Magic Labelings of Graphs

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Abstract

In this paper magic labelings of graphs are considered. These are labelings of the edges with integers such that the sum of the labels of incident edges is the same for all vertices. We particularly study positive magic labelings, where all labels are positive and different. A decomposition in terms of basis-graphs is described for such labelings. Basis-graphs are studied independently. A characterization of an algorithmic nature is given, leading to an integer linear programming problem. Some relations with other graph theoretical subjects, like vertex cycle covers, are discussed.

Keywords: graph, labeling, vertex cycle cover.
1992 Mathematics Subject Classification: 05C99

1 Introduction

We consider graphs $G = (V, E)$; $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set of G. We follow the terminology of Jeurissen [2]. A *labeling* of G is a mapping from E to $\mathbb{Z}$ such that for each $v \in V$ the sum of the labels of the edges incident with v is independent of v. This sum is called the *index* of the labeling. Every graph has the trivial zero-labeling. A labeling is called *pseudo-magic* if all labels are different; if moreover no label is negative, the labeling is called *magic*. Finally, a magic labeling is called *positive magic* if all labels are positive. A graph is called *pseudo-magic* if it has a pseudo-magic labeling; and similarly we define magic and positive magic graphs.

The problem to characterize the class of positive magic graphs may be due to the existence of magic squares. The 3×3-magic square is given in Figure 1.1.a. In Figure 1.1.b the graph $K_{3,3}$ has been drawn, one class representing the rows and one class representing the columns. The edges are labeled by the numbers in the magic square. In the square all row sums and column sums are 15. This translates into the index being 15 for all six vertices in the labeled

$K_{3,3}$. We do not give attention to the fact that also the sum of the numbers on the diagonals in the square is 15.

2	9	4
7	5	3
6	1	8

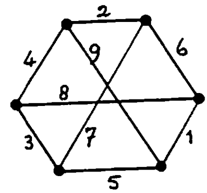

Figure 1.1

A characterization of magic graphs and positive magic graphs was given by Jeurissen [2].

The characterization has a structural flavour and the condition is not easy to check. The accent is on the existence of a labeling. However, once the existence of labelings is guaranteed, the next question is to construct a labeling and preferably one for which the index of the labeling is minimum. Given a graph G that satisfies the condition in the theorem, we have no clue to this construction. We therefore would like to have a characterization of a different, more algorithmic, nature, that enables us to calculate labelings. We will describe an algorithm for finding out whether a graph is pseudo-magic, magic or positive magic. Next to this we will discuss some features of magic labelings, the decomposition of labelings in terms of basis-graphs, describe a class of positive magic graphs, and point out a relation with vertex cycle covers.

2 Some features of labelings, basis-graphs and decompositions

Given a graph and a labeling, there are simple ways to change the labeling into a different one. The reader may easily check the following three lemmas, due to Stewart (see Jeurissen [3]).

Lemma 2.1. If a graph admits a labeling ℓ and contains a perfect matching M, then another labeling ℓ^* is obtained by

$$\begin{aligned} \ell^*(e) &= \ell(e) && \text{for } e \notin M, \\ \ell^*(e) &= \ell(e) + a && \text{for } e \in M, \end{aligned}$$

where a is an arbitrary integer. The index is changed by a.

Lemma 2.2. If a graph admits a labeling ℓ and contains a cycle C of even length, then another labeling ℓ^* is obtained by alternatingly changing the labels of the edges of C by a and $-a$, where a is an arbitrary integer. The index is not changed.

Lemma 2.3. If a graph admits a labeling ℓ and contains a k-factor F, then another labeling ℓ^* is obtained by

$$\begin{aligned} \ell^*(e) &= \ell(e) && \text{for } e \notin E(F), \\ \ell^*(e) &= \ell(e) + a && \text{for } e \in E(F), \end{aligned}$$

where a is an arbitrary integer. The index is changed by $k \cdot a$.

Let us consider a positive magic labeling ℓ of some graph G. A natural question is to ask for a decomposition of the labeling into labelings of subgraphs of G analogous to the decomposition of a vector in a vectorspace into basisvectors.

Let us assume that G is connected and that the index is s. Suppose G has an even cycle C with minimum label a on edge e. Applying Lemma 2.2 such that $\ell(e)$ is changed by $-a$, we obtain a labeling ℓ^* for G in which $\ell^*(e) = 0$. The index of ℓ^* is still s.

We now delete the edge e of G that has label 0 in ℓ^*. The resulting graph G' is a positively labeled graph with at least one even cycle less than G has. Repeating this procedure leads to a graph G^* that has a positive labeling with index s and that does not contain even cycles. Note that after the first step more edges that one may obtain label 0, and that by deletion of edges the graph may have become disconnected.

Suppose now that G^* has two or more odd cycles in one of its components. Then such a component contains a subgraph S consisting of two odd cycles connected by a path, possibly of length 0. Figure 2.1 shows an example.

