

Signless Laplacian spectral determinations of the starlike trees $ST(n, d_1)$ and the double starlike trees $H_n(p, p)$

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ABSTRACT

Two graphs are said to be Q -cospectral (respectively, A -cospectral) if they have the same signless Laplacian (respectively, adjacency) spectrum. A graph is said to be DQS (respectively, DAS) if there is no other non-isomorphic graphs Q -cospectral (respectively, A -cospectral) with it. A tree on n vertices with maximum degree d_1 is called starlike and denoted by $ST(n, d_1)$, if it has exactly one vertex with the degree greater than 2. A tree is called double starlike if it has exactly two vertices of degree greater than 2. If we attach p pendant vertices (vertices of degree 1) to each of pendant vertices of a path P_n , the resulting graph is called the double starlike tree $H_n(p, p)$. In this article, we prove that all double starlike trees $H_n(p, p)$ are DQS , where $p \geq 1, n \geq 2$ and p denotes . In addition, by a simple method, we show that all starlike trees are DQS excluding $K_{1,3} = ST(4, 3)$.

Keywords: (Double) starlike trees, signless Laplacian spectrum, line graph

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1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$, where v_1, v_2, \dots, v_n are indexed in the non-increasing order of degrees. Let $d_i = d_i(G) = d_G(v_i)$ be the degree of the vertex v_i , and $deg(G) = (d_1 = \Delta, d_2, \dots, d_n)$ the non-increasing degree sequence of G . We denote the line graph of G and the total number of triangles of G by G^L and $t(G)$, respectively.

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The adjacency matrix of G , denoted by $A(G)$, is the $n \times n$ matrix whose (i, j) -entry is 1 if v_i and v_j are adjacent and 0 otherwise. We call $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ the Laplacian matrix of G and the signless Laplacian spectrum, respectively, where $D(G)$ is the $n \times n$ diagonal matrix with d_1, d_2, \dots, d_n as diagonal entries. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the adjacency eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of G , respectively.

Denote by $\lambda_1(G) \geq \dots \geq \lambda_n(G)$, $\mu_1(G) \geq \dots \geq \mu_n(G)$ and $q_1(G) \geq \dots \geq q_n(G)$ the adjacency eigenvalues, the Laplacian eigenvalues and signless Laplacian eigenvalues of G , respectively. The multi-set of eigenvalues of $Q(G)$ (respectively, $L(G)$ and $A(G)$) is called the signless Laplacian (respectively, Laplacian and adjacency) spectrum of G . We shall denote the signless Laplacian spectrum of G by

$$\text{Spec}_Q(G) = \{[q_1(G)]^{m_1}, [q_2(G)]^{m_2}, \dots, [q_k(G)]^{m_k}\},$$

where m_i denotes the multiplicity of $q_i(G)$, and $q_i(G) \neq q_j(G)$ for all $i \neq j$. Two graphs are said to be Q -cospectral (respectively, L -cospectral and A -cospectral) if they have the same signless Laplacian (respectively, Laplacian and adjacency) spectrum. A graph is said to be DQS (respectively, DLS and DAS) if there is no other non-isomorphic graphs Q -cospectral (respectively, L -cospectral and A -cospectral) with it.

In this paper, let P_n , C_n and $K_{1,n-1}$ denote the path, the cycle and the star of order n , respectively. A tree is called starlike if it has exactly one vertex of degree greater than 2. We denote by $ST(n, d_1) = T(l_1, l_2, \dots, l_\Delta)$ the starlike tree with maximum degree Δ such that

$$T(l_1, l_2, \dots, l_\Delta) - v = P_{l_1} \cup P_{l_2} \cup \dots \cup P_{l_\Delta}, \quad (1)$$

where v is the vertex of degree Δ in the starlike tree, $l_1, l_2, \dots, l_\Delta$ are any positive integers. A starlike tree with maximum degree 3 is called a T -shape tree.

We call a tree double starlike if it has exactly two vertices of degree greater than two. Denote by $H_n(p, q)$ the double starlike tree obtained by attaching p pendant vertices (vertices of degree 1) to one pendant vertex of P_n and q pendant vertices to the other pendant vertex of P_n (shown in Figure 1). In this paper, we prove that any starlike tree $ST(n, d_1)$ and double starlike tree $H_n(p, p)$ is determined by its signless Laplacian spectrum.

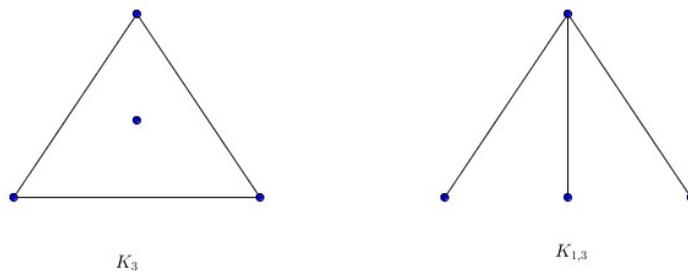


Fig. 1. Star graph $K_{1,3}$ is DLS but is not DQS . It is Q -cospectral with $K_3 \cup K_1$

The problem "which graphs are determined by their spectra?" originates from chemistry. Günthard and Primas [8] raised this question in the context of Hückel's theory.

