An Inequality Characterizing Chordal Graphs*

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Abstract

A simple inequality involving the number of components in an arbitrary graph becomes an equality precisely when the graph is chordal. This leads to a mechanism by which any graph parameter, if always at least as large as the number of components, corresponds to a subfamily of chordal graphs. As an example, the domination number corresponds to the well-studied family of P_4, C_4-free graphs.

Given any complete subgraph Q of G, define the *common neighborhood* of Q, denoted $N(Q)$, to be the subgraph induced by all vertices $v \in V(G)$ that are adjacent with every vertex in Q. Notice that $N(Q) \cap Q = \emptyset$. Define the common neighborhood of the *null subgraph* (the subgraph with no vertices) to be the entire graph G. For any vertex v of a graph G, set $N(v) = N(\{v\})$ and $N[v] = N(v) \cup \{v\}$, and let comp G count the number of connected components in G.

A graph is *chordal* when no cycle of length greater than three is an induced subgraph. Reference [3] contains a thorough survey of the theory and applications of chordal graphs (called "triangulated graphs there"), including most of the characterizations in the literature at that time; [7] is a more recent survey, from a different point of view. The following theorem presents a new characterization of chordal graphs that grew out of work in [8], attempting to identify interesting families of graphs that contain all chordal graphs.

Theorem 1 *For every graph G,*

$$\sum_{Q}[1 - \text{comp } N(Q)] \leq \text{comp } G \quad \textit{and} \quad \sum_{R}[1 - \text{comp } N(R)] \leq 1, \qquad (1)$$

where the first sum is taken over all nonempty complete subgraphs Q of G and the second sum is taken over all complete or null subgraphs R of G. Moreover, equality holds in either inequality if and only if G is chordal.

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Proof. We argue by induction on the order of G, noting that the result is immediate (and equality holds) when $G = K_1$; indeed, whenever G is edgeless. So suppose $v \in V(G)$ is not isolated. For each subgraph H of G, let $N_H(Q)$ denote the common neighborhood of Q in H and—in this proof only—let cN_HQ abbreviate comp $N_H(Q)$ and cN_Hv abbreviate comp $N_H(v)$. Then

$$\sum_{Q}[1-\text{comp } N(Q)] = \sum_{Q\subseteq G}[1-cN_GQ] = \sum_{\substack{Q\subseteq N[v]\\ Q\not\subseteq N(v)}}[1-cN_GQ] + \sum_{Q\subseteq G-v}[1-cN_GQ].$$

Each complete subgraph Q of $N[v]$ with $\{v\} \neq Q \not\subseteq N(v)$ has $N_G(Q) \subseteq N_G(v)$ and corresponds to $Q' = Q \cap N(v) \subseteq N(v)$ where $cN_GQ = cN_{N(v)}Q'$. Since each complete subgraph Q of $G - v$ with $Q \not\subseteq N(v)$ has $cN_GQ = cN_{G-v}Q$, $\sum_{Q\subseteq G}[1 - cN_GQ]$ equals

$$[1 - cN_Gv] + \sum_{Q\subseteq N(v)}[1 - cN_{N(v)}Q] + \sum_{Q\subseteq N(v)}[1 - cN_GQ] + \sum_{\substack{Q\subseteq G-v\\ Q\not\subseteq N(v)}}[1 - cN_{G-v}Q].$$

Since the last summation can be split into $\sum_{Q\subseteq G-v}[1 - cN_{G-v}Q]$ minus $\sum_{Q\subseteq N(v)}[1 - cN_{G-v}Q]$, we have that $\sum_{Q\subseteq G}[1 - cN_GQ]$ equals

$$1 - cN_Gv + \sum_{Q\subseteq N(v)}[1 - cN_{N(v)}Q - cN_GQ + cN_{G-v}Q] + \sum_{Q\subseteq G-v}[1 - cN_{G-v}Q].$$

But, for each complete subgraph Q of $N(v)$,

$$cN_GQ \geq cN_{G-v}Q - cN_{N(v)}Q + 1, \tag{2}$$

where $-cN_{N(v)}Q+1$ compensates for components of $N_{G-v}(Q)$ that combine when v is included, and the inequality becomes strict when components of $N_{N(v)}(Q)$ are joined by a path within $N_{G-v}(Q)$. Thus each term of the preceding sum over $Q \subseteq N(v)$ is nonpositive, and so

$$\sum_{Q\subseteq G}[1 - cN_GQ] \leq 1 - cN_Gv + \sum_{Q\subseteq G-v}[1 - cN_{G-v}Q].$$

Using the inductive hypothesis on the subgraph $G-v$, $\sum_{Q\subseteq G}[1-cN_GQ] \leq 1 - cN_Gv + \text{comp } (G - v)$, and that is less than or equal to comp G by the analogue of (2) where Q is the null subgraph.

