ARS COMBINATORIA 165 (2025) 79–89

Ars Combinatoria
www.combinatorialpress.com/ars

Combinatorial Press

# Some properties of generalized $(k,t)$-Jacobsthal $p$-sequences

Elahe Mehraban* and Evren Hincal

ABSTRACT

In this paper, we generalize the $k$-Jacobsthal sequences and call them the generalized $(k,t)$-Jacobsthal $p$-sequences. Also, we obtain combinatorial identities. Then, the generalized $(k,t)$-Jacobsthal $p$-matrix is used to factorize the Pascal matrix. Finally, using the Riordan method, we obtain two factorizations of the Pascal matrix involving the generalized $(k,t)$-Jacobsthal $p$-sequences.

*Keywords:* Jacobsthal number, Riordan arrays, Pascal matrix

*2020 Mathematics Subject Classification:* Primary: 11K31, 33C45, Secondary: 42C05.

## 1. Introduction

In Mathematics, sequences such as Fibonacci, Pell, Mersenne, etc., play an important role (see [1, 8, 15, 16, 21]). A Jacobsthal number is one of many sequences studied in mathematics and other fields. The Jacobsthal number $J_n$ is defined as

$$J_n = J_{n-1} + 2J_{n-2},\ n \geq 2,$$

with initial conditions $J_0 = 0$ and $J_1 = 1$ [11]. The Jacobsthal numbers have been generalized in several ways [4, 5, 6].

In 2008, the Jacobsthal Lucas $E$-matrix and $R$-matrix were given which are similar to the Fibonacci $Q$-matrix [12]. In [2], Gaussian Jacobsthal sequences were introduced and the corresponding generating functions were given. In 2016, upper and lower bounds were obtained on matrices whose elements are $k$-Jacobsthal sequences [19]. A definition of the adjacency-Jacobsthal sequence can be found in [7].

* Corresponding author.

Received 31 Jan 2025; Revised 20 Nov 2025; Accepted 29 Nov 2025; Published Online 25 Dec 2025.

DOI: 10.61091/ars165-06

In [14], obtained factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. In 2011, introduced new factorizations of the Pascal matrix via Fibonomial coefficients called the Fibo-Pascal matrix, involving the $k$-Fibonacci matrix and $k$-Pell matrix [18]. In [9], the $k$-Fibonacci matrix and the Pascal matrix were studied. In 2022, gave a factorization of the Pascal matrix involving the $t$-extension of the $p$-Fibonacci matrix [10]. In [13], the $(d, k)$-Fibonacci polynomial was introduced and a factorization of the Pascal matrix based on these sequences was presented.

Our motivation here is a generalization of the $k$-Jacobsthal sequence, which we used to factorize Pascal matrices, which can be used later in other fields.

Here, we introduce the generalized $k$-Jacobsthal sequence and obtain combinatorial identities. Also, we give a Factorization of the Pascal matrix involving these sequences.

The remainder of this paper is organized as follows. The Jacobsthal numbers are generalized in Section 2 and new sequences are obtained. In Section 3, we give three Factorization of the Pascal matrix involving the generalized $(k, t)$-Jacobsthal $p$-sequences.

## 2. Preliminaries

These are some definitions and concepts that will be useful during this process:

According to Jacobsthal's sequence, one of these generalizations can be stated as follows:

**Definition 2.1.** For $n \geq 0$ and $k \geq 2$, the generalized Jacobsthal sequence $\{J(k, n)\}$ is defined as

$$J(k, n) = kJ(k, n-1) + 2J(k, n-2) \text{ for } n \geq 2,$$

with initial conditions $J(k, 0) = 0$ and $J(k, 1) = 1$ [20].

For example, if $k = 3$, we have

$$J(3, n) = 3J(3, n-1) + 2J(3, n-2) \quad \text{for} \quad n \geq 2,$$

and thus $\{J(2, n)\}_0^\infty = \{0, 1, 3, 11, \ldots\}$.

**Definition 2.2.** The $n \times n$ lower triangular Pascal matrix, denoted by $P_n = [p_{ij}]$, is defined as follows [3]:

$$p_{ij} = \begin{cases} \dbinom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The Riordan group was introduced in [17] as follows.

