# The Concept of Diameter in Exponents of Symmetric Primitive Graphs

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ABSTRACT. A directed graph $G$ is primitive if there exists a positive integer $k$ such that for every pair $u$, $v$ of vertices of $G$ there is a walk from $u$ to $v$ of length $k$. The least such $k$ is called the exponent of $G$. The exponent set $E_n$ is the set of all integers $k$ such that there is a primitive graph $G$ on $n$ vertices whose exponent is $k$.

Let $G$ be a primitive directed graph on $n$ vertices. A well-known upper bound for the exponent of $G$ is $(n-1)^2+1$, due to H. Wielandt in 1950. If $G$ is symmetric, then an upper bound on its exponent is $2n-2$. In this paper we apply a recent diameter result to show the exponent of a symmetric primitive graph $G$ is at most $2d$ where $d$ is the diameter of $G$. We also characterize primitive symmetric graphs with exponent $2d$ and, as well, those on $n$ vertices with exponent at least $n$. Finally, we show how these characterizations can be used to obtain a complete description of the exponent set of this class of primitive graphs.

## 1 Preliminaries

### 1.1 Definitions and Notation

We generally follow the notation given in R.A. Brualdi and H.J. Ryser [1]. A *directed graph*, $G = (V, E)$, is a set $V$ of *vertices* and a set $E$ of ordered pairs $(u, v)$ of vertices of $G$ called *arcs*. If $(u, v)$ is an arc of $G$ we say there

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is an arc from $u$ to $v$. All of our directed graphs are finite and we allow loops but no multiple arcs.

A *walk*, $W$, from $u$ to $v$ (or an $u \to v$ walk) is a sequence of not necessarily distinct vertices $V(W) = (u, a_1, a_2, \ldots, v)$ and a set of arcs $E(W) = (u, a_1), (a_1, a_2) \ldots, (a_{i-1}, a_i), \ldots, (a_j, v)$. Since we do not allow multiple arcs, specifying the vertex sequence of a walk uniquely describes that walk. A *closed walk* is a walk where $u = v$. A *path* is a walk where all the vertices in the walk are distinct. A *cycle* is a closed walk where all the vertices except the first and last are distinct. We use the notation $u \xrightarrow{k} v$ to mean there is a walk from $u$ to $v$ of length $k$. The walk $W = A + B$ is obtained by identifying the final vertex of $A$ with the initial vertex of $B$.

The *length* of a walk $W$, denoted $|W|$, is the number of arcs in $W$. The *distance* from $u$ to $v$, denoted $d(u, v)$, is the length of a minimum $u \to v$ path. We define $d(u, u) = 0$. If there is no $u \to v$ path, then $d(u, v)$ is not defined.

If $u$ and $v$ are vertices on a walk $W$, then $W(u, v)$ denotes the portion of $W$ from $u$ to $v$. Here, when we speak of vertices $u$, $v$ on a walk $W$, we regard $u$ and $v$ as members of the sequence $V(W)$, even though, for convenience, we have dropped the subscripts. If $G$ is symmetric and $W(u, v)$ is a portion of a walk $W$ in $G$, then we define $W'(v, u)$ to be the $v \to u$ walk whose vertex sequence consists of the vertices of $W(u, v)$ listed in reverse order. If $u$ and $v$ are the initial and terminal vertices of $W$, then we write $W' = W'(v, u)$.

A graph, $G$, is said to be *strongly connected* (or strong) if there exists a path from $u$ to $v$ for all $u, v \in V(G)$. The *diameter* of a strongly connected graph $G$ is $\max\{d(u, v) \colon u, v \in VV(G)\}$. If $G$ is not strongly connected, then the diameter is not defined. Clearly, the diameter of a graph is at most $n - 1$. The *girth* of $G$ is the length of a shortest cycle in $G$. If $G$ has no cycles the girth is not defined. We define $G^k$ to be the directed graph with the same vertex set as $G$ and arcs $(u, v)$ if and only if $u \xrightarrow{k} v$.

A directed graph $G$ is *primitive* if there is a positive integer $k$ such that $u \xrightarrow{k} v$ for each pair $u$, $v$ of vertices of $G$. The least such $k$ is called the exponent of $G$, denoted $\exp(G)$.

We define $\exp(G; u)$ to be the least integer $k$ such that $u \xrightarrow{k} v$ for each vertex $v$ in $G$. Also, the notation $\exp(G; u, v)$ means the least integer $k$ such that $u \xrightarrow{m} v$ for all $m \geq k$. Clearly, $\exp(G; u) = \max_v\{\exp(G; u, v)\}$ and $\exp(G) = \max_u\{\exp(G; u)\}$.