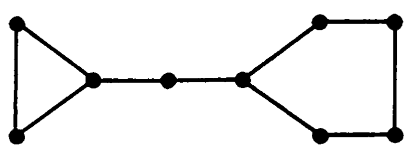

Figure 2.1

We may look upon S as an even cycle that has been "squeezed" by identifying certain pairs of edges. With the aid of these subgraphs, G^* may be further reduced.

Proposition 2.4. If a squeezed even cycle S has a positive labeling ℓ, then it has a non-negative labeling in which at least one edge has label 0.

Proof. We may assume that the two cycles in S are odd. Let m_c and m_p be the minimum of the edge-labels on the cycles and the connecting path, respectively. If $m_p > 2m_c$, we define $x = -m_c$, and $x = -m_p/2$ otherwise. Let e be any edge for which the minimum is attained.

S contains a closed walk of even length in which each cycle-edge is traversed once, and each edge of the connecting path twice. Along this walk, add x and $-x$ alternatingly to each edge-label, starting with $-x$ at e. By construction the edge-label of e will become 0. All other edges have non-negative labels. □

By repeatedly applying Proposition 2.4, we end up with a graph G^{**} in which each component contains at most one cycle, which moreover is odd. So a positive magic labeling can be decomposed into labelings with index 0 of even cycles and squeezed even cycles and positive labelings of unicyclic graphs and trees.

One remark should still be made. When using squeezed even cycles fractions may occur if an odd label in edge e' of the path is used for choosing $x = -\frac{b}{2}$ on e'. The labels on the cycles become fractions. Whenever this occurs we can multiply all labels of the graph, resulting after deleting the edges with label 0, by 2. This then gives a positive labeling with an index $2s$, if s was the index before, with integer labels. For the determination of the basis-graphs, the graphs into which we decompose the given positive magic graph, this choice of the labeling is irrelevant.

Our analysis leads to the study of positive labelings graphs with at most one odd cycle. Doob [1] has given an analysis of magic graphs in terms of chain groups. He already found the even cycles and squeezed even cycles as elements of his zerospace of magic graphs with index 0. His Corollary 2.10 states that magic labelings can be changed to magic labelings of a spanning tree with one chord added forming, one, odd cycle. We focus on positive magic labelings.

3 Basis-graphs

A different approach leading to basis-graphs is to determine the general labeling, in parameter form, and to choose values for the parameters such that a labeling of the desired type arises. We present an example. Consider the graph G_1 in Figure 3.1.

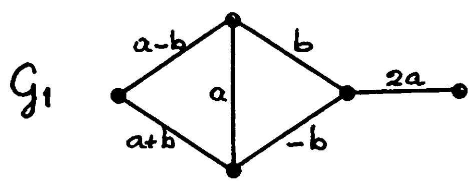

Figure 3.1

It can be easily verified that every labeling of G_1 can be written in the form indicated in Figure 3.1. Of course, other parametrizations are possible.

Since the labels are different *expressions* in a and b, it is possible to assign values to a and b such that the *values* of the labels are different too. Hence G_1 is pseudo-magic. Moreover, the occurrence of the labels b and $-b$ shows that G_1 is not magic.

This simple example also leads us to the concept of *basis-graphs.* Let G_2 be the subgraph of G_1 induced by the edges in which the parameter a occurs, and similarly G_3 for the parameter b; see Figure 3.2.

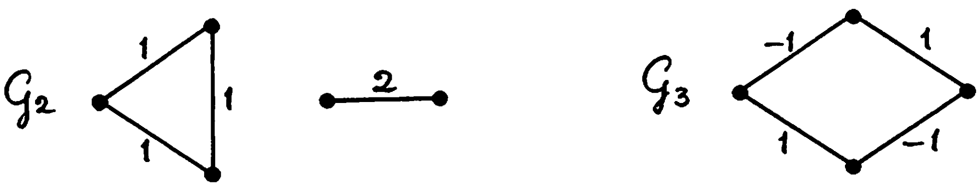

Figure 3.2

The labels in Figure 3.2 correspond to the coefficients of the parameters in Figure 3.1.

The graph G_2 is a spanning subgraph of G_1 and has index 2; G_3 is a non-spanning subgraph of G_1, and therefore has index 0. This inspired us to give the following definitions.

A *basis-graph of type I* is a graph that has a labeling of positive index without 0-labels, which is unique up to a constant factor.

A *basis-graph of type II* is a graph that has a labeling of index 0 without 0-labels, which is unique up to a constant factor.

The problem is "What do basis-graphs look like?"

For basis-graphs of type *I* we have the following result.

Proposition 3.1. If G and H are basis-graphs of type I with disjoint vertex sets, then $G \cup H$ is a basis-graph of type I.

Proof. Assuming that G and H satisfy the above condition, it is easily seen that $G \cup H$ has a labeling of positive index without 0-labels. And if $G \cup H$ had two such labelings, we could make the indices equal by a suitable multiplication. Since G and H are of type I, it follows that the two original labelings only differed by a constant factor. □

Since the converse of Proposition 3.1 is trivially true, we restrict our attention, for the time being, to connected graphs.