Since this problem is generally very difficult, van Dam and Haemers [17] proposed a more modest problem, that is "Which trees are determined by their spectra?" Here we introduce some results about spectral properties of the (double) starlike trees:

Lepović and Gutman proved that no two non-isomorphic starlike trees have the same adjacency spectrum (see [9]). Wang and Xu gave all T -shape trees which are determined by their adjacency spectra (see [19]), and proved that T -shape trees are DLS (see [18]). Omid and Tajbakhsh proved that starlike trees are DLS (see [15]). Ghareghani et al. gave a method to construct a graph which has the same adjacency spectrum with a starlike tree (see [7]). Omid gave all T -shape trees which are DQS (see [14]). Liu and et.al., proved that double starlike trees $H_n(p, p)$ ($n \geq 2, p \geq 1$) are DLS (see [12]). Omid and Vatandoost proved that starlike trees with maximum degree 4 are DQS (see [16]). Bu and Zhou proved that starlike trees whose maximum degree exceed 4 are DQS (see [2]). Lu and Liu proved that double starlike trees $H_n(p, q)$ are DLS (see [13]).

In this article, we prove that all double starlike trees $H_n(p, p)$ are DQS . In addition, by a simple method, we show that all starlike trees are DQS excluding $K_{1,3} = ST(4, 3)$.

2. Preliminaries

In this section, we give some lemmas playing an important role throughout this article.

Lemma 2.1 ([6]). *Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*

- (i) *the total number of vertices,*
- (ii) *the total number of edges.*

For the adjacency matrix, the following follows from the spectrum:

- (iii) *the total number of closed walks of any length.*
- (iv) *Being regular or not and the degree of regularity.*
- (v) *Being bipartite or not.*

For the Laplacian matrix, the following follows from the spectrum:

- (vi) *the total number of components.*

Lemma 2.2 ([1]). *Let G be a graph with n vertices, m edges and t triangles and vertex degrees d_1, d_2, \dots, d_n . Let $T_k = \sum_{i=1}^n (q_i(G))^k$. Then*

$$T_0 = n, T_1 = \sum_{i=1}^n d_i = 2m, T_2 = 2m + \sum_{i=1}^n d_i^2 \text{ and } T_3 = 6t + 3 \sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3.$$

Lemma 2.3 ([4]). *Let G be a simple graph. Then the total number of closed walks of length 4 is equal to twice the total number of edges plus four times of the total number of induced paths of length 2 plus eight times of the total number of 4-cycles.*

Lemma 2.4 ([10]). *Let G be a graph with $V(G) \neq \phi$ and $E(G) \neq \phi$. Then*

$$d_1 + 1 \leq \mu_1 \leq \max \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \mid v_i v_j \in E(G) \right\},$$

where μ_1 is the largest eigenvalue of $L(G)$, and m_i denotes the average of the degrees of the vertices adjacent to the vertex v_i in G .

Lemma 2.5 ([17, 5]). $\text{Spec}_Q(P_n)$ consists of 0 and $2 + 2 \cos \frac{\pi j}{n}$, for $j = 1, 2, \dots, n-1$. For a graph G , $0 < q_1(G) < 4$ if and only if all components of G are paths.

Lemma 2.6 ([6, 5]). For any bipartite graph G , $\text{Spec}_Q(G)$ coincides with the L -spectrum of G .

Lemma 2.7 ([6, 5]). For a connected graph G , $q_n(G) = 0$ if and only if the graph is bipartite. In this case 0 is a simple eigenvalue, i.e., $m_n = 1$. The multiplicity of 0 as the signless Laplacian eigenvalue is equal to the total number of bipartite components.

Lemma 2.8 ([20]). If two graphs G and H are Q -cospectral, then their line graphs are A -cospectral. The converse is true if G and H have the same number of vertices and edges.

Lemma 2.9 ([17]). (Interlacing theorem) Suppose that N is a symmetric $n \times n$ matrix with eigenvalues $a_1 \geq a_2 \geq \dots \geq a_n$. Then the eigenvalues $b_1 \geq b_2 \geq \dots \geq b_m$ of a principal submatrix of N of size m satisfy $a_i \geq b_i \geq a_{n-m+i}$, for $i = 1, 2, \dots, m$.

Lemma 2.10 ([11]). If G is a graph of order n , then $q_3(G) \geq d_3(G) - \sqrt{2}$.

Lemma 2.11 ([11]). If G is a graph of order n , then $q_2(G) \geq d_2(G) - 1$.

A connected graph with n vertices is said to be unicyclic if it has n edges. If the girth of a unicyclic graph is odd, then this unicyclic graph is said to be odd unicyclic.

Lemma 2.12 ([3]). Let G be a graph Q -cospectral with a tree T of order n . Then one of the following holds:

- (1) G is tree;
- (2) G is the union of a tree with f vertices and s odd unicyclic graphs, and $n = 4^s f$.

It is easy to see that $K_{1,3}$ and $K_3 \sqcup K_1$ are Q -cospectral, i.e., $\text{Spec}_Q(K_{1,3}) = \text{Spec}_Q(K_3 \sqcup K_1) = \{[4]^1, [1]^2, [0]^1\}$ (see Figure 1).

Remark 2.13. Note that:

- (1) $H_1(p, p) \cong K_{1,2p}$. Any star graph is DQS excluding $K_{1,3}$ [11].
- (2) $H_n(1, 1) \cong P_{n+2}$. Any disjoint union of the path graphs is DQS [17].

So, we always assume that $n, p \geq 2$ and the maximum degree of $H_n(p, p)$, $d_1(H_n(p, p))$, is greater than or equal to 3.

3. Main Results

3.1. Signless Laplacian determinations of the starlike trees $ST(n, d_1)$

In this subsection, by a simple method, we show that any starlike tree $ST(n, d_1)$ is *DQS* excluding $ST(4, 3)$. To do so, first we need the following observation.