## 1.2 Background

Let $A$ be the adjacency matrix of $G$. If $G$ is primitive, then we say $A$ is primitive. The following is well-known (for example. see R.A. Brualdi &

H.J. Ryser [1]).

**Theorem 1.1.** *The $(i,j)$ entry of $A^k$ is positive if and only if $u \xrightarrow{k} v$ in $G = G(A)$.*

Thus, $G$ has exponent $k$ if and only if $k$ is the least integer such that $A^k$ has all positive entries. If $A$ is symmetric, then we say $G$ is symmetric.

We note that the adjacency matrix of a symmetric directed graph $G$ is the same as the adjacency matrix of the undirected graph $G'$ obtained from $G$ by replacing each $(u,v)$, $(v,u)$ arc pair with an undirected edge $(u,v)$. Hence, we refer to symmetric directed graphs simply as symmetric graphs, and when we draw symmetric graphs we will draw them as undirected graphs.

The following upper bound on the exponent is due to H. Wielandt [13] in 1950.

**Theorem 1.2.** *The exponent of a $(0,1)$ primitive matrix of order $n$ is at most $(n-1)^2+1$. Moreover, the unique (up to simultaneous permutations of the rows and columns) matrix, $W_n$, for which $\exp(W_n) = (n-1)^2+1$ is given by*

$$W_1 = [1],\ W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ \textit{and}\ W_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & & 0 & 1 \\ 1 & 0 & \cdots & & 0 & 0 \end{bmatrix} \quad \textit{for } n \geq 3.$$

Since the bound $(n-1)^2+1$ is attained only for the Wielandt matrix, many attempts have been made to improve this bound, by introducing various parameters and by considering certain classes of graphs. In this paper, we consider the concept of diameter in conjunction with the class of symmetric graphs. We show that if $G$ is a symmetric primitive graph on $n$ vertices and with diameter $d$, then $\exp(G) \leq 2d$. We also characterize the graphs with exponent $2d$ and, as well, those with exponent at least $n$. Finally, we show how these results can be used to proved a complete description of the exponent set for this class of primitive graphs.

The notion of diameter has led to a recent refinement of the Wielandt bound by S. Neufeld [7] and independently by J. Shen [11]. R.E. Hartwig and M. Neumann [3] mention that this diameter bound was conjectured by R.E. Hartwig in an unpublished working paper.

**Theorem 1.3.** *If $G$ is a primitive directed graph with diameter $d$ then*

$$\exp(G) \leq d^2+1.$$

In proving Theorem 1.3, S. Neufeld [7] and J. Shen [10] also obtained the following useful result.

**Theorem 1.4.** *If $G$ is a primitive directed graph with diameter $d$, then the diameter of $G^k$ is at most $d$ for all positive integers $k$.*

To improve the Wielandt bound, the concept of girth was employed by A.L. Dulmagee & N.S. Mendelsohn [2] in a 1964 paper.

**Theorem 1.5.** *If $G$ is a primitive directed graph on $n$ vertices and with girth $s$, then* $\exp(G) \leq n + s(n-2)$.

J. Shen [12] has recently applied Theorems 1.3 and 1.4 to obtain a refinement of Theorem 1..5.

**Theorem 1.6.** *If $G$ is a primitive directed graph on $n$ vertices and with girth $s$ and diameter $d$, then*

$$\exp(G) \leq d + 1 + s(d-1) \leq n + s(n-2).$$

## 2 Symmetric Primitive Graphs

We first mention an upper bound on the exponent for symmetric primitive graphs due to J.C. Holladay & R.S. Varga [5].

**Theorem 2.1.** *If a primitive graph $G$ on $n$ vertices is symmetric, then* $\exp(G) \leq 2n - 2$.

B.R. Heap & M.S. Lynn [4] improved Thereom 2.1 as follows:

**Theorem 2.2.** *Suppose $G$ is a primitive symmetric graph on $n$ vertices. Let $d'$ be the diameter of $G^2$. Then, either*

$$\exp(G) = 2d' \leq 2n - 2 \text{ or } \exp(G) = 2d' - 1 \leq 2n - 3.$$

By Theorem 1.4, we know that $d' \leq d$. We use this fact in proving the next theorem, which is a refinement of Theorem 2.1.