**Definition 2.3.** Let $R' = [r_{ij}]_{i,j \geq 0}$ be an infinite matrix with complex entries. Let $C_i(t) = \sum\limits_{n \geq 0}^{\infty} r_{n,i} t^n$ be the generating function of the ith column of $R'$. We call $R'$ a Riordan matrix if $c_i(t) = g(t)[f(t)]^i$, where

$$g(t) = 1 + g_1 t + g_2 t^2 + g_3 t^3 + \cdots, \qquad f(t) = t + f_2 t^2 + f_3 t^3 + \cdots.$$

In this case we write $R = (g(t), f(t))$ and denote by $R$ the set of Riordan matrices. Then the set $R$ is a group under matrix multiplication $*$, with the following properties:
(i) $(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t)))$,
(ii) $I = (1, t)$ is the identity element,
(iii) the inverse of $R$ is given by $R^{-1} = (\frac{1}{g(\bar{f}(t))}, \bar{f}(t))$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$, that is,$f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

## 3. The Generalized $(k,t)$-Jacobsthal $p$-sequences

In this section, we define the generalized $(k,t)$-Jacobsthal $p$-sequences and some results are given which will be used later.

**Definition 3.1.** For integers $k \geq 1$, $p \geq 1$ and $t \geq 2$, the generalized $(k,t)$-Jacobsthal $p$-sequences denoted $\{J_n^p(k,t)\}$ are defined as

$$J_n^p(k,t) = kJ_{n-1}^p(k,t) + 2J_{n-p-1}^p(k,t) + \cdots + J_{n-p-t}^p(k,t),\ n \geq t+p+1, \tag{1}$$

where $J_0^p(k,t) = J_1^p(k,t) = \cdots = J_{t+p-2}^p(k,t) = 0$ and $J_{t+p-1}^p(k,t) = 1$.

**Example 3.2.** Let $p = 1$ and $k = 3$.
(i) If $t = 2$, according to Definition 3.1, we have

$$J_n^1(3,2) = 3J_{n-1}^1(3,2) + 2J_{n-2}^1(3,2) + J_{n-3}^1(3,2),\ n \geq 4. \tag{2}$$

Therefore, $\{J_n^1(3,2)\}_0^\infty = \{0, 0, 1, 3, 11, 40, 145, 526, \cdots\}$.
(ii) If $t = 3$ , according to Definition 3.1, we have

$$J_n^1(3,3) = 3J_{n-1}^1(3,3) + 2J_{n-2}^1(3,3) + J_{n-3}^1(3,3) + J_{n-4}^1(3,3),\ n \geq 5. \tag{3}$$

So, $\{J_n^1(3,3)\}_0^\infty = \{0, 0, 0, 1, 3, 11, 40, 146, 532, \cdots\}$.
(iii) If $t = 4$, according to Definition 3.1, we have

$$J_n^1(3,4) = 3J_{n-1}^1(3,4) + 2J_{n-2}^1(3,4) + J_{n-3}^1(3,4) + J_{n-4}^1(3,4) + J_{n-5}^1(3,4),\ n \geq 6. \tag{4}$$

So, $\{J_n^1(3,4)\}_0^\infty = \{0, 0, 0, 0, 1, 3, 11, 40, 146, 532, 1939, \cdots\}$.

**Lemma 3.3.** *Let $g_{J_n^p(k,t)}$ be the generating function of the generalized $(k,t)$-Jacobsthal $p$-numbers, then*

$$g_{J_n^p(k,t)} = \frac{x^{t+p-1}}{1 - kx - 2x^{p+1} - x^{p+2} - \cdots - x^{t+p}}. \tag{5}$$

**Proof.** We have

$$\begin{aligned} g_{J_n^p(k,t)} &= \sum_{n=1}^{\infty} J_n^p(k,t)x^n \\ &= J_1^p(k,t)x + J_2^p(k,t)x^2 + \cdots + J_{t+p-1}^p(k,t)x^{t+p-1} + \sum_{n=t+p}^{\infty} J_n^p(k,t)x^n \end{aligned}$$

$$= x^{t+p-1} + \sum_{n=t+p}^{\infty} kJ_{n-1}^p(k,t) + 2J_{n-p-1}^p(k,t) + J_{n-p-2}^p(k,t) + \cdots + J_{n-t}^p(k,t)x^n$$

$$= x^{t+p-1} + \sum_{n=t+p+1}^{\infty} (kJ_{n-1}^p(k,t)x^n + 2\sum_{n=t+p+1}^{\infty} J_{n-p-1}^p(k,t)x^n + \sum_{n=t+p+1}^{\infty} J_{n-p-2}^p(k,t)x^n$$