For the "moreover" part, first supose G is chordal. Then we can choose v to be a simplicial vertex, and $Q \subseteq N(v)$ implies that $cN_GQ = cN_{G-v}Q - cN_{N(v)}Q + 1$. Thus, using a perfect elimination ordering and induction,

$$\sum_{Q\subseteq G}[1 - cN_GQ] = 1 - cN_Gv + \sum_{Q\subseteq G-v}[1 - cN_{G-v}Q] = \text{comp } G.$$

If G is not chordal, then, somewhere along the line, $cN_GQ > cN_{G-v}Q - cN_{N(v)}Q + 1$, and so $\sum_{Q \subseteq G}[1 - cNQ_G] < \text{comp } G$. □

Corollary 2 *Suppose f is any graph parameter such that, for every graph H,* comp $H \le f(H)$. *Then*

$$\sum_Q [1 - f(N(Q))] \le f(G) \quad \textit{and} \quad \sum_R [1 - f(N(R))] \le 1, \tag{3}$$

where the first sum is taken over all nonempty complete subgraphs Q of G and the second sum is taken over all complete or null subgraphs R of G. Moreover, those graphs for which equality holds in either inequality are necessarily chordal.

Proof. The comp $H \le f(H)$ assumption implies the first inequality in

$$\sum_R [1 - f(N(R))] \le \sum_R [1 - \text{comp } N(R)] \le 1,$$

with the second from Theorem 1. Equality in (3) implies, by the preceding double inequality and Theorem 1, that G is chordal. □

Every choice of allowable parameter f in Corollary 2 produces an inequality, but the family of graphs for which equality holds—and here is where all the work is involved in proofs—is frequently uninteresting. For instance, if f counts the number of blocks, then equality holds if and only if every component of G is complete. If f counts the number of vertices (the order), then equality holds if and only if G is edgeless. (Both are important families of graphs, but ....) Other things can lead to a disappointing family, as illustrated at the end of this paper, but Theorem 3 involves a parameter that does lead to a satisfying family.

A set $S \subseteq V(G)$ *dominates* G whenever every vertex in $V(G) \backslash S$ has an adjacent vertex in S. The *domination number of* G, denoted $\gamma(G)$, is the cardinality of a smallest set that dominates G.

Theorem 3 *For every graph G,*

$$\sum_Q [1 - \gamma(N(Q))] \le \gamma(G) \quad \textit{and} \quad \sum_R [1 - \gamma(N(R))] \le 1, \tag{4}$$

where the first sum is taken over all nonempty complete subgraphs Q of G and the second sum is taken over all complete or null subgraphs R of G. Moreover, equality holds in either inequality if and only if G is a P_4-free chordal graph.

Proof. The inequalities (4) follow from Corollary 2.

For the "moreover" part, first suppose $\sum_R[1 - \gamma(N(R))] = 1$. Corollary 2 implies that G is chordal. We argue that G is P_4-free by induction on $|V(G)|$, with the result immediate when $|V(G)| \leq 2$. So suppose $|V(G)| > 2$ and $u \in V(G)$ is simplicial. Assume, without loss of generality, that G is connected.

Suppose $\sum_R[1 - \gamma(N(R))] = 1$ over G and consider what happens to this sum when u is removed:

(a) The term with $R = N[u]$ equals $1-0 = 1$ (since $N(R)$ will be empty) and any other term with $u \in R$ equals $1 - 1 = 0$ (since $N(R)$ will be dominated by one vertex); all these terms disappear when the sum is taken over $G - u$.

(b) The term with $R = N(u)$ increases by one when the sum is taken over $G - u$ (since u was an isolated vertex in $N(R)$ and so $\gamma(N(R))$ decreases by one).

(c) Any term with R a proper (possibly empty) subset of $N(u)$ either is unchanged when the sum is taken over $G - u$ (as happens when $N(u)$ has a vertex in common with some minimum dominating set of $N(R) - u$ and so u is not needed in a minimum dominating set of $N(R)$ in G) or increases by one (because u was needed).

(d) Any term with $R \not\subseteq N[v]$ is unchanged when the sum is taken over $G - u$.

Any increases in case (c) would make $\sum_R[1 - \gamma(N(R))] \geq 2$ in $G - u$, contradicting inequality (4). So there are no increases in case (c) and $\sum_R[1 - \gamma(N(R))] = 1$ in $G - u$. The inductive hypothesis implies that $G - u$ is P_4-free. Since $G - u$ is P_4, C_4-free, it is easy to show a vertex has maximum degree in $G - u$ if and only if that vertex dominates $G - u$, and so that the minimum dominating sets are precisely the dominating vertices. Suppose $uvwx$ is an induced path in G, arguing toward a contradiction. Put $R = N(u) \cap N(v) \cap N(w) \cap N(x)$. Since $v \notin R \subseteq N(u)$, we have that R is a proper subset of $N(u)$. Hence, since there are no increases in case (c), there is a dominating vertex d of $N(R) - u$ with $d \in N(u)$. Note that $d \in N(u)$ implies $d \neq u, w, x$, and so d is adjacent to u, w, and x. Also, d adjacent to x implies $d \neq v$, and so d is adjacent to v. But that makes $d \in R$, contradicting that $d \in N(R)$. Therefore G is P_4-free.