From the definition of a basis-graph of type I and Lemma 2.1 if follows that such graphs contain no even cycle. Hence the blocks are P_2's and odd cycles. It also follows from the definition that there is at most one cycle. So we can distinguish two subtypes in the class of basis-graphs of type I: *trees* and *unicyclic graphs* (in which the cycle is odd). We call these graphs ***basis-trees*** and ***basis-rings*** respectively. First we consider the basis-trees.

Proposition 3.2. A tree has a non-trivial labeling if and only if it is balanced.

Proof. (only if) Let G be a bipartite graph with classes of n_1 and n_2 points. Then in each labeling of G with index s, the sum of the edge-labels is $n_1 s$ in one class and $n_2 s$ in the other class. Since the edges are the same, we have $n_1 s = n_2 s$, i.e. either $n_1 = n_2$, or $s = 0$. Hence if the bipartite graph G (in particular: a tree) has a non-trivial labeling, then it is balanced.

(if) Let T be a balanced tree. We construct a labeling of index 1 as follows. Choose an arbitrary vertex r of T as its root. Each vertex has a level: the distance to r. First label the edges incident with the vertices at the highest level. At the next level, each vertex is incident with exactly 1 unlabeled edge. Calculate the label of that edge, remembering that the index is 1, and so on. It is not possible to get stuck because the only vertex where this might happen is r, i.e. only one vertex. Now apply the fact that T is balanced. If the sum is 1 in all vertices with at most one exception, then it must be 1 in that vertex too! □

However, not every balanced tree is a basis-tree. The simplest example is P_4: the central edge has label 0 in each labeling. The following simple result handles the general case.

Proposition 3.3. A balanced tree T is a basis-tree if and only if for all $e \in E(T)$ neither of the two components of $T - e$ is a basis-tree.

Proof. Suppose T is a basis-tree. From the definition it follows that it has a positive labeling. If for some edge e, some component U of $T - e$ is a basis-tree, then the labeling of U can be extended to T. Since labelings in trees are unique up to a constant factor, e will necessarily get label 0; a contradiction.

Conversely, if T is such that for all $e \in E(T)$ neither of the two components of $T - e$ is a basis-tree, then a non-trivial labeling of T can be constructed, and no edge-label will be 0, or else we have a subtree that is a basis-tree. □

Propositions 3.2 and 3.3 imply that all basis-trees can be obtained by applying a *sieve process* for the balanced trees. We first have to generate the balanced trees. This can be done as follows.

Let T be a balanced tree. Color the vertices with 2 colors. Choose a pair of differently colored vertices (in every possible way) and attach an edge and a vertex to each of these. The generation process starts with P_2.

The question whether all balanced trees on $2n$ vertices are generated in this way can be answered affirmatively, with the following argument. If all endvertices have the same color, then all n vertices of the other color have degree ≥ 2, which requires $\geq 2n$ edges, a contradiction. Hence in each balanced tree there exist endvertices of both colors.

In Figure 3.3 we have pictured the first few balanced trees. They are ordered according to the number of vertices, otherwise more or less arbitrarily.

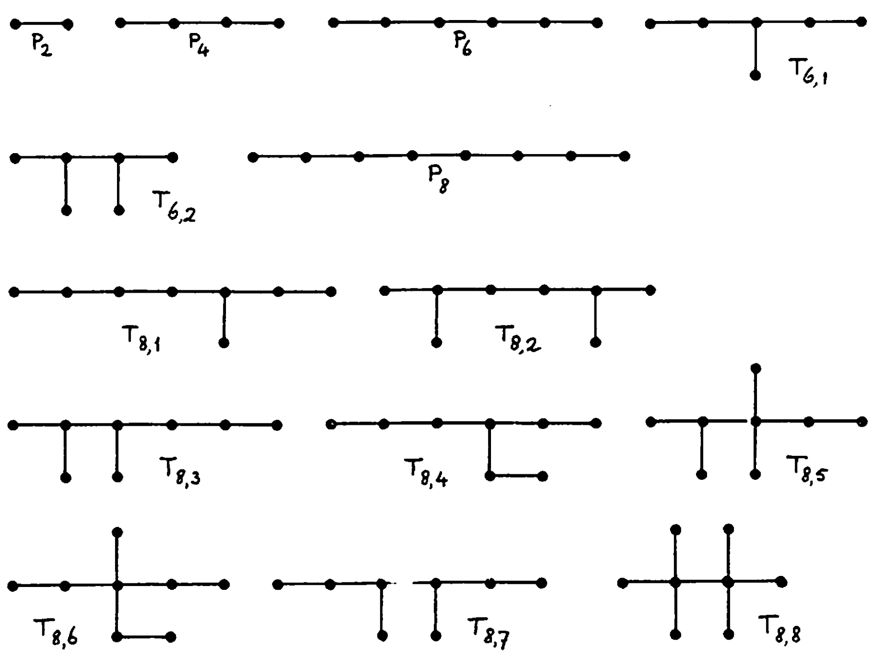

Figure 3.3
The balanced trees on 8 vertices or less

Having obtained the balanced trees, the sieve process proceeds as follows.

