Z_n$  is called *row latin matrix* if each row of  $A$  is a permutation of  $Z_n$ .

Let  $n$  be a positive integer,  $n \equiv 5 \pmod{6}$ . A row latin matrix  $A = (a_{i,j})_{3 \times n}$  based on  $Z_n$  is called *S-array* if the following conditions hold:

(i)

$$\sum_{i=0}^2 a_{i,j} = \sum_{i=0}^2 a_{2-i, n-1-j}, \quad 0 \leq j < (n-1)/2.$$

(ii)

$$\sum_{i=0}^2 a_{i,j+i} = \sum_{i=0}^2 a_{2-i, \frac{2n-4}{3}-j-i}, \quad 0 \leq j \leq n-1.$$

(iii)

$$\sum_{i=0}^2 a_{i,j-i} = \sum_{i=0}^2 a_{2-i, \frac{n-2}{3}+i-j}, \quad 0 \leq j \leq n-1.$$

(iv)

$$\sum_{i=0}^2 a_{i, \frac{j-i}{3}} = \sum_{i=0}^2 a_{2-i, \frac{n-8}{9} - \frac{j-i}{3}}, \quad 0 \leq j \leq n-1.$$

**Example 2.1.** There exists a  $3 \times 11$  S-array.

**Proof.** Let

$$A = \begin{pmatrix} 8 & 6 & 4 & 2 & 0 & 9 & 7 & 5 & 3 & 1 & 10 \\ 5 & 3 & 1 & 10 & 8 & 6 & 4 & 2 & 0 & 9 & 7 \\ 4 & 8 & 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 0 \end{pmatrix}.$$

Clearly,  $A$  is a  $3 \times 11$  row latin matrix based on  $Z_{11}$ . It is readily checked that the conditions (i)-(iv) hold. So,  $A$  is a  $3 \times 11$  S-array.  $\square$

An S-array can be used to construct a regular pandiagonal sparse magic square. We have the following.

**Lemma 2.2.** *Let  $n$  be a positive integer,  $n > 6$  and  $n \equiv 5 \pmod{6}$ . If there exists a  $3 \times n$  S-array based on  $Z_n$ , then there exists a regular PSMS( $n, 6$ ).*

**Proof.** Suppose that  $A = (a_{i,j})_{3 \times n}$  is a  $3 \times n$  S-array based on  $Z_n$ . Let  $k = \frac{n+1}{3}$ . We shall construct a regular PSMS( $n, 6$ ) by the following three steps.

*Step 1.* Construct an  $n \times n$  matrix  $B = (b_{i,j})$  based on  $Z_n$  such that for each  $j$ ,  $0 \leq j \leq n-1$ ,

$$\sum_{i=0}^{n-1} b_{i,j} = \sum_{i=0}^{n-1} b_{i,j+i} = \sum_{i=0}^{n-1} b_{i,j-i} = \sum_{i=0}^{n-1} b_{i, \frac{j-i}{3}} = 3(n-1).$$

Let  $B = (b_{i,j})_{n \times n}$ , where

$$\begin{aligned} b_{i,j} &= a_{i,j}, \quad 0 \leq i \leq 2, \quad 0 \leq j \leq n-1, \\ b_{k+i,j} &= n-1 - a_{2-i, n-1-j}, \quad 0 \leq i \leq 2, \quad 0 \leq j \leq n-1, \\ b_{i,j} &= 0, \quad i \in Z_n \setminus \{0, 1, 2, k, k+1, k+2\}, \quad 0 \leq j \leq n-1. \end{aligned}$$

