Bipartite Graphs with Balanced (a, b)-Partitions

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Abstract. In this paper, we will be concerned with graphs G satisfying (i) G is isometrically embeddable in a hypercube; and (ii) $|C(a,b)| = |C(b,a)|$ for every edge $[a,b]$ of G, where $C(a,b)$ is the set of vertices nearer to a than to b. Some properties of such graphs are shown, in particular, it is shown that all such graphs G are 3-connected if G has at least two edges and G is not a cycle.

1 Introduction

Throughout this paper, graphs are finite, undirected and have neither loops nor multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. As usual we use $d(u,v)$, $d(u)$ and $N(u)$ to denote the distance of vertices $u,v \in V(G)$, the degree of a vertex u and the set of neighbors of u in G, respectively. For a subset $W \subseteq V(G)$, we denote by $< W >$ the subgraph of G induced by W. The subgraph $< V(G) - W >$ is simply denoted by $G - W$.

Let G be a connected graph. A subset W of $V(G)$ is *convex* in G if for all $u,v \in W$ all shortest (u,v)-paths are contained in $< W >$. For each edge $[a,b]$, we define $C(a,b) = \{x \in V(G) : d(a,x) < d(b,x)\}$ and $S(a,b)$ $(= S(b,a)) = \{x \in V(G) : d(a,x) = d(b,x)\}$. The triple $\{C(a,b), C(b,a), S(a,b)\}$ is called the (a,b) *-partition* of G. Note that if G is bipartite, then we have always $S(a,b) = \emptyset$.

A nontrivial connected graph G is *even* if for any vertex v of G there exists a unique vertex $\overline{v}$ such that $d(v,\overline{v}) = diam(G)$, the diameter of G. An even graph G is called *harmonic* if $[\overline{u},\overline{v}] \in E(G)$ whenever $[u,v] \in E(G)$, and is called *symmetric* if $d(u,v) + d(u,\overline{v}) = diam(G)$ for all $u,v \in V(G)$, see [4]. (Symmetric-even graphs are equivalent to 'antipodal graphs', see [1].)

Lemma 1.1. *If a graph G is symmetric-even, G satisfies*
(B1) $|C(a,b)| = |C(b,a)|$ *for all* $[a,b] \in E(G)$.

Proof. Let $[a,b] \in E(G)$ and let $z \in C(a,b)$. If $\overline{z} \in C(a,b)$, then $diam(G) = d(z,a) + d(a,\overline{z}) < d(z,b) + d(b,\overline{z}) = diam(G)$, a contradiction.

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If $\bar{z} \in S(a,b)$, then $d(a,\bar{z}) = d(b,\bar{z})$. Since $diam(G) = d(a,z) + d(a,\bar{z}) = d(b,z) + d(b,\bar{z})$, we have $d(a,z) = d(b,z)$, i.e., $z \in S(a,b)$, a contradiction. Hence, $|C(a,b)| = |C(b,a)|$ holds. ∎

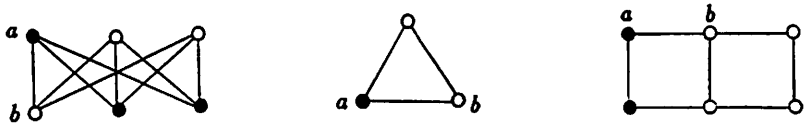
Harmonic-even graphs do not always satisfy (B1), see [4, Fig.2 H_3]. However, by [3, Prop.4.1;Thm.4.2], if a harmonic-even graph is isometrically embeddable in a hypercube, then it is symmetric. Hence, all such harmonic-even graphs also satisfy (B1). (*Note.* In [3], harmonic-evenness and isometric-embeddability in hypercube have been used as properties of 'tope graphs' of oriented matroids.)

As a characterization of graphs isometrically embeddable in a hypercube, the following theorem is well-known:

Theorem 1.2. (Djoković [2]). *A connected graph G is isometrically embeddable in a hypercube if and only if*

(B2) *G is bipartite, and*
(B3) *$C(a,b)$ is convex for all $[a,b] \in E(G)$.*

In this paper, we will be concerned with connected graphs satisfying the conditions (B1) $\sim$ (B3). As seen in Fig.1, the conditions (B1), (B2) and (B3) are independent of each other. The vertices depicted by • belong to $C(a,b)$ for the edge $[a,b]$ of each figure.