The first tree in the list, P_2, is a basis-tree. We then strike out all trees T that contain an edge e such that some component of $T - e$ is P_2, i.e. we strike out $P_4, P_6, T_{6,1}, P_8, T_{8,1}, T_{8,3}, T_{8,4}, T_{8,5}, T_{8,6}$ and $T_{8,7}$. The first remaining tree after P_2 is $T_{6,2}$, and *hence* it is a basis-tree. With $T_{6.2}$ we strike out $T_{8,3}$ and $T_{8,5}$. (It so happens that these have already been stroken out.) The next remaining tree is $T_{8.2}$: our third basis-tree.

In this short example nothing is stroken out by $T_{8.2}$. Finally we find the basis-tree $T_{8,8}$. In Figure 3.4 we present the basis-trees on 10 vertices.

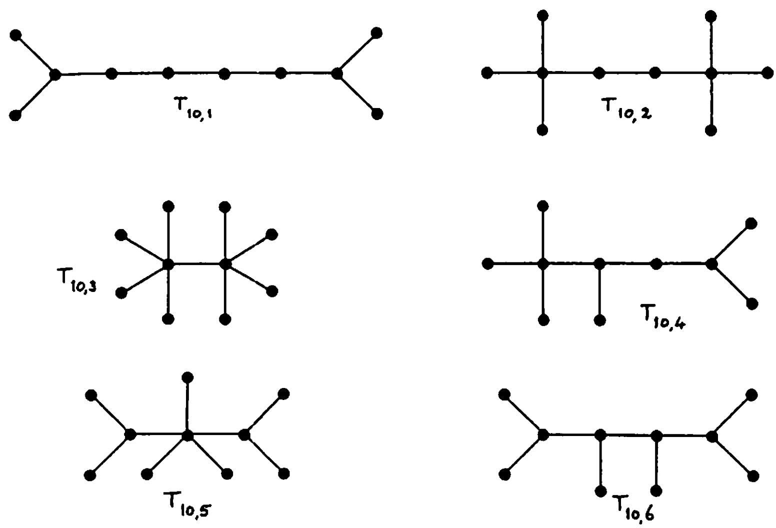

Figure 3.4
The basis-trees on 10 vertices

Beside this systematic approach of generating basis-trees, there are ad hoc methods. First we introduce a useful parameter.

Definition 3.4. Let T be a rooted tree with root r. Starting at the edges incident with end-vertices, we label the edges such that the sum of the labels of the edges incident with every vertex with the exception of vertex r, is 1. (This is possible, see the proof of Proposition 3.2.) Then the *value* $v(T)$ of T is the sum of the labels of the edges incident with r. The resulting function $E(T) \to \mathbb{Z}$ will be called the *quasi-labeling* of T.

Note: If the free tree corresponding to T is a basis-tree, then $v(T) = 1$; the converse is not true.

One ad hoc method to produce basis-trees is as follows. Take a rooted tree T and assume its quasi-labeling has no 0-labels. Then if $v(T) \leq 0$, attach $v(T)+1$ new edges to the root with label 1, and if $v(T) \geq 2$, attach $v(T) - 1$ "forks" to the root (i.e. take $v(T) - 1$ copies of $K_{1,3}$ and identify an end-vertex of each with the root), and assign labels 1 to the end-edges and labels -1 to the edges of attachment. If $v(T) = 1$, then attach e.g. one edge and one fork.

A second ad hoc method also starts with a rooted tree T with root r, and all edge-labels different from 0. Now let T' with root r' be isomorphic to T. If $v(r) \neq 1$, then $T \cup T' \cup \{rr'\}$ is a basis-tree.

We now investigate which unicyclic graphs occur as basis-rings.

Let C_{2n+1} be an odd cycle on the vertices $p_0, p_1, \dots, p_{2n}$, in natural order, and let T_i be a rooted tree with root r_i and value v_i $(i = 0, 1, \dots, 2n)$.
A unicyclic graph G is obtained by identifying p_i and r_i $(i = 0, 1, \dots, 2n)$. Now assign the label x_i to the edge $p_{i-1}p_i$ of C_{2n+1}, where the indices are taken modulo $(2n+1)$. Then an easy calculation shows that the quasi-labelings of the T_i together with $x_0, \dots, x_{2n}$ give a labeling of G (with index 1) if

$$2x_i = 1 - \sum_{j=0}^{2n} (-1)^j v_{i+j}.$$

In order that G be a basis-ring, all labels must be different from 0.

We now turn to the basis-graphs of type II. Examples of these graphs are even cycles and the squeezed even cycles we met in Section 2. Note that the analogue of Proposition 3.1 is false. In fact, all basis-graphs of type II are connected, as is easily seen.