Lemma 3.1. $q_2(ST(n, d_1)) < 4$.

Proof. Let v_1 be the unique vertex of $ST(n, d_1)$ of degree d_1 . Let Y_1 be the $(n-1) \times (n-1)$ principal submatrix of the signless Laplacian matrix of $ST(n, d_1)$ obtained by removing the row and the column corresponding to v_1 . Then by (1) and Lemma 2.5, the largest signless Laplacian eigenvalues of Y_1 is less than 4 (In fact, the several blocks of Y_r are principal submatrices of $L(P_{l_{i+1}}), \dots, L(P_{l_{d_1+1}})$ which are nonnegative and irreducible). So, by Lemma 2.9,

$$4 > q_1(Y_r) \geq q_2(ST(n, d_1)). \quad (2)$$

□

Proposition 3.2. Let G be a connected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(ST(n, d_1))$. Then $G \cong ST(n, d_1)$.

Proof. It is clear by Lemmas 2.1 and 2.7 G has a bipartite component, n vertices and $n-1$ edges. So, G is a tree. We know that $ST(n, d_1)$ has one vertex of degree $d_1 > 2$, d_1 vertices of degree 1 and $n-d_1-1$ vertices of degree 2. Set $F = \max \left\{ d_1 + \frac{7}{d_1+2}, d_1 + \frac{8}{d_1+2}, d_1 + 1 \right\}$. It follows from Lemmas 2.4 and 2.6 that $\mu_1(G) = q_1(G) \leq F$ (see Figure 2). Therefore, $d_1(G) + 1 \leq \mu_1(G) = q_1(G) \leq F$. Hence $d_1(G) \leq F - 1 < d_1 + 1$. So, $d_1(G) \leq d_1$.

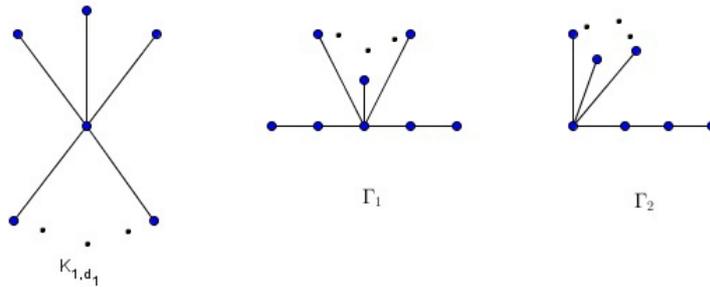


Fig. 2. Star graph $K_{1,d_1} = ST(d_1 + 1, d_1)$ and starlike trees Γ_1 and Γ_2

Denote by N_k the total number of vertices of degree $k \in \{1, 2, 3, \dots, d_1(G) \leq d_1\}$ in G . By Lemma 2.1,

$$\sum_{k=1}^{d_1(G)} N_k = n, \quad \sum_{k=1}^{d_1(G)} kN_k = 2(n-1), \quad \sum_{k=1}^{d_1(G)} k^2N_k = d_1^2 + d_1 + 4(n-d_1-2).$$

Therefore,

$$\sum_{k=1}^{d_1(G)} (k^2 - 3k + 2)N_k = d_1^2 - 3d_1 - 2. \quad (3)$$

By Lemma 2.8, $ST(n, d_1)^L$ and G^L are A -cospectral. In addition, by (iii) of Lemma 2.1, they have the same number of triangles (six times the total number of closed walks of length 3). In other words,

$$\sum_{i=1}^{d_1(G)} \binom{k}{3} N_k = \binom{d_1}{3}.$$

We aim to show that $N_{d_1} = 1$. Suppose that $N_{d_1} = 0$. To put that another way $d_1(G) < d_1$. Then:

$$\binom{d_1}{3} = \sum_{k=1}^{d_1(G)} \binom{k}{3} N_k < \frac{d_1}{6} \sum_{k=1}^{d_1(G)} (k-1)(k-2)N_k.$$

Thus

$$(d_1^2 - 3d_1 - 2) < \sum_{k=1}^{d_1(G)} (k^2 - 3k + 2)N_k,$$

a contradiction to (3). Therefore, $N_{d_1} \geq 1$. Up to now, we have shown that G has at least one vertex with maximum degree d_1 , say v_1 . It follows from Lemma 3.1 that $q_2(G) < 4$. So by Lemma 2.11, $d_2(G) < 5$. Therefore, $d_2(G) \in \{1, 2, 3, 4\}$. Assume that there are a vertices of degree 1, b vertices of degree 2, c vertices of degree 3, d vertices of degree 4 between $v_2 \leq v_i \leq v_n$.

$$\begin{cases} a + b + c + d = n - 1, \\ a + 2b + 3c + 4d = 2(n - 1) - d_1, \\ a + 4b + 9c + 16d = d_1^2 + d_1 - d_1^2 + 4(n - d_1 - 1) = d_1 + 4(n - d_1 - 1), \\ a + 8b + 27c + 64d = d_1^3 + d_1 - d_1^3 + 8(n - d_1 - 1) = d_1 + 8(n - d_1 - 1). \end{cases}$$

The roots are $a = d_1$, $b = n - d_1 - 1$ and $c = d = 0$. Therefore, G is a tree with exactly one vertex of degree more than two. So, $G \cong ST(n, d_1)$. This completes the proof. \square

Proposition 3.3. *Let G be a disconnected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(ST(n, d_1))$. Then G has no triangles.*

Proof. By Lemma 2.7, G has only one bipartite component. Consider the following two cases:

Case 1. $\delta = d_n < 1$; i.e, $\delta = d_n = 0$. In this case, the bipartite component of G is only K_1 , since G has only one bipartite component (see Lemma 2.7) and the other components of G are odd unicyclic graphs. Therefore, $G = K_1 \cup H_1 \cup \dots \cup H_c$, ($1 \leq i \leq c$), where H_i 's are odd unicyclic graphs. By notations of Lemma 2.12, $f = 1$. If $c = 1$, it follows from Lemma 2.12 that $n = 4$ and so $G = K_1 \cup H_1$. By Lemma 2.12, $|V(ST(n, d_1))| = 4$.