It is readily checked that  $B = (b_{i,j})$  is an  $n \times n$  matrix based on  $Z_n$ . Note that the conditions (i)-(iv) hold, we have

$$\begin{aligned} \sum_{i=0}^{n-1} b_{i,j} &= \sum_{i=0}^2 b_{i,j} + \sum_{i=0}^2 b_{k+i,j} = \sum_{i=0}^2 a_{i,j} + \sum_{i=0}^2 (n-1 - a_{2-i, n-1-j}) \\ &= 3(n-1) + \sum_{i=0}^2 a_{i,j} - \sum_{i=0}^2 a_{2-i, n-1-j} = 3(n-1). \end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{n-1} b_{i,j+i} &= \sum_{i=0}^2 b_{i,j+i} + \sum_{i=0}^2 b_{k+i,j+k+i} = \sum_{i=0}^2 a_{i,j+i} + \sum_{i=0}^2 (n-1 - a_{2-i,n-1-(j+k+i)}) \\
&= 3(n-1) + \sum_{i=0}^2 a_{i,j+i} - \sum_{i=0}^2 a_{2-i,\frac{2n-4}{3}-j-i} = 3(n-1). \\
\sum_{i=0}^{n-1} b_{i,j-i} &= \sum_{i=0}^2 b_{i,j-i} + \sum_{i=0}^2 b_{k+i,j-k-i} = \sum_{i=0}^2 a_{i,j-i} + \sum_{i=0}^2 (n-1 - a_{2-i,n-1-(j-k-i)}) \\
&= 3(n-1) + \sum_{i=0}^2 a_{i,j-i} - \sum_{i=0}^2 a_{2-i,\frac{n-2}{3}-j} = 3(n-1). \\
\sum_{i=0}^{n-1} b_{i,\frac{j-i}{3}} &= \sum_{i=0}^2 b_{i,\frac{j-i}{3}} + \sum_{i=0}^2 b_{k+i,\frac{j-k-i}{3}} = \sum_{i=0}^2 a_{i,\frac{j-i}{3}} + \sum_{i=0}^2 (n-1 - a_{2-i,n-1-\frac{j-k-i}{3}}) \\
&= 3(n-1) + \sum_{i=0}^2 a_{i,\frac{j-i}{3}} - \sum_{i=0}^2 a_{2-i,\frac{n-8}{9}-\frac{j-i}{3}} = 3(n-1).
\end{aligned}$$

*Step 2.* Construct an  $n \times n$  matrix  $D = (d_{i,j})$  based on  $\{0, 1, 2, \dots, 6n\}$  such that for each  $0 \leq j \leq n-1$ ,

$$\sum_{i=0}^{n-1} d_{i,j} = \sum_{i=0}^{n-1} d_{i,j+i} = \sum_{i=0}^{n-1} d_{i,j-i} = \sum_{i=0}^{n-1} d_{i,\frac{j-i}{3}} = 18n + 3.$$

Let  $C = (c_{i,j})_{n \times n}$ , where

$$\begin{aligned}
c_{i,j} &= i + 1, 0 \leq i \leq 2, 0 \leq j \leq n-1, \\
c_{k+i,j} &= i + 4, 0 \leq i \leq 2, 0 \leq j \leq n-1, \\
c_{i,j} &= 0, i \in Z_n \setminus \{0, 1, 2, k, k+1, k+2\}, 0 \leq j \leq n-1.
\end{aligned}$$

Let  $D = (d_{i,j}) = 6B + C$ , where  $d_{i,j} = 6b_{i,j} + c_{i,j}$ ,  $0 \leq i, j \leq n-1$ . Obviously,  $D = (d_{i,j})$  is an  $n \times n$  matrix based on  $\{0, 1, 2, \dots, 6n\}$  and it is readily checked that