Proposition 3.5. The even cycles and the squeezed even cycles containing two odd cycles are the only basisgraphs of type II.

Proof. Let G be a basisgraph of type II. Then all degrees in G are at least 2. Also, G contains at most one even cycle, squeezed or not. Let p be the number of vertices of degree at least 3 in G. We successively consider the cases $p = 0, 1, 2, 3$, and finally $p \geq 4$.
$\underline{p = 0}$. G is a cycle, and if it is an odd cycle, then each labeling of index 0 will contain a 0-label. Hence G is an even cycle.
$\underline{p = 1}$. G is a union of cycles which have one vertex in common. If the number of cycles is more than 2, then G must contain at least two (squeezed) even cycles. Hence G is the union of two cycles which have one vertex in common. If the cycle lengths are of different parity, each labeling of index 0 will contain a 0-label. It is not possible for both cycles to be even, hence G is a squeezed even cycle (in which the path connecting the two odd cycles has length 0).
$\underline{p = 2}$. G is homeomorf to a pseudograph G^* on two vertices with degrees at least 3. It is easily seen that the number of loops in G^* is at most 2. In each of the subcases to be considered, one quickly obtains either a contradiction or the conclusion that G is a squeezed even cycle.
$\underline{p = 3}$. G is homeomorph to a pseudograph G^* for which the underlying (simple) graph is P_3 or C_3. Again, the number of loops in G^* can be at most 2. Whatever the position of the loops and the multiple lines in G^*, at least three cycles are present in G, at most one of which can be even. But the number of odd cycles cannot be larger than 1 either, and we have a contradiction.

$p \geq 4$. G is homeomorf to a pseudograph G^* in which all degrees are at least 3. If G^* contains a cut-vertex, then a contradiction is obtained in the same way as in the case $p = 3$. If G^* contains no cut-vertex, then G^* contains a *subgraph* isomorphic to one of the three multigraphs shown in Figure 3.5.

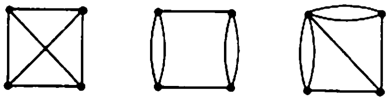

Figure 3.5

If in K_4 all triangles are odd, i.e. correspond to odd cycles in G, then all 4-cycles correspond to even cycles, a contradiction. Hence there exist an even triangle. Then all other triangles must be odd, which still leads to an even quadrangle.

In the second graph of Figure 3.5, the two 2-cycles cannot both be even, nor can both be odd. Hence one is odd, the other is even. In the odd one, we have an even segment and an odd segment. The horizontal paths now either have equal or different parity. In each case we have a second even cycle, a contradiction.

Finally consider the third graph in Figure 3.5. Suppose one of the 2-cycles is odd. Then an even triangle can be formed. This means that there are at least two even cycles, a contradiction. □

4 An algorithmic approach to characterizing positive magic graphs

In this section we will give an algorithmic characterization of positive magic graphs by describing in its general form the algorithm used in the following example.

We consider the class of wheels $W_{n+1} = K_1 \circ C_n$. The wheel W_{n+1} has $K_{1,n}$ as a spanning tree. The spokes of the wheel are its branches and the rim edges of the wheel its chords. Let v_0 be the central vertex, and $v_1, v_2, \ldots, v_n$ the other vertices, in natural order. Let the edges $v_o v_i$ have the variable x_i, the edge $v_{i-1}v_i$ the variable a_i $(i = 2, \ldots, m)$, and $v_n v_1$ the variable a_1. Then, assuming the index is s, the above assignment is a labeling if and only if the following equations are satisfied.

$$\begin{cases} x_1 &= s - a_1 - a_2, \\ x_2 &= s - a_2 - a_3, \\ & \ldots \\ x_{n-1} &= s - a_{n-1} - a_n, \\ x_n &= s - a_n - a_1, \end{cases}$$
$$x_1 + x_2 + \ldots + x_n = s.$$

From this one easily obtains $s = \frac{2\sum_{1}^{n} a_i}{n-1}$ and hence linear expressions for $x_1, \ldots, x_n$ in terms of $a_1, \ldots, a_n$, with rational coefficients.

A few features should be stressed now. First, the way to use a spanning tree to introduce labels x, that act as unknowns. Second, the fact that the chords are given labels a, that act as independent variables. Third, the fact that we can start constructing a solution for s and the x's by choosing the a's in the appropriate way. Fourth, the possibility that non-integer coefficients occur in the solution, whereas we are looking for solutions in $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{N}^+$. Especially the fourth point asks for consideration. In the present example, it suffices to choose $a_1, \ldots, a_n$ such that $n - 1$ divides $2\sum_{1}^{n} a_i$, or to multiply all labels by $n - 1$.

The labelings given for W_5 and W_6 in Figure 4.1 illustrate our results so far. Note that in the labelings for W_6, all expressions have been divided by 2.