Obviously, $ST(n, d_1) = K_{1,3}$ or $ST(n, d_1) = P_4$ (there are two types trees of order 4). But, if $\text{Spec}_Q(G) = \text{Spec}_Q(P_4)$, then G is DQS and $G \cong P_4$. In this case G is connected, which is impossible. Hence we must have $\text{Spec}_Q(G = K_1 \cup H_1) = \text{Spec}_Q(K_{1,3})$ or $G = K_1 \cup K_3$. If $t(G) \geq 2$, then we have two components which are odd unicyclic graphs and so $q_2(G) \geq 4$, a contradiction with Lemma 3.1.

Case 2. $\delta = d_n \geq 1$. By Lemma 2.12, if $t(G) = 1$, then $G = Y \cup T$ (*), where Y and T are a connected graph consisting of a unique triangle and a tree on at least two vertices, respectively. This means that G must have at least a vertex with degree 3 or 4. Otherwise, $G = kP_t \cup lC_r$ and by (*) we get $G = C_3 \cup P_{n-4}$. Clearly, $q_1(G = C_3 \cup P_{n-4}) = 4$. On the other hand, since $\text{Spec}_Q(G) = \text{Spec}_Q(ST(n, d_1))$, thus $ST(n, d_1)$ has $K_{1,3}$ as its proper induced subgraph and so $q_1(G) = q_1(ST(n, d_1)) > q_1(K_{1,3}) = 4$, a contradiction.

By using the previous notations, we get

$$\begin{cases} a + b + c + d = n - 1, \\ a + 2b + 3c + 4d = 2(n - 1) - d_1, \\ a + 4b + 9c + 16d = d_1 + 4(n - 1 - d_1) - d_1^2, \\ a + 8b + 27c + 64d = d_1 + 8(n - 1 - d_1) - d_1^3 - 6. \\ 6c + 24d = d_1(d_1 - 1)(d_1 - 2) - d_1(d_1 - 1)(d_1 - 2), \end{cases}$$

Therefore, $d = -1$, a contradiction. □

Proposition 3.4. *Let G be a disconnected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(ST(n, d_1))$. Then $G \cong K_3 \cup K_1$.*

Proof. By Lemma 2.7, G has only one bipartite component. Consider the following two cases:

Case 1. $\delta = d_n < 1$; i.e, $\delta = d_n = 0$. In this case, the bipartite component of G is only K_1 , since G has only one bipartite component (see Lemma 2.7) and the other components of G are odd unicyclic graphs. Therefore, $G = K_1 \cup H_1 \cup \dots \cup H_c$, ($1 \leq i \leq c$) are odd unicyclic graphs. By notations of Lemma 2.12, $f = 1$. If $c = 1$, it follows from Lemma 2.12 that $n = 4$ and so $G = K_1 \cup H_1$. By Lemma 2.12, $|V(ST(n, d_1))| = 4$. Obviously, $ST(n, d_1) = K_{1,3}$ or $ST(n, d_1) = P_4$ (there are two types trees of order 4). But, if $\text{Spec}_Q(G) = \text{Spec}_Q(P_4)$, then G is DQS and $G \cong P_4$. In this case G is connected, which is impossible. Hence we must have $\text{Spec}_Q(G = K_1 \cup H_1) = \text{Spec}_Q(K_{1,3})$ or $G = K_1 \cup K_3$. If $c \geq 2$, we have two components which are odd unicyclic graphs and so $q_2(G) \geq 4$, a contradiction with Lemma 3.1.

Case 2. $\delta = d_n \geq 1$. By Lemma 2.12, if $c = 1$, then $G = Y \cup T$, where Y and T are a connected graph consisting of a unique cycle of order at least 5 and a tree on at least two vertices, respectively. This means that G must have at least a vertex with degree 3 or 4. By using the previous notations, we get

$$\begin{cases} a + b + c + d = n - 1, \\ a + 2b + 3c + 4d = 2(n - 1) - d_1, \\ a + 4b + 9c + 16d = d_1 + 4(n - d_1 - 1), \\ a + 8b + 27c + 64d = d_1 + 8(n - d_1 - 1). \end{cases}$$

Therefore, $c = d = 0$, a contradiction. \square

Combining Proposition 3.2 and Proposition 3.4 we have the following theorem.

Theorem 3.5. *Let $\text{Spec}_Q(G) = \text{Spec}_Q(ST(n, d_1))$. Then G is either $ST(n, d_1)$ or $K_3 \cup K_1$.*

In [20, Theorem 3.3], it was shown that for $\Delta \geq 12$, the line graph of starlike trees are DAS. Now, we present the following corollary.

Now, we present a corollary which immediately follows from Lemma 2.8 and Proposition 3.2.