$$\begin{aligned}
\sum_{i=0}^{n-1} d_{i,j} &= 6 \sum_{i=0}^{n-1} b_{i,j} + \sum_{i=0}^{n-1} c_{i,j} = 6 \cdot 3(n-1) + 21 = 18n + 3, \\
\sum_{i=0}^{n-1} d_{i,j+i} &= 6 \sum_{i=0}^{n-1} b_{i,j+i} + \sum_{i=0}^{n-1} c_{i,j+i} = 6 \cdot 3(n-1) + 21 = 18n + 3, \\
\sum_{i=0}^{n-1} d_{i,j-i} &= 6 \sum_{i=0}^{n-1} b_{i,j-i} + \sum_{i=0}^{n-1} c_{i,j-i} = 6 \cdot 3(n-1) + 21 = 18n + 3, \\
\sum_{i=0}^{n-1} d_{i,\frac{j-i}{3}} &= 6 \sum_{i=0}^{n-1} b_{i,\frac{j-i}{3}} + \sum_{i=0}^{n-1} c_{i,\frac{j-i}{3}} = 6 \cdot 3(n-1) + 21 = 18n + 3.
\end{aligned}$$

*Step 3.* Construct an  $n \times n$  matrix  $E = (e_{i,j})$  by putting  $d_{i,j}$  in the position  $(i+j, 2j)$ ,  $0 \leq i, j \leq n-1$ , i.e.,  $e_{i+j,2j} = d_{i,j}$  or  $e_{i,j} = d_{i-\frac{j}{2},\frac{j}{2}}$ ,  $0 \leq i, j \leq n-1$ . We shall show that  $E$  is a regular PSMS( $n, 6$ ).

It is easy to see that there are exactly 6 non-zero elements in each row, each column and each main diagonal of  $E$ . In fact, for each  $j$ ,  $0 \leq j \leq n-1$ ,  $j$ -th column of  $E$  is just  $\frac{j}{2}$ -th column of  $D$ , so there are exactly 6 non-zero elements in each column of  $E$ .

Evidently, for each  $i$ ,  $0 \leq i \leq n-1$ ,

$$\begin{aligned} e_{i,j} \neq 0 &\iff d_{i-\frac{j}{2},\frac{j}{2}} \neq 0 \iff (i - \frac{j}{2}) \pmod{n} \in \{0, 1, 2, k, k+1, k+2\} \\ &\iff j \in \{2i, 2(i-1), 2(i-2), 2(i-k), 2(i-k-1), 2(i-k-2)\} \pmod{n}, \end{aligned}$$

which means that there are exactly 6 non-zero elements in each row of  $E$ . Similarly, one can check that there are exactly 6 non-zero elements in each main diagonal of  $E$ .

Now we show that the sum of all elements in each row, in each column and in each broken diagonal of  $E$  is the same.

For each  $i$ ,  $0 \leq i \leq n-1$ , by the definition of  $E$ , we have

$$\sum_{j=0}^{n-1} e_{i,j} = \sum_{j=0}^{n-1} d_{i-\frac{j}{2},\frac{j}{2}} = \sum_{u=0}^{n-1} d_{u,i-u} = 18n + 3,$$

noting that  $\{u \mid u = i - \frac{j}{2}, 0 \leq j \leq n-1\} = Z_n$ .

For each  $j$ ,  $0 \leq j \leq n-1$ , we have

$$\sum_{i=0}^{n-1} e_{i,j} = \sum_{i=0}^{n-1} d_{i-\frac{j}{2},\frac{j}{2}} = \sum_{u=0}^{n-1} d_{u,\frac{j}{2}} = 18n + 3,$$

noting that  $\{u \mid u = i - \frac{j}{2}, 0 \leq i \leq n-1\} = Z_n$ .

$$\sum_{i=0}^{n-1} e_{i,j+i} = \sum_{i=0}^{n-1} d_{i-\frac{j+i}{2},\frac{j+i}{2}} = \sum_{i=0}^{n-1} d_{\frac{i-j}{2},\frac{j+i}{2}} = \sum_{u=0}^{n-1} d_{u,j+u} = 18n + 3,$$

noting that  $\{u \mid u = \frac{i-j}{2}, 0 \leq i \leq n-1\} = Z_n$ .

$$\sum_{i=0}^{n-1} e_{i,j-i} = \sum_{i=0}^{n-1} d_{i-\frac{j-i}{2},\frac{j-i}{2}} = \sum_{u=0}^{n-1} d_{u,\frac{j-u}{3}} = 18n + 3,$$

noting that  $\{u \mid u = i - \frac{j-i}{2}, 0 \leq i \leq n-1\} = Z_n$ .