(i) (B1) (B2) ~~(B3)~~ (ii) (B1) ~~(B2)~~ (B3) (iii) ~~(B1)~~ (B2) (B3)

Fig. 1.

Our main theorem says:

Theorem 1.3. *If a connected graph G with $|E(G)| \geq 2$ satisfies* (B1) $\sim$ (B3) *and G is not a cycle, then G is 3-connected.*

By this theorem, we know that every harmonic-even graph isometrically embeddable in a hypercube is 3-connected unless G is a cycle or K_2. Note that this fact also follows from the following result by Göbel and Veldman: every symmetric-even graph is 3-connected unless G is a cycle or K_2, see [4, Thm.17].

The graph in Fig.2 shows that connected graphs with (B1) $\sim$ (B3) are not necessarily even. The labels of vertices are from those of the hypercube on the 5-element set $\{1, 2, ..., 5\}$. For example, the label 145 denotes the subset $\{1, 4, 5\}$.

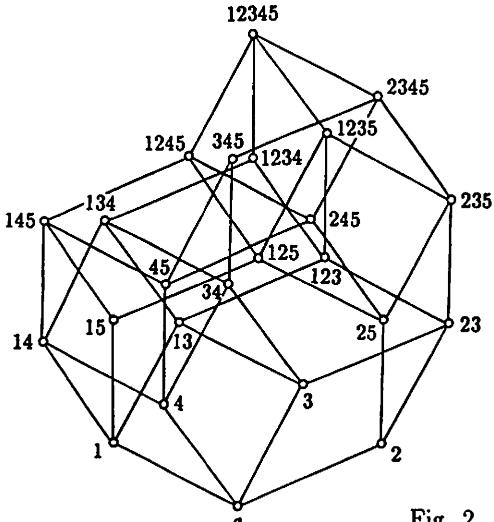


Fig. 2.

2 Proof of the main theorem

Throughout this section, let G be a connected graph with n vertices. First we will investigate some properties of connected graphs with (B1) and (B2), and then we will prove the main theorem.

Lemma 2.1. *If G satisfies* (B1) *and* $|E(G)| \geq 2$, *then G is 2-connected.*

Proof. Suppose there is a vertex a such that the subgraph $G - a$ is disconnected. Let G_1 denote a (connected) component of $G - a$ and let G_2 denote the graph formed by the remaining components. If $b \in V(G_1) \cap N(a)$, then $V(G_2) \cup a \subseteq C(a, b)$, and hence $|V(G_2)| + 1 \leq n/2$. If $c \in V(G_2) \cap N(a)$, then $V(G_1) \cup a \subseteq C(a, c)$, and hence $|V(G_1)| + 1 \leq n/2$. Now we have $n + 1 = |V(G_1)| + |V(G_2)| + 2 \leq n$, which is a contradiction. ∎

Lemma 2.2. $[a, b], [u, v] \in E(G)$, $u \in C(a, b)$ *and* $v \in C(b, a)$ *imply* $d(a, u) = d(b, v)$.

Proof. Since $d(b,v) < d(a,v) \leq d(a,u) + d(u,v)$, we have $d(b,v) \leq d(a,u)$. On the other hand, since $d(a,u) < d(b,u) \leq d(b,v) + d(v,u)$, we have $d(a,u) \leq d(b,v)$. Hence, the equation $d(a,u) = d(b,v)$ holds. ∎

We write $G_1 \cong G_2$ if graphs G_1 and G_2 are isomorphic, and denote by $\delta(G)$ the minimum degree of a graph G. For $k \geq 3$, C_k denotes the cycle of length k. For every edge $[a,b]$ in G, we define $\omega(a,b) = \{[u,v] : u \in C(a,b) \text{ and } v \in C(b,a)\}$ and $U(a,b) = \{x \in C(a,b) : x \text{ is an endpoint of some edge in } \omega(a,b)\}$.