$$+ \cdots + \sum_{n=t+p+1}^{\infty} J_{n-t-p}^p(k,t))x^n$$

$$= x^{t+p-1} + kx\sum_{n=1}^{\infty} J_n^p(k,t)x^n + 2x^{p+1}\sum_{n=1}^{\infty} J_n^p(k,t)x^n + \cdots + x^{t+p}\sum_{n=1}^{\infty} J_n^p(k,t)x^n$$

$$= x^{t+p-1} + xkg_{J_n^p(k,t)} + 2x^{p+1}g_{J_n^p(k,t)} + \cdots + x^{t+p}g_{J_n^p(k,t)}.$$

Thus,

$$g_{J_n^p(k,t)} = \frac{x^{t+p-1}}{1 - kx - 2x^{p+1} - x^{p+2} - \cdots - x^{t+p}}.$$

□

**Lemma 3.4.** *The generating function of the generalized $(k,t)$-Jacobsthal $p$-numbers has the following exponential representation*

$$g_{J_n^p(k,t)} = x^{t+p-1} \exp\sum_{i=1}^{\infty} \frac{x^i}{i}(k + 2x^{p-2} + x^{p-3} + \cdots + x^{t+p-1})^i,$$

*where $t \geq 2$.*

**Proof.** From Eq. (5), we have

$$\ln\frac{g_{J_n^p(k,t)}}{x^{t+p-1}} = -\ln(1 - kx - 2x^{p+1} - x^{p+2} - \cdots - x^{t+p}).$$

$$-\ln(1 - kx - 2x^{p+1} - x^{p+2} - \cdots - x^{t+p}) = -[-x(k + 2x^p + \cdots + x^{t+p-1})$$
$$-\frac{1}{2}x^2(k + 2x^p + \cdots + x^{t+p-1})^2 - \cdots - \frac{1}{n}x^n(k + 2x^p + \cdots + x^{t+p-1})^n - \cdots],$$

which gives the result. □

## 4. Factorization of the Pascal matrix

In this section, we obtain the inverse of the generalized $(k,1)$-Jacobsthal 1-matrix. Also, we give a factorization of generalized $(k,1)$-Jacobsthal 1-matrix and get some results from it. First, we define the generalized $(k,t)$-Jacobsthal $p$-matrix.

**Definition 4.1.** The $n \times n$ generalized $(k,t)$-Jacobsthal $p$-matrix $p \geq 1$, denoted by $M_{(n,k)}^{p,t} = [m_{(k,ij)}^{p,t}]$, is defined as follows:

$$m_{(k,ij)}^{p,t} = J_{i-j+1}^p(k,t).$$

For example, suppose that $n=7,\ k=2,\ t=1$ and $p=1$, we have

$$M^{1,1}_{(7,2)}=\begin{bmatrix} J^1_1(2,1) & J^1_0(2,1) & J^1_{-1}(2,1) & J^1_{-2}(2,1) & J^1_{-3}(2,1) & J^1_{-4}(2,1) & J^1_{-5}(2,1)\\ J^1_2(2,1) & J^1_1(2,1) & J^1_0(2,1) & J^1_{-1}(2,1) & J^1_{-2}(2,1) & J^1_{-3}(2,1) & J^1_{-4}(2,1)\\ J^1_3(2,1) & J^1_2(2,1) & J^1_1(2,1) & J^1_0(2,1) & J^1_{-1}(2,1) & J^1_{-2}(2,1) & J^1_{-3}(2,1)\\ J^1_4(2,1) & J^1_3(2,1) & J^1_2(2,1) & J^1_1(2,1) & J^1_0(2,1) & J^1_{-1}(2,1) & J^1_{-2}(2,1)\\ J^1_5(2,1) & J^1_4(2,1) & J^1_3(2,1) & J^1_2(2,1) & J^1_1(2,1) & J^1_0(2,1) & J^1_{-1}(2,1)\\ J^1_6(2,1) & J^1_5(2,1) & J^1_4(2,1) & J^1_3(2,1) & J^1_2(2,1) & J^1_1(2,1) & J^1_0(2,1)\\ J^1_7(2,1) & J^1_6(2,1) & J^1_5(2,1) & J^1_4(2,1) & J^1_3(2,1) & J^1_2(2,1) & J^1_1(2,1) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 0 & 0 & 0 & 0\\ 6 & 2 & 1 & 0 & 0 & 0 & 0\\ 16 & 6 & 2 & 1 & 0 & 0 & 0\\ 44 & 16 & 6 & 2 & 1 & 0 & 0\\ 120 & 44 & 16 & 6 & 2 & 1 & 0\\ 328 & 120 & 44 & 16 & 6 & 2 & 1 \end{bmatrix}.$$