Thus,  $E$  is a regular PSMS( $n, 6$ ). □

To illustrate the construction described in the proof of Theorem 2.2, an example is given bellow.

**Example 2.3.** There exists a regular PSMS(11, 6).

**Proof.** Let  $A$  be a  $3 \times 11$  S-array based on  $Z_{11}$  listed in Example 2.1. By the method

described in the proof of Theorem 2.2, we take

$$B = \begin{pmatrix} 8 & 6 & 4 & 2 & 0 & 9 & 7 & 5 & 3 & 1 & 10 \\ 5 & 3 & 1 & 10 & 8 & 6 & 4 & 2 & 0 & 9 & 7 \\ 4 & 8 & 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 3 & 7 & 0 & 4 & 8 & 1 & 5 & 9 & 2 & 6 \\ 3 & 1 & 10 & 8 & 6 & 4 & 2 & 0 & 9 & 7 & 5 \\ 0 & 9 & 7 & 5 & 3 & 1 & 10 & 8 & 6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then

$$D = 6B + C = \begin{pmatrix} 49 & 37 & 25 & 13 & 1 & 55 & 43 & 31 & 19 & 7 & 61 \\ 32 & 20 & 8 & 62 & 50 & 38 & 26 & 14 & 2 & 56 & 44 \\ 27 & 51 & 9 & 33 & 57 & 15 & 39 & 63 & 21 & 45 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 64 & 22 & 46 & 4 & 28 & 52 & 10 & 34 & 58 & 16 & 40 \\ 23 & 11 & 65 & 53 & 41 & 29 & 17 & 5 & 59 & 47 & 35 \\ 6 & 60 & 48 & 36 & 24 & 12 & 66 & 54 & 42 & 30 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$E = (e_{i,j}) = (d_{i-\frac{j}{2}, \frac{j}{2}}) = \begin{pmatrix} 49 & 17 & 0 & 34 & 0 & 0 & 0 & 45 & 0 & 44 & 12 \\ 32 & 66 & 37 & 5 & 0 & 58 & 0 & 0 & 0 & 3 & 0 \\ 27 & 0 & 20 & 54 & 25 & 59 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 51 & 0 & 8 & 42 & 13 & 47 & 0 & 40 & 0 \\ 64 & 0 & 0 & 0 & 9 & 0 & 62 & 30 & 1 & 35 & 0 \\ 23 & 0 & 22 & 0 & 0 & 0 & 33 & 0 & 50 & 18 & 55 \\ 6 & 43 & 11 & 0 & 46 & 0 & 0 & 0 & 57 & 0 & 38 \\ 0 & 26 & 60 & 31 & 65 & 0 & 4 & 0 & 0 & 0 & 15 \\ 0 & 39 & 0 & 14 & 48 & 19 & 53 & 0 & 28 & 0 & 0 \\ 0 & 0 & 0 & 63 & 0 & 2 & 36 & 7 & 41 & 0 & 52 \\ 0 & 10 & 0 & 0 & 0 & 21 & 0 & 56 & 24 & 61 & 29 \end{pmatrix}.$$

It is readily checked that there are exactly 6 non-zero elements in each row, each column and each main diagonal of  $E$ , and its row-sums, column-sums, and broken diagonal-sums is 201. So  $E$  a regular PSMS(11, 6).  $\square$

### 3. The proof of Theorem 1.2

For integer  $n \equiv 5 \pmod{6}$ , by Lemma 2.2, to construct a regular PSMS( $n, 6$ ), it suffices to construct a  $3 \times n$  S-array on  $Z_n$ . In this section, we shall show that such a  $3 \times n$  S-array always exists whenever  $n > 6$ ,  $n \equiv 5 \pmod{6}$  and the proof of Theorem 1.2 is followed.