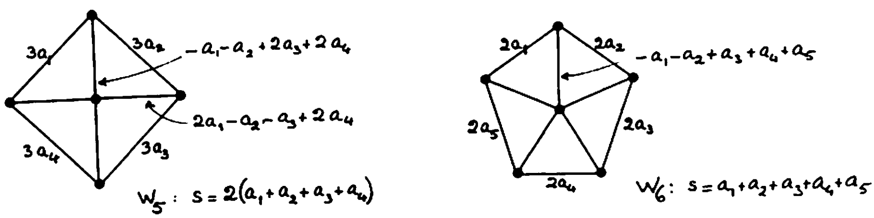

Figure 4.1

As noted before, the fact that the expressions for the labels are different, allows us to choose values for $a_1, \ldots, a_n$ such that the resulting values of the expressions are different. Hence all wheels are pseudo-magic. Moreover, since the sum of the coefficients of the a_i in each x_j is $\frac{2}{n-1} \cdot n - 2 = \frac{2}{n-1} > 0$, we have the following result.

Proposition 4.1. All wheels are positive magic.

Remark. When n is even, an entirely different labeling for W_n can be given: the spoke $v_0 v_i$ gets label a_i and the edge $v_i v_{i+1}$ gets label

$$\sum_{j=1}^{\frac{n}{2}-1} a_{i+2j},$$

where the indices are taken modulo $n-1$. Note that all coefficients of the a_i occurring in the labels are equal to 1. □

By $s(v)$ we denote the sum of the labels of edges incident with vertex v.

Proposition 4.2. A connected graph G with $\delta \geq 2$, $|V| = n$, $|E| = m$ has a positive magic labeling if and only if the variables x_i and s occurring in the following algorithm A satisfy the resulting set of equations with different and positive integers.

Algorithm A

1. Find a spanning tree T of G. Give the chords labels $a_1, \ldots, a_{m-n+1}$.
2. Choose a leaf v_1 of T, and orient T as an outtree with root v_1.
3. Give the branch incident with v_1 the label s minus the sum of the labels of the chords incident with v_1.
4. Determine the successor v_2 of v_1. Give all but one of the branches following this successor labels x_1, x_2 etc. Give the last branch a label so that $s(v_2) = s$.
5. Determine the successors of each of the out-neighbours of v_2 in the oriented tree. If a neighbour, say v_3, is reached by an edge, in most cases carrying a label x, label all but one of its incident branches with x's again and label the last edge so that $s(v_3) = s$. (Note that possibly incident chords have to be taken into account).
6. Continue this process of labeling by x's and adjusting one of the edges to the successors until all branches have obtained a label.
7. Derive equations by demanding $s(v) = s$ for all leaves of T but v_1.

Proof. If G has a positive magic labeling, this determines the a's, the x's and s in such a way that the equations are satisfied. If the equations have a solution for the a's, the x's and s in $\mathbb{N}^+$, G has a positive magic labeling of index S as $s(v) = s$ for all $v \in V(G)$, either by the labeling procedure for all inner vertices of T and v_1 or by the equations demanded in 7. □

Proposition 4.2 leaves open the question about the form of the equations. Can the equations be solved for the x's and s in term of linear expressions of the a's with integer coefficients? This question is less important than we think at first sight. The elimination process leads to linear expressions with rational coefficients in general, and by suitable multiplication of all a's all x's and s are multiplied with the same factor and are then linear expressions in the a's with integer coefficients.

A more serious problem is caused by the occurrence of negative coefficients. If all coefficients of a's are integer and all labels are different expressions in the a's, we have seen that we can construct a pseudo-magic labeling. The problem is, as for the W_5 in the example, whether we can express the labels as different linear expressions in the a's with non-negative coefficients, so that we are sure to get a positive magic labeling.

Let us reconsider the equations for W_5 in Figure 4.1.a. The eight labels are

$$\begin{aligned}
\ell_1 &= 3a_1 \\
\ell_2 &= 3a_2 \\
\ell_3 &= 3a_3 \\
\ell_4 &= 3a_4 \\
\ell_5 &= -a_1 - a_2 + 2a_3 + 2a_4 \\
\ell_6 &= 2a_1 - a_2 - a_3 + 2a_4 \\
\ell_7 &= 2a_1 + 2a_2 - a_3 - a_4 \\
\ell_8 &= -a_1 + 2a_2 + 2a_3 - a_4,
\end{aligned}$$

while $s = 2a_1 + 2a_2 + 2a_3 + 2a_4$.