Corollary 3.6. *Let G be a connected graph such that $\text{Spec}_A(G^L) = \text{Spec}_A(ST(n, d_1)^L)$ and $|V(G)| = |V(ST(n, d_1))|$. Then $G \cong ST(n, d_1)$ and so $G^L \cong ST(n, d_1)^L$.*

3.2. Signless Laplacian determinations of the double starlike trees $H_n(p, p)$

In this subsection, we show that any double starlike tree $H_n(p, p)$ is DQS. To do so, first we need the following observation.

Lemma 3.7. $q_3(H_n(p, p)) < 4$.

Proof. Let v_1 and v_2 be the vertices of $H_n(p, p)$ with the degree $p + 1$. Let $Y_{1,2}$ be the $(2p + n - 2) \times (2p + n - 2)$ principal submatrix of the signless Laplacian matrix of $H_n(p, p)$ obtained by removing the rows and columns corresponding to v_1 and v_2 . Then by (1) and Lemma 2.5, the largest signless Laplacian eigenvalue of $Y_{1,2}$ is less than 4. By Lemma 2.9, $q_3(H_n(p, p)) \leq q_1(Y_{1,2}) < 4$. \square

Proposition 3.8. *Let G be connected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$. Then G is a tree with exactly two vertices of degree $p + 1$.*

Proof. It is clear by Lemma 2.7 that G has a bipartite component, $n + 2p$ vertices and $n + 2p - 1$ edges. So, G is a tree, since G is connected. By Lemmas 2.4 and 2.6, we have $d_1 + 1 \leq q_1(G) = \mu_1(G) \leq p + 2 + \frac{1}{p + 2}$ and so $\Delta = d_1(G) \leq p + 1$. Now, we show that $d_1 = d_2 = p + 1$. Denote by N_k the total number of vertices of degree $k \in \{1, 2, 3, \dots, d_1 \leq p + 1\}$ in G . By Lemma 2.1,

$$\sum_{k=1}^{d_1} N_k = n + 2p, \quad \sum_{k=1}^{d_1} kN_k = 2(n + 2p - 1), \quad \sum_{k=1}^{d_1} k^2 N_k = 2(p + 1)^2 + 2p + 4(n - 2).$$

Therefore, By adding up the first, second and third Equations with coefficients 2, -3 , 1, respectively we get:

$$\sum_{k=1}^{d_1} (k^2 - 3k + 2)N_k = 2[(p + 1)^2 - 3(p + 1) + 2]. \quad (4)$$

By Lemma 2.8, $H_n(p, p)^L$ and G^L are A -cospectral. In addition, by Lemma 2.1 (iii), they have the same number of triangles (six times the total number of closed walks of length 3). In other words,

$$\sum_{i=1}^{d_1} \binom{k}{3} N_k = 2 \binom{p+1}{3}.$$

We aim to show that $N_{p+1} = 2$. Consider the following cases:

Case 1. $N_{p+1} = 0$. To put that another way $d_1 < p + 1$. Then:

$$2 \binom{p+1}{3} = \sum_{k=1}^{d_1} \binom{k}{3} N_k < \frac{p+1}{6} \sum_{k=1}^{d_1} (k-1)(k-2)N_k.$$

Thus

$$2[(p+1)^2 - 3(p+1) + 2] < \sum_{k=1}^{d_1} (k^2 - 3k + 2)N_k,$$

a contradiction to (4).

Case 2. $N_{p+1} = 1$. To put that another way, there exists only one vertex with degree $d_1 = p + 1$. So

$$2 \binom{p+1}{3} = \sum_{i=1}^{d_1} \binom{k}{3} N_k.$$

i.e.,

$$2 \frac{p+1}{6} [(p+1)^2 - 3(p+1) + 2] = \frac{p+1}{6} [(p+1)^2 - 3(p+1) + 2] + \sum_{i=1}^p \frac{k}{6} (k-1)(k-2)N_k,$$

i.e.,

$$\frac{p+1}{6} [(p+1)^2 - 3(p+1) + 2] = \sum_{i=1}^p \frac{k}{6} (k-1)(k-2)N_k < \frac{p+1}{6} \sum_{i=1}^p (k-1)(k-2)N_k,$$

or

$$[(p+1)^2 - 3(p+1) + 2] < \sum_{i=1}^p (k-1)(k-2)N_k. \quad (5)$$

Consequently, by (5) we get

$$\begin{aligned}
2[(p+1)^2 - 3(p+1) + 2] &= [(p+1)^2 - 3(p+1) + 2] + [(p+1)^2 - 3(p+1) + 2] \\
&< [(p+1)^2 - 3(p+1) + 2] + \sum_{i=1}^p (k-1)(k-2)N_k \\
&= \sum_{i=1}^{p+1} (k-1)(k-2)N_k \\
&= \sum_{i=1}^{d_1} (k^2 - 3k + 2)N_k,
\end{aligned}$$

a contradiction to (4). So, one may deduce that $N_{p+1} = 2$. \square

Lemma 3.9. *Let G be a connected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$. Then G is a double starlike tree with $\text{deg}(G) = (p+1, p+1, \underbrace{2, \dots, 2}_{n-2 \text{ times}}, \underbrace{1, \dots, 1}_{2p \text{ times}})$.*