Let  $a, n$  be integers,  $[a]_n$  be the smallest nonnegative integer such that  $a \equiv [a]_n \pmod{n}$ , i.e.,  $[a]_n = r$  if  $a = kn + r$ , where  $k, r$  are integers and  $0 \leq r < n$ . Let  $a \in Z_n$ , if there exists  $b \in Z_n$ , such that  $ab \equiv 1 \pmod{n}$ , then  $b$  is called *the inverse of  $a$*  in  $Z_n$ , and denoted by  $[\frac{1}{a}]_n$ .

**Lemma 3.1.** *There exists a  $3 \times n$  S-array for any positive integer  $n \equiv 5 \pmod{6}$  and  $n > 6$ .*

**Proof.** Suppose that  $n = 6m + 5$ . We shall construct a  $3 \times n$  S-array based on  $Z_n$ . Let  $A = (a_{i,j})_{3 \times n}$ , where

$$\begin{aligned} a_{0,j} &= \left[ \frac{3j-1}{4} \right]_n, \quad 0 \leq j \leq n-1, \\ a_{1,j} &= \left[ \frac{3j-2}{4} \right]_n, \quad 0 \leq j \leq n-1 \\ a_{2,j} &= \left[ \frac{-3j-3}{2} \right]_n, \quad 0 \leq j \leq n-1. \end{aligned}$$

One can check directly that  $A$  is a  $3 \times n$  row Latin matrix and the conditions (i)-(iv) hold. In fact, for each  $i$ ,  $0 \leq i \leq 2$ , if  $a_{i,j} = a_{i,j'}$  for some  $j, j' \in Z_n$ , we have  $3j \equiv 3j' \pmod{n}$  by definition, which follows  $j = j'$  noting that  $\gcd(3, n) = 1$ . So, each row of  $A$  is a permutation of  $Z_n$ , i.e.,  $A$  is a  $3 \times n$  row latin matrix.

Now, we show that the condition (i) holds. For  $j \in Z_n$ ,  $j < \frac{n-1}{2}$ , we can write  $j = 4t + r$ ,  $r \in \{0, 1, 2, 3\}$ . Denote  $U_j = \sum_{i=0}^2 a_{i,j}$ ,  $V_j = \sum_{i=0}^2 a_{i, n-1-j}$ . We shall prove that  $U_j = V_j$ .

*Case 1.*  $m$  is odd. In this case,  $[\frac{1}{2}]_n = \frac{n+1}{2}$ ,  $[\frac{1}{4}]_n = \frac{n+1}{4}$ .

If  $r = 0$ , we have

$$\begin{aligned} a_{0,j} &= \left[ \frac{3j-1}{4} \right]_n = \left[ 3t - \frac{1}{4} \right]_n = \begin{cases} n + 3t - \frac{n+1}{4}, & \text{if } 3t < \frac{n+1}{4}, \\ 3t - \frac{n+1}{4}, & \text{if } 3t \geq \frac{n+1}{4}. \end{cases} \\ a_{1,j} &= \left[ \frac{3j-2}{4} \right]_n = \left[ \frac{3t-1}{2} \right]_n = n + 3t - \frac{n+1}{2}. \\ a_{2,j} &= \left[ \frac{-3j-3}{2} \right]_n = \left[ -6t - 1 - \frac{n+1}{2} \right]_n = \begin{cases} n - 6t - 1 - \frac{n+1}{2}, & \text{if } 3t < \frac{n+1}{4}, \\ 2n - 6t - 1 - \frac{n+1}{2}, & \text{if } 3t \geq \frac{n+1}{4}. \end{cases} \end{aligned}$$