We have to choose positive a's in such a way that ℓ_5, ℓ_6, ℓ_7 and ℓ_8 are positive. So next to $a_i > 0$, for $i = 1, 2, 3, 4$, we should have

$$\begin{aligned}
-a_1 - a_2 + 2a_3 + 2a_4 &> 0 \\
2a_1 - a_2 - a_3 + 2a_4 &> 0 \\
2a_1 + 2a_2 - a_3 - a_4 &> 0 \\
-a_1 + 2a_2 + 2a_3 - a_4 &> 0.
\end{aligned}$$

A purely algorithmic approach can be based on the fact that, in this case, the sum of the coefficients of the a's is positive, namely 2 in all four inequalities. To prove the existence of a positive magic labeling we put.

$$a_i = M + \Delta_i, \quad i = 1, \ldots, 8,$$

where M is some large positive integer, chosen so large in fact that we can adjust the Δ_i so that the a_i become different. We obtain

$$\begin{aligned}
2M &+ (-\Delta_1 - \Delta_2 + 2\Delta_3 + 2\Delta_4) > 0 \\
2M &+ (\ 2\Delta_1 - \Delta_2 + \Delta_3 + 2\Delta_4) > 0 \\
2M &+ (\ 2\Delta_1 + 2\Delta_2 + \Delta_3 + \Delta_4) > 0 \\
2M &+ (-\Delta_1 - 2\Delta_2 + 2\Delta_3 + \Delta_4) > 0,
\end{aligned}$$

next to $3M + (3\Delta_i) > 0$, $i = 1, 2, 3, 4$, as conditions to be fulfilled. The expressions between parentheses can be made to have different values and we have reached our goal, a positive magic labeling for W_5.

The technique to put $a_i = M + \Delta_i$, with large M, does not work in case we have sums of coefficients that are negative or zero. In these cases there is the problem to modify the technique, if we do not want to refer to integer programming techniques. Suppose in some case the index condition is met and the conditions on the labels are

$$\begin{array}{lllllll}
a_1 & > & 0 & (1), & a_1 + 2a_2 + a_3 - a_4 & > & 0 \quad (6), \\
a_2 & > & 0 & (2), & 2a_1 + a_2 + a_3 - a_5 & > & 0 \quad (7), \\
a_3 & > & 0 & (3), & 2a_1 + 2a_2 + 2a_3 - a_4 - a_5 & > & 0 \quad (8), \\
a_4 & > & 0 & (4), & -a_1 - a_2 + a_4 + a_5 & > & 0 \quad (9). \\
a_5 & > & 0 & (5), & & &
\end{array}$$

The last inequality has zero as sum of the coefficients. The variable a_3 does not occur with a negative coefficient. So whatever the values of the other variables, a_3 can be chosen so large that all inequalities in which it occurs, are satisfied. In the remaining inequalities, the variable a_4 does not occur with negative coefficients. The remaining a_i can be given suitable values and after that we give suitable values to a_4 and a_3, respectively.

This method does not work if at any stage in the procedure there is no a_i that has non-negative coefficients only. If this occurs we can still use integer programming.

An interesting problem is to determine a labeling with minimum value of the index, once the existence of, say, a positive magic labeling is established. Jeurissen [3] mentions a smallest positive magic index for the Petersen graph of 26, and a smallest magic index for the Petersen graph of 23 and for the K_5 a smallest magic index of 20. The general setting that we have developed by Proposition 4.2 has translated this problem into minimizing s under m conditions $\ell_i \geq 0$ respectively $\ell_i > 0$, $i = 1, \ldots, m$, for s and the labels ℓ_i as expressions in terms of the $m-n+1$ a's. This is an integer linear programming problem, the solution of which gives both the existence and the minimum index, in case a labeling exists.

5 Vertex cycle covers

In the preceding section we have met the problem to prove that the labels of the edges can be made positive by proper choice of the a's. Another way to achieve positive labels would be to raise labels in such a way that terms with negative coefficients disappear, while the numbers $s(v)$ are kept equal. Along this line of thought we hit upon a connection with vertex cycle covers.

A *λ-vertex cycle cover* of a graph G, $\lambda \in \mathbb{N}^+$, is a family of cycles in G such that every vertex of G belong to exactly λ cycles.

Lemma 5.1. If a graph G has a p-vertex cycle cover, then it has a labeling with index $2p$.

Proof. Consider a cycle C of the cover. Give weight 1 to every edge of C. Do the same for all other cycles in the cover. The label of an arbitrary edge e will be the sum of its weights. Since each vertex is incident with p cycles, we obtain $s(v) = 2p$, hence the result. □

Lemma 5.1 gives the opportunity to raise all the labels of edges belonging to one of the cycles in the cover by an integer of arbitrary magnitude, e.g. by a multiple of $\sum_{i=1}^{m-n+1} a_i$. This enables us to derive linear expressions for x's and s with positive coefficients.

An example of a 4-vertex cycle cover is shown in Figure 5.1. It so happens that all cycles have length 4 but this is not essential.