Lemma 2.3. *Let G satisfy* (B1) *and* (B2), *and let $G \not\cong C_n$ and $|E(G)| \geq 2$. Then we have*

(i) $\delta(G) \geq 3$; *and*
(ii) *for every edge $[a,b]$ in G, the set $\omega(a,b)$ contains a matching with cardinality ≥ 3.*

Proof. (i) By Lemma 2.1, $\delta(G) \geq 2$. Suppose there is a vertex $b \in V(G)$ with degree 2, i.e., $d(b) = 2$. It suffices to show the equation $d(a) = 2$ for any neighbor a of b, because then by the connectiveness of G, we have $d(x) = 2$ for all $x \in V(G)$.

Now by Lemma 2.1, the graph obtained from G by deleting the edge $[a,b]$ is connected, and hence there is an edge $[u,v]$ in G such that $a \neq u \in C(a,b)$ and $b \neq v \in C(b,a)$. In fact, such an edge $[u,v]$ is uniquely determined, which is showed as follows. First, consider the set $C(c,b)$, where c is the other neighbor of b. Clearly, $C(b,a) - b \subseteq C(c,b)$, and by Lemma 2.2, $U(a,b) - a \subseteq C(c,b)$. Since $|C(c,b)| = n/2$ by (B1) and (B2), we know the fact $U(a,b) = \{a,u\}$. Next, consider the set $C(u,v)$. By Lemma 2.2, $C(a,b) \subseteq C(u,v)$, and so we know there is no vertex in the set $(N(u) - v) \cap C(b,a)$. Hence we have $U(b,a) = \{b,v\}$, which means $\omega(a,b) = \{[a,b],[u,v]\}$.

Put $k = d(a,u)$. Then by Lemma 2.2, $d(b,v) = k$. Choose a vertex $f \in N(a) - b$. For all $x \in N(v) - u$, $d(b,x) = k-1$ (if $d(b,x) = k+1$, then $a, u \in C(v,x)$, i.e., $C(a,b) \subset C(v,x)$, a contradiction). Hence $d(a,x) = k$ and $d(f,x) = k+1$, and so we have $C(b,a) - v \subseteq C(a,f) \ni a$. This implies $C(a,f) = (C(b,a) - v) \cup \{a\}$, and hence we have $N(a) = \{b,f\}$, i.e., $d(a) = 2$.

(ii) By Lemma 2.1, $\omega(a,b)$ contains a matching $\{[a,b],[a',b']\}$ of cardinality 2, where $a' \in C(a,b)$ and $b' \in C(b,a)$. Put $k = d(a,a') = d(b,b')$. It suffices to show that $|U(a,b)| \geq 3$ and $|U(b,a)| \geq 3$. Suppose $|U(a,b)| = 2$, that is, $U(a,b) = \{a,a'\}$. Then for all $x \in N(a') \cap C(a,b)$, $d(a,x) = k-1$ (if $d(a,x) = k+1$, then $C(b,a) \cup a \subseteq C(a',x)$, a contradiction). Hence

$d(b,x) = k$. Also, for all $c \in N(b) - a$, $d(c,x) = k+1$ (if $d(c,x) = k-1$, then $d(c,a') = k-2$, which contradicts $d(b,a') = k+1$). Hence $C(a,b) - a' \subseteq C(b,c) \ni b$. Moreover by (i), there is $g \in N(b) - \{a,c\}$ such that $g \in C(b,c)$, which shows $|C(b,c)| > n/2$, a contradiction. Thus we have $|U(a,b)| \geq 3$. By symmetry, we have also $|U(b,a)| \geq 3$. ∎

Lemma 2.4. *Let G satisfy* (B1) *and* (B2) *and let $|E(G)| \geq 2$. Then for all adjacent vertices $a,b \in V(G)$, the subgraph $G - \{a,b\}$ is connected.*