Proof. It is clear by Lemma 2.1, G has a bipartite component, $n+2p$ vertices and $n+2p-1$ edges. So, G is a tree, since it is connected. Now, by Lemma 3.7, $q_3(G) < 4$. By Lemma 2.10, $d_3 - \sqrt{2} \leq q_3(G) < 4$ and so $d_3 < 4 + \sqrt{2} \approx 5.42$. Therefore, $0 < d_3 \in \{1, 2, 3, 4, 5\}$. Obviously, $d_{n+2p}(G) = 1$, since any tree has at least two vertices of degree 1. We know that $d_1 = d_2 = p+1$. On the other hand, it follows from Lemmas 2.8 and 2.1 that $t(G^L) = t(H_n(p, p)^L) = 2 \binom{p+1}{3} = \sum_{k=1}^{d_1} \binom{k}{3} N_k$. Assume that there are a vertices of degree 1, b vertices of degree 2, c vertices of degree 3, d vertices of degree 4 and e vertices of degree 5 between $v_2 \leq v_i \leq v_{n+2p}$.

$$\left\{ \begin{array}{l}
a + b + c + d + e = n + 2p - 2, \\
a + 2b + 3c + 4d + 5e = 2(n + 2p - 1) - x, \\
a + 4b + 9c + 16d + 25e = 2(p + 1)^2 + 2p + 4(n - 2) - y, \\
a + 8b + 27c + 64d + 125e = 2(p + 1)^3 + 2p + 8(n - 2) - w, \\
6c + 24d + 60e = 2p(p - 1)(p + 1) - z,
\end{array} \right.$$

where

$$\begin{aligned}
x &= d_1 + d_2 = 2p + 2, \\
y &= d_1^2 + d_2^2 = 2(p + 1)^2, \\
w &= d_1^3 + d_2^3 = 2(p + 1)^3, \\
z &= d_1(d_1^2 - 3d_1 + 2) + d_2(d_2^2 - 3d_2 + 2) = 2p(p - 1)(p + 1).
\end{aligned}$$

It is easy to see that $e = d = c = 0$. Therefore, by the first and the second equations, we get $b = n + 2p - x = n + 2p - (2p + 2) = n - 2$ and $a = x - 2 = 2p + 2 - 2 = 2p$. This

means that the connected tree G has exactly two vertices of degree more than two. So, G is a double starlike tree with $\deg(G) = (p+1, p+1, \underbrace{2, \dots, 2}_{n-2 \text{ times}}, \underbrace{1, \dots, 1}_{2p \text{ times}})$. \square

Proposition 3.10. *The degrees sequence of the line graph of $H_n(p, p)$ is*

$$(p+1, p+1, \underbrace{p, \dots, p}_{2p \text{ times}}, \underbrace{2, \dots, 2}_{(n-3) \text{ times}}).$$

Proof. Note that if uv is an edge belonging to $H_n(p, p)$, then the vertex degree of uv in the line graph of $H_n(p, p)$ is $\deg(u) + \deg(v) - 2$. Now, since $\deg(H_n(p, p)) = (p+2, p+2, \underbrace{2, \dots, 2}_{n-2 \text{ times}}, \underbrace{1, \dots, 1}_{2p \text{ times}})$ the proof is straightforward. \square

Proposition 3.11. *Let G be a connected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$.*

Proof. Let $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$. By Lemma 3.9, G is a double starlike tree with $\deg(G) = (p+2, p+2, \underbrace{2, \dots, 2}_{n-2 \text{ times}}, \underbrace{1, \dots, 1}_{2p \text{ times}})$. We aim to show that $G \cong H_n(p, p)$. To do

so, we consider the following two cases:

Case 1. Two vertices of degree greater than two are adjacent in G . Assume that there exist $p-a$ (respectively, $p-b$) pendant vertices adjacent to the vertices of degree $p+1$ in G , where a and b are non-negative integers satisfying that $0 \leq a, b \leq p$. Let $m_{G^L}^i$ be the total number of vertices with degree i in G^L . Therefore, $m_{G^L}^1 = a+b$, $m_{G^L}^{2p} = 1$, $m_{G^L}^{p+1} = b$, $m_{G^L}^p = p-b$, $m_{G^L}^p = p-a$, $m_{G^L}^{p+1} = a$, $m_{G^L}^2 = n-2-a-b$ and $m_{G^L}^k = 0$ for $k \notin \{1, 2, p, p+1, 2p\}$. By Lemma 2.8, $H_n(p, p)^L$ and G^L are A -cospectral. $H_n(p, p)^L$ and G^L have the same number of edges and the same number of closed walks of length 4. Moreover, they have the same number of 4-cycles. Lemma 2.1 implies that $H_n(p, p)^L$ and G^L have the same number of induced paths of length 2, that is,

$$\begin{aligned} 2 \binom{p+1}{2} + 2p \binom{p}{2} + (n-3) \binom{2}{2} &= \binom{2p}{2} + b \binom{p+1}{2} + (p-b) \binom{p}{2} \\ &\quad + a \binom{p+1}{2} + (p-b) \binom{p}{2} \\ &\quad + (n-2-a-b) \binom{2}{2}. \end{aligned}$$

It follows that $(p-1)^2 + (a+b)(p-1) = 0$, a contradiction, since $0 \leq a, b \leq p$ and $p \geq 2$.

Case 2. Two vertices of degree greater than two are non-adjacent in G . As before, assume that there exist $p-a$ (respectively, $p-b$) pendant vertices adjacent to the vertices of degree $p+1$ in G , where a and b are non-negative integers such that $0 \leq a, b \leq p$.