Then

$$U_j = \sum_{i=0}^2 a_{i,j} = \frac{7n-9}{4}.$$

Meanwhile

$$\begin{aligned} a_{0, n-1-j} &= \left[ \frac{3(n-1-j)-1}{4} \right]_n = \left[ \frac{-3j-4}{4} \right]_n = [-3t-1]_n = n - 3t - 1, \\ a_{1, n-1-j} &= \left[ \frac{3(n-1-j)-2}{4} \right]_n = \left[ \frac{-3j-5}{4} \right]_n = \left[ -3t-1-\frac{1}{4} \right]_n = n - 3t - 1 - \frac{n+1}{4}, \end{aligned}$$

$$a_{2,n-1-j} = \left[ \frac{-3(n-1-j)-3}{2} \right]_n = [6t]_n = 6t.$$

It follows

$$V_i = \sum_{i=0}^2 a_{i,n-1-j} = \frac{7n-9}{4}.$$

Thus

$$U_j = V_j.$$

Similarly, one can calculate

$$U_j = V_j = \begin{cases} \frac{7n-9}{4}, & \text{if } r = 1, 3, \\ \frac{3n-9}{4}, & \text{if } r = 2. \end{cases}$$

It means that the condition (i) holds when  $m$  is odd.

*Case 2:  $m$  is even.* In this case,  $[\frac{1}{2}]_n = \frac{n+1}{2}$ ,  $[\frac{1}{4}]_n = \frac{3n+1}{4}$ . As above, one can calculate  $U_j$  and  $V_j$ , respectively, and get

$$U_j = V_j = \begin{cases} \frac{5n-9}{4}, & \text{if } r = 0, 2, 3, \\ \frac{9n-9}{4}, & \text{if } r = 1. \end{cases}$$

Which indicates that the condition (i) also holds when  $m$  is even. In a similar way, one can directly check that the conditions (ii)-(iv) hold.  $\square$

To illustrate the proof of Theorem 2.2, an example is given bellow.

**Example 3.2.** There exists a  $3 \times 17$  S-array based on  $Z_{17}$ .

**Proof.** For  $n = 17$ ,  $[\frac{1}{2}]_{17} = 9$ ,  $[\frac{1}{4}]_{17} = 13$ . By means of the construction described in the proof of Theorem 2.2, for  $0 \leq j \leq 16$ , let

$$\begin{aligned} a_{0,j} &= \left[ \frac{3j-1}{4} \right]_{17} = [13(3j-1)]_{17} = [39j-13]_{17} = [5j+4]_{17}, \\ a_{1,1} &= \left[ \frac{3j-2}{4} \right]_{17} = [13(3j-2)]_{17} = [39j-26]_{17} = [5j+8]_{17}, \\ a_{2,j} &= \left[ \frac{-3j-3}{2} \right]_{17} = [9(-3j-3)]_{17} = [-27j-27]_{17} = [7j+7]_{17}. \end{aligned}$$

Then

$$A = (a_{i,j}) = \begin{pmatrix} 4 & 9 & 14 & 2 & 7 & 12 & 0 & 5 & 10 & 15 & 3 & 8 & 13 & 1 & 6 & 11 & 16 \\ 8 & 13 & 1 & 6 & 11 & 16 & 4 & 9 & 14 & 2 & 7 & 12 & 0 & 5 & 10 & 15 & 3 \\ 7 & 14 & 4 & 11 & 1 & 8 & 15 & 5 & 12 & 2 & 9 & 16 & 6 & 13 & 3 & 10 & 0 \end{pmatrix}.$$

It is easy to check that  $A$  is a  $3 \times 17$  S-array based on  $Z_{17}$ .  $\square$

By the way, the  $3 \times 11$  S-array based on  $Z_{11}$  listed in Example 2.1 is also just constructed by the method described in the proof of Theorem 2.2.