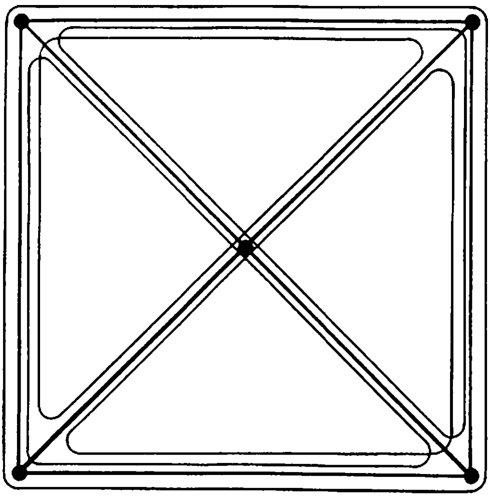

Figure 5.1

Vertex cycle covers seem not to have been studied in the literature. Well-known are edge cycle covers.

A *λ-edge cycle cover* is a family of cycles such that every edge is contained in exactly λ cycles. It has been conjectured that all bridgeless graphs have a double edge cycle cover.

The vertex version of the cycle cover problem is interesting in itself.

Let a graph G have n vertices. Does it admit a λ-vertex cycle cover, $\lambda \in \mathbb{N}^+$? If the graph is λ-regular and the 2-edge cycle cover conjecture is true, there are $\binom{\lambda}{2}$ cycles containing an arbitrary vertex and the graph has a $\binom{\lambda}{2}$-vertex cycle cover. A cycle with one subdivided chord has a 2-edge cycle cover, consisting of three cycles. The two vertices of degree 3 are covered by three and the remaining vertices of degree 2 are covered by two cycles. There is no λ-vertex cycle cover for $\lambda > 0$.

The wheel W_5 is not regular either, but it has as we have seen, a 4-vertex cycle cover. We propose the following problem.

Problem: Characterize the class of graphs that admit a λ-vertex cycle cover for some $\lambda \in \mathbb{N}^+$.

An approach to this problem is the following.

If a graph G has a λ-vertex cycle cover, then let C be the family of cycles. Each vertex determines a set of λ elements from C, that we shall call an intersection. Let the cycles be denoted by $c_1, \ldots, c_c$, then the cover determines n intersections of λ elements c. If $|c_i|$ denotes the length of c_i we must have

$$\sum_{i=1}^{c} |c_i| = n \cdot \lambda,$$

as c_i occurs in $|c_i|$ intersections. If e.g. $n = 5$ and $\lambda = 3$, then the total length of the cycles should be 15. Let the vertices be labeled $1, 2, 3, 4$ and 5. Then there are only C_3's, C_4's and C_5's that can be used as cycles in a cover. Two intersection schemes are

$$\begin{aligned} B_1 &= (c_1, c_2, c_3) \\ B_2 &= (c_1, c_2, c_3) \\ B_3 &= (c_1, c_2, c_3) \\ B_4 &= (c_1, c_2, c_3) \\ B_5 &= (c_1, c_2, c_3) \quad \text{(a)} \end{aligned} \qquad \text{and} \qquad \begin{aligned} B_1 &= (c_1, c_2, c_4) \\ B_2 &= (c_1, c_3, c_4) \\ B_3 &= (c_1, c_3, c_5) \\ B_4 &= (c_2, c_3, c_5) \\ B_5 &= (c_2, c_4, c_5) \quad \text{(b)} \end{aligned}$$

where c_1, c_2 and c_3 are contain all five vertices each in case (a) and $C_1, \ldots, C_5$ each contain three vertices in case (b). Graphs that admit these covers can be constructed from those intersection schemes. C_5 or K_5 in case (a) and the graph K_5 in case (b) are examples. The study of λ-vertex cycle covers can be carried out by surveying all possible intersection schemes. Only those graphs with cycles according to the intersection scheme can have a vertex cycle cover.

$$\begin{aligned} B_1 &= (c_1, c_2, c_3) \\ B_2 &= (c_1, c_2, c_3) \\ B_3 &= (c_1, c_2, c_4) \\ B_4 &= (c_1, c_3, c_4) \\ B_5 &= (c_2, c_3, c_4) \end{aligned}$$

gives e.g. a 3-vertex cycle cover solution for a graph like in Figure 5.2,

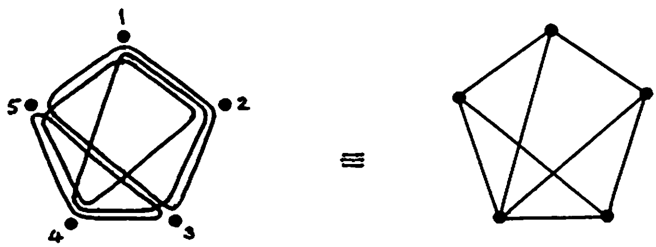

Figure 5.2

which is the wheel W_5, for which graph we already found a 4-vertex cycle cover before.

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References

[1] M. Doob, *Generalization of magic graphs*, J. Comb. Theory **B17** (205–217), 1974.

[2] R.H. Jeurissen, *Magic graphs, a characterization*, Report 8201, Mathematisch Instituut, Katholieke Universiteit Nijmegen, 1982.

[3] R.H. Jeurissen, *Pseudo-magic graphs*, Report 7922, Mathematisch Instituut, Katholieke Universiteit Nijmegen, 1979.