We show that $a = b = 0$. Then, by counting the total number of vertices in G , we get $K + \sum_{j=1}^a K_j' + \sum_{l=1}^b K_l'' + 2p = n + 2p$ (see Figure 3). Hence $K + a + b = n$. On the other hand, $m_{G^L}^1 = a + b$, $m_{G^L}^{p+1} = b + 1$, $m_{G^L}^p = p - b$, $m_{G^L}^{p+1} = a + 1$, $m_{G^L}^p = p - a$, $m_{G^L}^2 = n - 3 - a - b$ and $m_{G^L}^k = 0$ for $k \notin \{1, 2, p, p + 1\}$. Note that $H_n(p, p)^L$ and G^L are A -cospectral. By Lemma 2.1, $H_n(p, p)^L$ and G^L have the same number of edges and the same number of closed walks of length 4. Moreover, they have the same number of 4-cycles. Lemma 2.8 implies that $H_n(p, p)^L$ and G^L have the same number of induced paths of length 2, that is,

$$\begin{aligned} 2 \binom{p+1}{2} + 2p \binom{p}{2} + (n-3) \binom{2}{2} &= (b+1) \binom{p+1}{2} + (p-b) \binom{p}{2} \\ &\quad + (a+1) \binom{p+1}{2} + (p-a) \binom{p}{2} \\ &\quad + (n-3-a-b) \binom{2}{2}. \end{aligned}$$

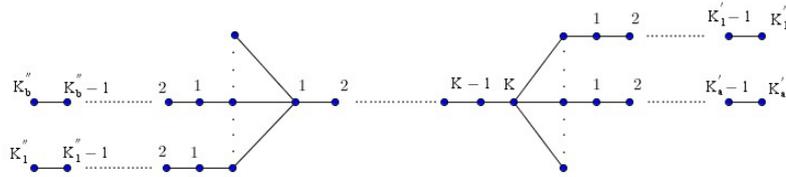


Fig. 3. The graph G in the proof of Proposition 3.11

By a simple computation, we get $(a + b)(p - 1) = 0$ and so $a = b = 0$. Therefore, $K = n$. This means that $G \cong H_n(p, p)$ and the proof is completed. \square

Proposition 3.12. *Let G be a disconnected graph such that $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$. Then G has no triangles.*

Proof. By Lemma 2.7, G has one bipartite component. We aim to show that $t(G) = 0$. Suppose by the contrary that $t(G) \geq 1$. Obviously $t(G) \leq 2$, otherwise since G is disconnected, by Lemma 2.12, G has at least three odd unicyclic components that any of them has a triangle. So $q_1(G), q_2(G), q_3(G) \geq 4$, a contradiction with Lemma 3.7. Suppose that $t(G) = 1$. Consider the following two cases:

Case 1. $\delta = d_n < 1$; i.e, $\delta = d_n = 0$. Since, G has only one bipartite component, so one may deduce that G has only one isolated vertex.

Subcase 1.1. By Lemma 2.12, $G = K_1 \cup H_1$, where H_1 is an odd unicyclic graph consisting of a triangle. By Lemma 2.8,

$$t(G^L) = t(H_n(p, p)^L) = 2 \binom{p+1}{3}.$$

Assume that there are a vertices of degree 1, b vertices of degree 2, c vertices of degree 3, d vertices of degree 4 and e vertices of degree 5 between $v_3 \leq v_i \leq v_{n+2p-1}$.

$$\begin{cases} a + b + c + d + e = n + 2p - 3, \\ a + 2b + 3c + 4d + 5e = 2(n + 2p - 1) - x, \\ a + 4b + 9c + 16d + 25e = 2(p + 1)^2 + 2p + 4(n - 2) - y, \\ a + 8b + 27c + 64d + 125e = 2(p + 1)^3 + 2p + 8(n - 2) - 6 - w, \\ 6c + 24d + 60e = 2p(p - 1)(p + 1) - z, \end{cases}$$

where $x = d_1 + d_2$, $y = d_1^2 + d_2^2$, $w = d_1^3 + d_2^3$ and $z = d_1(d_1^2 - 3d_1 + 2) + d_2(d_2^2 - 3d_2 + 2)$. By a simple computation, $d = \frac{-48(2x - 3y + w - z + 6)}{12} < 0$, a contradiction, since $2x - 3y + w - z = 0$.

Subcase 1.2. Let $t(G) = 2$, by a similar argument, and by using the previous notations, we get $G = K_1 \cup H_1 \cup H_2$ and so

$$\begin{cases} a + b + c + d + e = n + 2p - 3, \\ a + 2b + 3c + 4d + 5e = 2(n + 2p - 1) - x, \\ a + 4b + 9c + 16d + 25e = 2(p + 1)^2 + 2p + 4(n - 2) - y, \\ a + 8b + 27c + 64d + 125e = 2(p + 1)^3 + 2p + 8(n - 2) - 12 - w, \\ 6c + 24d + 60e = 2p(p - 1)(p + 1) - z. \end{cases}$$

By a simple computation, $d = \frac{-48(2x - 3y + w - z + 12)}{12} < 0$, which is a contradiction (since $2x - 3y + w - z = 0$).

Case 2. $\delta = d_n \geq 1$. By Lemma 2.12, if $t(G) = 1$, then $G = Y \cup T$, where Y and T are a connected graph consisting of a triangle and a tree on at least two vertices, respectively. Consider the following two cases:

Subcase 2.1. By using the previous notations, we get

$$\begin{cases} a + b + c + d + e = n + 2p - 2, \\ a + 2b + 3c + 4d + 5e = 2(n + 2p - 1) - x, \\ a + 4b + 9c + 16d + 25e = 2(p + 1)^2 + 2p + 4(n - 2) - y, \\ a + 8b + 27c + 64d + 125e = 2(p + 1)^3 + 2p + 8(n - 2) - 6 - w, \\ 6c + 24d + 60e = 2p(p - 1)(p + 1) - z. \end{cases}$$

Therefore, $d = \frac{-48(2x - 3y + w - z + 6)}{12} < 0$, which is a contradiction (since $2x - 3y + w - z = 0$).