Proof. Suppose $G - \{a,b\}$ is disconnected. Then by Lemma 2.1, at least one of the subgraphs $< C(a,b) - a >$ and $< C(b,a) - b >$ is disconnected. We may assume that $< C(a,b) - a >$ is disconnected without loss of generality. Now, since $G - a$ is connected, there must be an edge between each component of $< C(a,b) - a >$ and $C(b,a)$. This implies that $< C(b,a) - b >$ is also disconnected. Hence any component of $G - \{a,b\}$ can be denoted by $< V_1 \cup V_2 \cup ... \cup V_s \cup W_1 \cup W_2 \cup ... \cup W_t >$, where V_i $(1 \leq i \leq s)$ and W_j $(1 \leq j \leq t)$ are vertex sets of components of $< C(a,b) - a >$ and $< C(b,a) - b >$, respectively. Now for $x \in N(a) \cap V_1$, $V(G) - \{V_1 \cup ... \cup V_s \cup W_1 \cup ... \cup W_t\} \subseteq C(a,x)$. Hence $|V_1 \cup ... \cup V_s \cup W_1 \cup ... \cup W_t| \geq n/2$, that is, the number of vertices of any component of $G - \{a,b\}$ is $\geq n/2$. This is a contradiction. ∎

For vertices u,v of a graph G, we denote $I(u,v) = \{w \in V(G) :$ w lies on a shortest (u,v)-path in $G\}$ and call each set $I(u,v)$ an *interval* in G.

Lemma 2.5. *Let G satisfy* (B1) *and* (B2), *and let $G \not\cong C_n$ and $|E(G)| \geq 2$. Suppose that the subgraph $G - \{a,b\}$ is disconnected, and choose such vertices a and b so that the distance $d(a,b)$ is minimum. Let $c \in N(a) \cap I(a,b)$. Then*

(i) *both subgraphs $< C(a,c) - a >$ and $< C(c,a) - b >$ are disconnected; and*

(ii) *the number of components of $G - \{a,b\}$ is exactly 2, and they can be written as follows:*

$$G_1 \equiv < V_1 \cup ... \cup V_s \cup W_1 \cup ... W_t > \quad (W_1 \ni c),$$
$$G_2 \equiv < V_{s+1} \cup ... \cup V_p \cup W_{t+1} \cup ... W_q > \quad (s < p,\ t < q),$$

where $V_1, ..., V_p$ are the vertex sets of components of $< C(a,c) - a >$ and $W_1, ..., W_q$ those of $< C(c,a) - b >$. Moreover, for every $x \in N(b) \cap W_j$ $(t < j \leq q)$, $V(G_2) \supseteq C(x,b)$ holds.

Proof. By Lemma 2.4, $d(a,b) \geq 2$. Put $k = d(c,b) = d(a,b) - 1$.

(i) By Lemma 2.3 (ii), at least one of $< C(a,c) - a >$ and $< C(c,a) - b >$ is disconnected. We distinguish two cases and derive contradictions.

Case (1): $< C(a,c) - a >$ is connected. Denote the vertex sets of components of $< C(c,a) - b >$ by W_1 $(\ni c), W_2, ..., W_q$. Since G is 2-connected, there is an edge between each W_j and $C(a,c)$. Here note that, for $j \geq 2$, such an edge does not have a as an endpoint. Hence the components of $G - \{a,b\}$ are exactly $< \{C(a,c) - a\} \cup W_2 \cup ... \cup W_q >$ and $< W_1 >$. This fact shows $[a,c]$ is the only edge between $C(a,c)$ and W_1. Moreover, we have $|W_1| \geq 2$, because $|N(c) - a| \geq 2$ by Lemma 2.3 (i). Thus $G - \{b,c\}$ is disconnected, which contradicts the minimality of $d(a,b)$.

Case (2): $< C(c,a) - b >$ is connected. Denote the vertex sets of components of $< C(a,c) - a >$ by $V_1, V_2, ..., V_p$. Since $G - \{a,b\}$ is disconnected, for some i, $N(b) \cap V_i \neq \emptyset$ and there is no edge between V_i and $C(c,a) - b$. Now, let $x \in N(b) \cap V_i$. Then $C(c,a) \subseteq C(b,x)$, and hence $N(b) \cap C(a,c) = \{x\}$. Since $|N(x) - b| \geq 2$ and since $[x,b]$ is the only edge between V_i and $C(c,a)$, we know that $G - \{a,x\}$ is disconnected, which contradicts the minimality of $d(a,b)$. (Note that $d(a,x) = d(c,b)$.)