**Remark 4.2.** Using Definition 2.1, for $n<0$, $J^P_n(k,t)=0$. Set $M_{(n,k)}:=M^{1,1}_{(n,k)}$ and $J_n(k):=J^1_n(k,1)$.

**Theorem 4.3.** *For the inverse of the generalized $(k,1)$-Jacobsthal 1-matrix, denoted by $(M_{(n,k)})^{-1}=[m'_{(ij,k)}]$, we have*

$$m'_{ij}(k)=\begin{cases} 1, & \text{if } i=j,\\ -k, & \text{if } j=i-1,\\ -2, & \text{if } j=i-2,\\ 0, & \text{otherwise.}\end{cases}$$

**Proof.** To find the inverse of inverse of the generalized $(k,1)$-Jacobsthal 1-matrix, we define the $n\times n$ matrix $F_{(k,n)}=[f^k_{ij}]$ as follows:

$$F_{(k,n)}=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0\\ J_2(k) & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ J_{n-1}(k) & 0 & 0 & \dots & 1 & 0\\ J_n(k) & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Clearly, $F_{(k,n)}$ is invertible and

$$(F_{(k,n)})^{-1}=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0\\ -J_2(k) & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -J_{n-1}(k) & 0 & 0 & \dots & 1 & 0\\ -J_n(k) & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Hence,

$$M_{(n,k)} = F_{(k,n)} \times (I_1 \oplus F_{(k,n-1)}) \times (I_2 \oplus F_{(k,n-2)}) \times \cdots \times (I_{n-2} \oplus F_{(2,2)}),$$

where $I_j$ is an identity matrix. Since $(I_t \oplus F_{(k,n-t)})^{-1} = I_t \oplus (F_{(k,n-t)})^{-1}$, we have

$$(M_{(n,k)})^{-1} = (I_{n-2} \oplus (F_{(k,2)})^{-1}) \times \cdots \times (I_1 \oplus (F_{(k,n-1)})^{-1}) \times (F_{(k,n)})^{-1}.$$

Therefore,

$$m'_{ij}(k) = \begin{cases} 1, & \text{if } i = j, \\ -k, & \text{if } j = i-1, \\ -2, & \text{if } j = i-2, \\ 0, & \text{otherwise.} \end{cases}$$

□

**Example 4.4.** For $n = 4$, we have

$$F_{(k,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ J_2(k) & 1 & 0 & 0 \\ J_3(k) & 0 & 1 & 0 \\ J_4(k) & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ k^2+2 & 0 & 1 & 0 \\ k^3+4k & 0 & 0 & 1 \end{bmatrix}.$$

$$M_{(k,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ J_2(k) & 1 & 0 & 0 \\ J_3(k) & J_2(k) & 1 & 0 \\ J_4(k) & J_3(k) & J_2(k) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ k^2+2 & k & 1 & 0 \\ k^3+4k & k^2+2 & k & 1 \end{bmatrix}.$$

$$I_1 \oplus F_{(k,3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & k^2+2 & k & 1 \end{bmatrix}.$$

$$I_2 \oplus F_{(2,k)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}.$$

$$M_{(4,k)} = F_{(4,k)} \times (I_1 \oplus F_{(3,k)}) \times (I_2 \oplus F_{(2,k)}).$$

Therefore, for $k \geq 1$,

$$(M_{(4,k)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ 0 & -k & 1 & 0 \\ -2 & 0 & -k & 1 \end{bmatrix}.$$

Here, we give a factorization of the the generalized $(k,1)$-Jacobsthal 1-matrix. First, we introduce the matrix $V_n^k$.

**Definition 4.5.** Entries of the $n \times n$ matrix $V_n^k = [v_{ij}^k]$ are defined as following:

$$v_{ij}^k = \binom{i-1}{j-1} - k\binom{i-2}{j-1} - 2\binom{i-3}{j-1}. \tag{6}$$

For $i, j \geq 2$, using relation (6), we can write

$$v_{ij}^k = v_{i-1j-1}^k + v_{i-1j}^k,$$

where $v_{11}^k = 1$, $v_{1j}^k = 0$, $j \geq 2$.