Subcase 2.2. Suppose that $t = 2$, so by Lemma 2.12, $G = T \cup Y_1 \cup Y_2$, where Y_i denotes a connected graph consisting of a triangle and T is a tree on at least two vertices. By using the previous notations,

$$\begin{cases} a + b + c + d + e = n + 2p - 2, \\ a + 2b + 3c + 4d + 5e = 2(n + 2p - 1) - x, \\ a + 4b + 9c + 16d + 25e = 2(p + 1)^2 + 2p + 4(n - 2) - y, \\ a + 8b + 27c + 64d + 125e = 2(p + 1)^3 + 2p + 8(n - 2) - 12 - w, \\ 6c + 24d + 60e = 2p(p - 1)(p + 1) - z. \end{cases}$$

Therefore, $d = \frac{-48(2x - 3y + w - z + 12)}{12} < 0$, which is a contradiction (since $2x - 3y + w - z = 0$). \square

Proposition 3.13. *There is no disconnected graph Q -cospectral with $H_n(p, p)$, ($n, p \geq 2$).*

Proof. Let $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$ and G be disconnected. By Proposition 3.12, G has no triangles, i.e., $t(G) = 0$. Similar to Proposition 3.12 we have the following two case:

Case 1. $d_1(G) = 0$. By Lemma 2.12, if $s = 1$, then $G = Y \cup T$, where Y is a connected graph consisting of a unique cycle of order at least 5 and $T = K_1$. On the other hand, Lemma 2.12 implies that $H_n(p, p)$ is either $K_{1,3}$ or P_4 , a contradiction. So, let $s = 2$. In this case $G = Y_1 \cup Y_2 \cup K_1$, where Y_i is a connected graph consisting of a unique cycle of order at least 5. It is clear that $|V(G)| = 16$ and $|E(G)| = 15$. Since $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$, $|V(G)| = |V(H_n(p, p))| = n + 2p$. Therefore, $n + 2p = 16$ and $(n, p) \in \{(2, 7), (4, 6), (6, 5), (8, 4), (10, 3), (12, 2)\}$. Applying the previous notations for $(n, p) = (2, 7)$ we get

$$\begin{cases} a + b + c + d + e = 13, \\ a + 2b + 3c + 4d + 5e = 2(2 + 14 - 1) - x, \\ a + 4b + 9c + 16d + 25e = 2(2 + 1)^2 + 4(2 - 2) + 14 - y, \\ a + 8b + 27c + 64d + 125e = 2(2 + 1)^3 + 14 + 8(2 - 2) - w, \\ 6c + 24d + 60e = 2(8)(7)(6) - z. \end{cases}$$

(Note that the degree of one of vertices is $\delta = 0$ and the two others degrees are d_1, d_2 . We can subtract these three vertices from 16. So, the total number of vertices with degrees 1, 2, 3, 4 and 5 is 13.). Then $e = \frac{12(2x - 3y + w - z + 640)}{12}$ and so $e = 640$, a contradiction.

By solving the system of equations for $(10, 3)$, $e = \frac{12(2x - 3y + w - z + 1948)}{12}$ and so $e = 1948$, a contradiction. In a similar way, one can show that by replacing any of the above cases in the system of equations, we will have a contradiction. If $s \geq 3$, then $q_1(G), q_2(G), q_3(G) \geq 4$, a contradiction with Lemma 3.7.

Case 2. $d_1(G) \geq 1$. Applying the previous notations we get

$$\begin{cases} a + b + c + d + e = n + 2p - 2, \\ a + 2b + 3c + 4d + 5e = 2(n + 2p - 1) - x, \\ a + 4b + 9c + 16d + 25e = 2(p + 1)^2 + 2p + 4(n - 2) - y, \\ a + 8b + 27c + 64d + 125e = 2(p + 1)^3 + 2p + 8(n - 2) - 12 - w, \\ 6c + 24d + 60e = 2p(p - 1)(p + 1) - z. \end{cases}$$

Since G is disconnected, by Lemma 2.12, G consists of at least an odd unicyclic component such that the order of its odd cycle is greater than or equal to 5. This means that G has at least one vertex of degree either 3 or 4 or 5. But, by solving the system of equations, we get $c = e = d = 0$, which is impossible. \square

Combining Proposition 3.11 and Proposition 3.13 we have the following results.

Corollary 3.14. *Any double starlike tree $H_n(p, p)$, ($n, p \geq 2$) is DQS.*

Theorem 3.15. *Let $\text{Spec}_A(G^L) = \text{Spec}_A(H_n(p, p)^L)$ and $|V(G)| = |V(H_n(p, p))|$. Then $G^L \cong H_n(p, p)^L$.*

Proof. By Lemma 2.8, $\text{Spec}_A(G^L) = \text{Spec}_A(H_n(p, p)^L)$ implies that $\text{Spec}_Q(G) = \text{Spec}_Q(H_n(p, p))$ and by Lemma 2.10, $G \cong H_n(p, p)$. Therefore, $G^L \cong H_n(p, p)^L$. \square

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