(ii) Let $V_1, ..., V_p$ be the vertex sets of components of $< C(a,c) - a >$, and let W_1 $(\ni c), ..., W_q$ be those of $< C(c,a) - b >$. First, we show that any component of $G - \{a,b\}$ is a subgraph induced by at least one V_i and at least one W_j. Now, suppose some $< V_i >$ forms a component in $G - \{a,b\}$. Then, since $G - a$ is connected, there must be an edge $[x,b]$ between V_i and b. For this vertex x, we can easily show that $G - \{a,x\}$ is disconnected, which contradicts the minimality of $d(a,b)$. Next, suppose some $< W_j >$ forms a component in $G - \{a,b\}$. Since $G - b$ is connected, we know $j = 1$. Let y be any neighbor of c in $< W_j >$. If $d(y,b) = k+1$, then $C(a,c) \cup \{b\} \subseteq C(c,y)$, a contradiction. Hence, we know $d(y,b) = k - 1$, that is, $y \in I(c,b)$. This means $G - \{b,c\}$ is disconnected, which contradicts the minimality of $d(a,b)$ again.

Now, we can denote the component of $G - \{a,b\}$ containing c by $G_1 \equiv < V_1 \cup ... \cup V_s \cup W_1 \cup ... W_t >$, and the second component by $G_2 \equiv < V_{s+1} \cup ... \cup V_l \cup W_{t+1} \cup ... W_m >$ $(s < l \leq p,\ t < m \leq q)$. To show the equations $l = p$ and $m = q$, it is sufficient to show that $V(G_2) \supseteq C(x,b)$ for every $x \in N(b) \cap W_j$ $(t < j \leq m)$. Because then we have $|V(G_2)| \geq n/2$. Now choose any vertex $x \in N(b) \cap W_j$ $(t < j \leq m)$. By $d(b,a) = k+1$, either $d(x,a) = k$ or $= k+2$. If $d(x,a) = k$, then every shortest (x,a) -path arrives at the vertex a through an edge $[u,v]$ between W_j and some V_i $(s < i \leq l)$, where $u \in W_j$ and $v \in V_i$. Hence we have $d(x,a) > d(v,a) = d(c,u) > d(c,b) = k$, a contradiction. So $d(x,a) = k+2$ holds, and so a belongs to $C(b,x)$. Hence we have $V(G) - V(G_2) \subseteq C(b,x)$, that is, $V(G_2) \supseteq C(x,b)$. This completes the proof. ∎

Proof of Theorem 1.3. Under the same assumptions as in Lemma 2.5, we will derive a contradiction. We use the notations in Lemma 2.5. Put $k = d(c, b)$ and let $y \in N(a) \cap V_1$. Then either $d(y, b) = k$ or $= k + 2$ holds.

If $d(y, b) = k$, then every shortest (y, b) -path arrives at the vertex b through an edge $[u, v]$ between V_1 and $W_l \cup b$ for some l $(1 \leq l \leq t)$, where $u \in V_1$ and $v \in W_l \cup b$. Now let $[e, f]$ be any edge such that $e \in V_i$ $(s < i \leq p)$ and $f \in W_j$ $(t < j \leq q)$. Then by (B3), $d(v, f) = d(v, b) + d(b, f) < k + (d(c, f) - d(c, b)) = d(c, f) = d(a, e)$. Also by (B2) and (B3), $d(v, f) = d(u, e) = d(u, a) + d(a, e) > d(a, e)$. This is a contradiction.

On the other hand, if $d(y, b) = k + 2$, then $b \in C(a, y)$. Hence, $V(G_2) \cup \{a, b\} \subseteq C(a, y)$ and we have $|V(G_2)| < n/2$, which contradicts the second statement of Lemma 2.5 (ii). ∎

In Theorem 1.3, it may not need for G to satisfy (B3), which we will leave as an open question:

Question. *If G satisfies* (B1) *and* (B2) *and if* $G \not\cong C_n$ *and* $|E(G)| \geq 2$, *then G is* 3*-connected?*

Acknowledgement: I would like to thank K. Fukuda for his valuable comments, and S. Honiden for his great encouragement.

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