For $k = 1$ and $n = 4$, we have

$$V_4^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-k & 1 & 0 & 0 \\ -1-k & 2-k & 1 & 0 \\ -1-k & 1-2k & 3-k & 1 \end{bmatrix}.$$

By the above information, we prove the following theorem.

**Theorem 4.6.** *For the Pascal matrix $P_n$, we have $P_n = M_{(n,k)} V_n^k$.*

**Proof.** The matrix $M_{(n,k)}$ is invertible. If we get $(M_{(n,k)})^{-1} P_n = V_n^k$, then Theorem is proved. Let $(M_{(n,k)})^{-1} P_n = A_n$ where $A_n = (a_{i,j})_{1 \leq i,j \leq n}$, i.e.,

$$a_{i,j} = \sum_{u=j}^{i} m'_{iu}(k) v_{uj}^k.$$

Since $(M_{(n,k)})^{-1}$ and $P_n$ are lower triangular matrices, by the definition of $(M_{(n,k)})^{-1}$, we have

$$\begin{aligned} a_{i,j} &= \sum_{u=j}^{i} m'_{iu}(k)\binom{u-1}{j-1} \\ &= m'_{ii-2}(k)\binom{i-3}{j-1} + m'_{ii-1}(k)\binom{i-2}{j-1} + m'_{ii}(k)\binom{i-1}{j-1} \\ &= -2\binom{i-3}{j-1} - k\binom{i-2}{j-1} + \binom{i-1}{j-1} = (v_{ij}^k)_{1 \leq i,j \leq n}. \end{aligned}$$

□

**Corollary 4.7.** *For $t, u \in \mathbb{N}$,*

$$\binom{t-1}{u-1} = P_{tu} = \sum_{j=u}^{t} m_{tj}(k) v_{ju}^k = m_{t1}(k) v_{1u}^k + m_{t2}(k) v_{2u}^k + \cdots + m_{tt-1}(k) v_{t-1u}^k + m_{tt}(k) v_{tu}^k.$$

*For $u = 1$, we have*

$$P_{t1} = \sum_{j=u}^{t} m_{tj}(k) v_{ju}^k = m_{t1}(k) v_{11}^k + m_{t2}(k) v_{21}^k + \cdots + m_{tt-1}(k) v_{t-11}^k + m_{tt}(k) v_{t1}^k.$$

**Proof.** By Theorem 4.6, we have $P_n = M_{(n,k)}V_n^k$. Then

$$P_n = v_{11} + J_{t-1}(k)v_{21} + \cdots + J_2(k)v_{t-11} + J_1(k)v_{t1}.$$

Let $u = 1$. Since

$$v_{i1} = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{if } i \leq 3, \\ -1-k, & \text{if } i \geq 4, \end{cases} \tag{7}$$

we have the result. □

Here, we define an infinite Generalized $(k, 1)$-Jacobsthal 1-matrix.

**Definition 4.8.** For $k \geq 1$,the Generalized $(k, 1)$-Jacobsthal $p$-matrix, denoted by $O(x) = [J_n(k)]$, is defined as follows:

$$F(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ k & 1 & 0 & 0 & 0 & 0 & \cdots \\ k^2+2 & k & 1 & 0 & 0 & 0 & \cdots \\ k^3+4k & k^2+2 & k & 1 & 0 & 0 & \cdots \\ k^4+6k^2+4 & k^3+4k & k^2+2 & k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = (g_{O(x)}(t), f_{O(x)}(t)).$$

The matrix $O(x)$ is an element of the set of Riordan matrices. Since the first column of $O(x)$ is

$$(1, k, k^2+2, k^3+4k, k^4+6k^2+4, \cdots)^T.$$

Then it is obvious that $g_{O(x)}(t) = \sum_{n=0}^{\infty} J_n(k)t^n = \frac{t}{1-kt-2t^2}$. In the matrix $O(x)$ each entry has a rule the upper two rows, that is,

$$J_n(k) = \begin{cases} 0, & n < 1, \\ 1, & n = 1, \\ kJ_{n-1}(k) + 2J_{n-2}(k), & n > 1. \end{cases}$$