Conversely, suppose G is a P_4-free chordal graph. We argue by induction on the order of G, with the result immediate when $|V(G)| \leq 2$. So suppose $|V(G)| > 2$ and $u \in V(G)$ is a simplicial vertex. Assume, without loss of generality, that G is connected, but not complete.

The inductive hypothesis implies that $\sum_R[1 - \gamma(N(R))] = 1$ in the P_4-free chordal graph $G - u$. Consider what happens to this sum when u is introduced to form G (paralleling the four previous cases):

(a') The term with $R = N[u]$ equals one and any other term with $u \in R$ equals zero; all these terms are introduced into the sum.

(b') The term with $R = N(u)$ decreases by one.

(c') Any term with R a proper (possibly empty) subset of $N(u)$ either is unchanged or decreases by one, with a decrease if and only if $N(u)$ is disjoint from every minimum donimating set of $N(R) - u$.

(d') Any term with $R \not\subseteq N[v]$ is unchanged.

Being P_4, C_4-free implies that maximum degree vertices, dominating vertices, and minimum dominating sets are all the same thing. Thus any decreases in case (c') would mean that, for some proper subset R of $N(u)$, no dominating vertex of $N(R) - u$ is in $N(u)$. Yet any vertex d of maximum degree in G dominates G, and so dominates $N(R) - u$. Since we are assuming G is not complete, $d \neq u$ and so $d \in N(u)$, a contradiction. Thus there are no decreases in case (c'), and therefore $\sum_R [1 - \gamma N(R))] = 1$ in G. □

This family of P_4-free chordal graphs is also well-studied, having been introduced as P_4, C_4-free graphs and as the comparability graphs of trees in [11, 12] and studied as "trivially-perfect graphs" in [2], "nested interval graphs" in [10] (see also [9]), and "domination reducible graphs" in [6]. In particular, it is easy to see that P_4-free chordal graphs are precisely those in which every vertex of maximum degree dominates its component, as was used in the preceding proof.

There are many "nice" families of graphs in between P_4-free chordal graphs and chordal graphs—interval graphs for one. While there is no reason to expect that every such family will correspond to a graph parameter, those parameters in between the domination number and the number of components are the ones to inspect.

Corollary 4 *Suppose f is any graph parameter such that, for every graph H,* comp $H \leq f(H) \leq \gamma(H)$. *Then the inequalities* (3) *hold, with equality holding for, at least, all P_4-free chordal graphs.*

Proof. The comp $H \leq f(H) \leq \gamma(H)$ assumption implies the first inequailty in

$$\sum_R [1 - \gamma(N(R))] \leq \sum_R [1 - f(N(R))] \leq 1,$$

with the second from Corollary 2. If G is P_4-free and chordal, then $\sum_R [1 - \gamma(N(F))] = 1$ by Theorem 3, so equality will hold in (3). □

It is mentioned in [5] that lower bounds for $\gamma(G)$ are relatively rare, and several parameters that are mentioned there fail to satisfy the comp $H \leq f(H)$ requirement when $H = K_1$. The natural candidate from the literature is the "irredundance number" from [1]. A subset $S \subseteq V(G)$ is an *irredundant set in* G if, for each $v \in S$, $N[v]$ adds a vertex to $\bigcup\{N[x] : x \in S\}$ that is contributed by no other vertex of S. The *irredundance number of* G, denoted $\mathrm{ir}(G)$, is the cardinality of a smallest maximal irredundant set in G, and [1] contains that, for every graph H, $\mathrm{ir}(H) \leq \gamma(H)$. The following is then disappointing.

Theorem 5 *The inequalities* (3) *hold for* $f(H) = \mathrm{ir}(H)$, *with equality if and only if* G *is a* P_4*-free chordal graph.*

Proof. Using Corollary 4, all that remains to show is that $\sum_R[1-\mathrm{ir}(N(R))] = 1$ implies G is P_4-free and chordal. The proof is the same as for the corresponding part of the proof of Theorem 3 with two modifications:
(1) In case (c), there is an increase if and only if $N(u)$ is disjoint from every minimum maximal irredundant set of $N(R) - u$.
(2) Observe that, within $G - u$, the minimum maximal irredundant sets are precisely the dominating vertices. □

There are generalized domination and irredundance parameters $\gamma_n(G)$ and $\mathrm{ir}_n(G)$ in [4] that satisfy the hypothesis of Corollary 4, but the families for which equality holds in (3) are not even closed under taking induced subgraphs. It is intriguing to wonder whether there are parameters that lead to other natural families of chordal graphs.

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