**Proof of Theorem 1.2.** Combine Lemma 2.2 and Lemma 3.1, the proof of Theorem 1.2 is obtained.  $\square$

**Remark 3.3.** Take some row permutation and column permutation on  $E$  given in the proof of Theorem 2.2, one can get a new regular PSMS( $n, 6$ ),  $H = (h_{i,j})$ , with the property that  $h_{i,j} + h_{n-1-i,n-1-j} = 6n + 1$  for all  $0 \leq i, j \leq n - 1$  whenever  $h_{i,j} \neq 0$ . A regular PSMS( $n, d$ ),  $H = (h_{i,j})$ , with such a property is called *central complementary-symmetry*. We have the following result.

**Theorem 3.4.** *There exists a central complementary-symmetry regular PSMS( $n, 6$ ) for all positive integer  $n \equiv 5 \pmod{6}$  and  $n > 6$ .*

**Proof.** By Lemma 3.1, we can suppose that  $A = (a_{i,j})$  is a  $3 \times n$  S-array based on  $Z_n$ . Let  $B, C, D$  and  $E$  be the same as those in the proof of Lemma 2.2, where  $E = (e_{i,j}) = (d_{i-\frac{j}{2}, \frac{j}{2}})$  is a regular PSMS( $n, 6$ ) of order order  $n \equiv 5$  with density 6.

Let  $H = (h_{i,j})$ , where  $h_{i,j} = e_{i+\frac{n+7}{6}, j+\frac{n-1}{2}}$ ,  $0 \leq i, j \leq n - 1$ . It is readily checked that  $H$  is a regular PSMS( $n, 6$ ), and  $h_{i,j} + h_{n-1-i,n-1-j} = 6n + 1$  for all  $0 \leq i, j \leq n - 1$  whenever  $h_{i,j} \neq 0$ . So,  $H$  is a central complementary-symmetry regular PSMS( $n, 6$ )  $\square$

To illustrate the construction provided in the proof of Theorem 3.4, an example is given bellow.

**Example 3.5.** There exists a central complementary-symmetry regular PSMS(11, 6).

**Proof.** Let  $n = 11$ ,  $E = (e_{i,j})$  be a PSMS(11, 6) based on  $Z_{11}$  listed in Example 2.3. Here,  $\frac{n+7}{6} = 3$ ,  $\frac{n-1}{2} = 5$ ,  $6n + 1 = 67$ . Take

$$H = (h_{i,j}) = (e_{i+3,j+5}) = \begin{pmatrix} 42 & 13 & 47 & 0 & 40 & 0 & 0 & 0 & 51 & 0 & 8 \\ 0 & 62 & 30 & 1 & 35 & 0 & 64 & 0 & 0 & 0 & 9 \\ 0 & 33 & 0 & 50 & 18 & 55 & 23 & 0 & 22 & 0 & 0 \\ 0 & 0 & 0 & 57 & 0 & 38 & 6 & 43 & 11 & 0 & 46 \\ 0 & 4 & 0 & 0 & 0 & 15 & 0 & 26 & 60 & 31 & 65 \\ 19 & 53 & 0 & 28 & 0 & 0 & 0 & 39 & 0 & 14 & 48 \\ 2 & 36 & 7 & 41 & 0 & 52 & 0 & 0 & 0 & 63 & 0 \\ 21 & 0 & 56 & 24 & 61 & 29 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 45 & 0 & 44 & 12 & 49 & 17 & 0 & 34 & 0 \\ 58 & 0 & 0 & 0 & 3 & 0 & 32 & 66 & 37 & 5 & 0 \\ 59 & 0 & 16 & 0 & 0 & 0 & 27 & 0 & 20 & 54 & 25 \end{pmatrix}.$$

$\square$

It is readily checked that  $H$  is a regular PSMS(11, 6) and  $h_{i,j} + h_{10-i,10-j} = 67$  for all  $0 \leq i, j \leq 10$  whenever  $h_{i,j} \neq 0$ . So,  $H$  is a central complementary-symmetry regular PSMS(11, 6).

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