Then $J_{O(x)}(t) = t$, that is

$$O(x) = (g_{O(x)}(t), f_{O(x)}(t)) = (\frac{1}{1-kt-2t^2}, t),$$

hence $O(x)$ is $R$. For these factorization, we need to define $V_n^k(x) = (v_{ij}^k)$, as follows:

$$v_{ij}^k(x) = \binom{i-1}{j-1} - k\binom{i-2}{j-1} - 2\binom{i-3}{j-1}, \tag{8}$$

we have the infinite matrix $V_n^k(x)$ as follows:

$$V_n^k(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1-k & 1 & 0 & 0 & 0 & \cdots \\ -1-k & 2-k & 1 & 0 & 0 & \cdots \\ -1-k & 1-2k & 3-k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{9}$$

**Theorem 4.9.** *Let $V_n^k(x)$ be the infinite matrix as (4) and $O(x)$ be the infinite generalized $(k,1)-$Jacobsthal $1-$ matrix . Then $P(x) = O(x) * V_n^k(x)$, where $P$ is the Pascal matrix.*

**Proof.** From the definitions of the infinite Pascal matrix and the infinite the generalized $(k,1)-$Jacobsthal $1-$ matrix we have the following Riordan representing

$$P(x) = (\frac{1}{1-t}, \frac{t}{1-t}), \quad O(x) = (\frac{1}{1-kt-2t^2}, t).$$

Now we can find the Riordan representation of infinite matrix

$$V_n^k(x) = (g_{V_n^k(x)}(t), f_{V_n^k(x)}(t)),$$

as follows:

$$V_n^k(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1-k & 1 & 0 & 0 & 0 & \cdots \\ -1-k & 2-k & 1 & 0 & 0 & \cdots \\ -1-k & 1-2k & 3-k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

From the first column of the matrix $V_n^k(x)$ we obtain $V_n^k(x) = O(x)^{-1} * P(x)$ and

$$O(x)^{-1} = (g_{O(x)}(t), f_{O(x)}(t))^{-1} = (1-kt-2t^2, t),$$

we have

$$V_n^k(x) = (\frac{1-kt-2t^2}{1-t}, \frac{t}{1-t}),$$

which completes the proof. □

Now we define the $n \times n$ matrix $B(x) = (b_{ij}(x))$ as follows

$$b_{ij}(x) = \binom{i-1}{j-1} - k\binom{i-1}{j} - 2\binom{i-1}{j+1},$$

we have the infinite matrix $B(x)$ as follows

$$B(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-k & 1 & 0 & 0 & \cdots \\ -2k-1 & 1-k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{10}$$

**Lemma 4.10.** *Let $B(x)$ be the matrix in (10). Then we have $P(x) = B(x) * F(x)$.*

**Proof.** The proof is similar to that of Theorem 4.3. □

## 5. Conclusion

Our approach was to introduce generalized $(k,t)$-Jacobsthal $p$-sequences. The combinatorial identities we obtained were also obtained. Then, using generalized $(k,t)$-Jacobsthal $p$-matrices, we factorized the Pascal matrix. Finally, by using the Riordan method, we got two factorizations of the Pascal matrix involving generalized $(k,t)$-Jacobsthal $p$-numbers.

## Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References


[1] H. Akkuş and E. Özkan. Copper fibonacci, copper lucas polynomials and their some special transformations and hyperbolic quaternions. *Proceedings of the Indian National Science Academy*, 2025. https://doi.org/10.1007/s43538-025-00403-4.

[2] M. Asci and E. Gurel. Gaussian jacobsthal and gaussian jacobsthal lucas polynomials. *Notes on Number Theory and Discrete Mathematics*, 19(1):25–36, 2013.

[3] R. Brawer and M. Pirovino. The linear algebra of the pascal matrix. *Linear Algebra and its Applications*, 174:13–23, 1992. https://doi.org/10.1016/0024-3795%2892%2990038-C.

[4] D. Bród and A. Michalski. On generalized jacobsthal and jacobsthal-lucas numbers. *Annales Mathematicae Silesianae*, 36(32):115–128, 2022. https://doi.org/10.2478/amsil-2022-0011.

[5] D. Bród and A. Szynal-Liana. On $J(r,n)$-jacobsthal quaternions. *Pure and Applied Mathematics Quarterly*, 14(3–4):579–590, 2018. https://doi.org/10.4310/PAMQ.2018.v14.n3.a7.

[6] A. Daşdemir. On the jacobsthal numbers by matrix method. *SDU Journal of Science (E-Journal)*, 7(1):69–76, 2012.

[7] Ö. Deveci and G. Artun. On the adjacency-jacobsthal numbers. *Communications in Algebra*, 47(11):4520–4532, 2019. https://doi.org/10.1080/00927872.2018.1541464.

[8] E. Eser, B. Kuloğlu, and E. Özkan. Extending mersenne numbers to bicomplex $(p,q)$-mersenne quaternions with catalan transformations. *Equations*, 5:29–47, 2025. https://doi.org/10.37394/232021.2025.5.5.

[9] S. Falcon. The $k$-fibonacci matrix and the pascal matrix. *Central European Journal of Mathematics*, 9(6):1403–1410, 2011. https://doi.org/10.2478/s11533-011-0089-9.

[10] M. Hashemi and E. Mehraban. Factorization of the $t$-extension of the $p$-fibonacci matrix and pascal matrix. *Journal of Mathematical Modeling*, 10(1):161–172, 2022. https://doi.org/10.22124/jmm.2021.18678.1597.

[11] A. F. Horadam. Jacobsthal and pell curves. *The Fibonacci Quarterly*, 26(1):79–83, 1988. https://doi.org/10.1080/00150517.1988.12429664.

[12] F. Koken and D. Bozkurt. On the jacobsthal-lucas numbers by matrix method. *International Journal of Contemporary Mathematical Sciences*, 3(13):605–614, 2008.

[13] B. Kuloğlu and E. Özkan. Matrix representation of the $(d,k)$-fibonacci polynomials. *Bulletin of the Transilvania University of Brasov Series III: Mathematics and Computer Science*, 4(66)(2):185–200, 2024. https://doi.org/10.31926/but.mif.2024.4.66.2.11.

[14] G. Y. Lee, J. S. Kim, and S. G. Lee. Factorizations and eigenvalues of fibonacci and symmetric fibonacci matrices. *The Fibonacci Quarterly*, 40:203–211, 2002. https://doi.org/10.1080/00150517.2002.12428645.

[15] E. Mehraban, T. A. Gulliver, S. M. Boulaaras, K. Hosseini, and E. Hincal. New sequences from the generalized pell $p$-numbers and mersenne numbers and their application in cryptography. *AIMS Mathematics*, 9(5):13537–13552, 2024. https://doi.org/10.3934/math.2024660.

[16] E. Mehraban, T. A. Gulliver, and E. Hincal. An rsa cryptosystem based on lehmer sequences in some classes of groups. *Advances in Mathematics of Communications*, 19(4):1026–1040, 2025. https://doi.org/10.3934/amc.2024039.

[17] L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson. The riordan group. *Discrete Applied Mathematics*, 34(1–3):229–239, 1991. https://doi.org/10.1016/0166-218X(91)90088-E.

[18] D. Tasci, N. Tuglu, and M. Asci. On fibo-pascal matrix involving $k$-fibonacci and $k$-pell matrices. *Arabian Journal for Science and Engineering*, 36:1031–1037, 2011. https://doi.org/10.1007/s13369-011-0091-3.

[19] S. Uygun. Some bounds for the norms of circulant matrices with the $k$-jacobsthal and $k$-jacobsthal lucas numbers. *Journal of Mathematics Research*, 8(6):133–138, 2016. https://doi.org/10.5539/jmr.v8n6p133.

[20] Ş. Uygun and H. Eldogan. Properties of $k$-jacobsthal and $k$-jacobsthal lucas sequences. *General Mathematics Notes*, 36(1):34–47, 2016.

[21] M. Uysal and E. Özkan. On pell functions and pell numbers. *Discrete Mathematical Structures*, 10(3):1–9, 2023.

Elahe Mehraban
Mathematics Research Center, Near East University TRNC, Mersin 10, 99138 Nicosia, Turkey
Department of Mathematics, Near East University TRNC, Mersin 10, 99138 Nicosia, Turkey
E-mail e.mehraban.math@gmail.com

Evren Hincal
Mathematics Research Center, Near East University TRNC, Mersin 10, 99138 Nicosia, Turkey
Department of Mathematics, Near East University TRNC, Mersin 10, 99138 Nicosia, Turkey
Research Center of Applied Mathematics, Khazar University, Baku, Azerbaijan
E-mail evren.hincal@neu.edu.tr