**Theorem 2.3.** *Suppose $G$ is a primitive symmetric graph with diameter $d$. Then*

$$\exp(G) \leq 2d.$$

**Proof:** The graph $G^2$ has a loop at every vertex and by Theorem 1.4, $G^2$ has diameter at most $d$. Therefore, $\exp(G^2) \leq d$ and hence $\exp(G) \leq 2d$. □

If a symmetric primitive graph $G$ contains at least one loop some improvements can be made on the upper bound given in Theorem 2.1. The following result is due to M. Lewin [6].

**Theorem 2.4.** *If a primitive symmetric graph $G$ on $n$ vertices has $k > 0$ loops, then* $\exp(G) \leq \max(n-1, 2(n-k))$.

Note that if $G$ is a symmetric graph and $W$ is a $u \to v$ walk in $G$, then there is a $v \to u$ walk $W'$ in $G$ whose vertex sequence consists of the vertices of $W$ listed in reverse order. In the following theorem we use this fact and Thereom 1.4 to obtain a refinement of Theorem 2.4.

**Theorem 2.5.** *Suppose $G$ is a symmetric primitive graph on $n$ vertices with $k > 0$ loops and diameter $d$. Then,*

$$\exp(G) \leq \max\{d, \min(2d, 2(n-k))\}.$$

**Proof:** We note $\exp(G) \geq d$ and by Theorem 2.3, $\exp(G) \leq 2d$. Let $u, v \in V(G)$. Let $P$ be a shortest $u \to v$ path. If $P$ intersects a vertex with a loop, then $u \xrightarrow{k} v$ for every $k \geq |P|$. Suppose $P$ does not intersect a vertex with a loop. Let $Q$ be a shortest path from $P$ to a vertex with a loop. Let $Q'$ be the path $Q$ with the vertices in reverse order. Then $|Q| \leq n - k - |P|$. Therefore, there is a $u \to v$ walk which intersects a vertex with a loop and which is of length at most $|P| + |Q| + |Q'| \leq |P| + 2(n-k-|P|) = 2(n-k) - |P| \leq 2(n-k)$ and hence there is a $u \to v$ walk of length exactly $2(n-k)$. Therefore, since $u$ and $v$ are arbitrary, $\exp(G) \leq \max\{d, \min(2d, 2(n-k))\}$. □

## 2.1 A Characterization of Symmetric Primitive Graphs with Exponent $2d$

J. Shao [9] showed that $\exp(G) = 2n-2$ if and only if (up to isomorphism) $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $E = \{(i, i+1) \colon 1 \leq i \leq n-1\} \cup \{(n, n)\}$. In this section we generalize J. Shao's result by providing necessary and sufficient conditions for $\exp(G) = 2d$. When $d = n-1$, this gives the extremal graph found by J. Shao.

We begin with several lemmas.

**Lemma 2.6.** *let $G$ be a symmetric primitive graph and let $u, v, \in V(G)$. If there are $u \to v$ walks $P$ and $Q$ of opposite parity, then* $\exp(G; u, v) \leq \max\{|P|, |Q|\} - 1$.

**Proof:** With no loss of generality suppose $|P| < |Q|$. Since every vertex of $G$ is on a 2-cycle and $P$ and $Q$ have opposite parity, there is a $u \to v$ walk of length $|P| + 2k = |Q| - 1$ for some non-negative integer $k$. Moreover, for each $m \geq |Q| - 1$, there is a $u \xrightarrow{m} v$ walk where $m = |P| + 2j$ or $m = |Q| + 2l$ for non-negative integers $j$ and $l$. Thus, $\exp(G; u, v) \leq \max\{|P|, |Q|\} - 1$. □

In Lemma 2.7 we find, given a positive integer $m \geq 2$ and a vertex $u \in V(G)$, an upper bound on the length of a shortest closed walk containing $u$ which is not divisible by $m$.

**Lemma 2.7.** *Let $G$ be a primitive directed graph with diameter $d$. Let $u \in G$ and let $m \geq 2$ be a positive integer. Then $u$ is contained in a closed walk $W$ where $|W| \leq 2d+1$ and $|W|$ is not divisible by $m$.*

**Proof:** Let $W$ be a shortest walk among all the closed walks in $G$ which contain $u$ and have lengths not divisible by $m$. Suppose, contrary to the statement of the Lemma, $|W| > 2d+1$. Let $W = W(u,v) + W(v,u)$ and choose $v \in W$ so that $|W(u,v)|$ and $|W(v,u)|$ both exceed $d$. Let $P_1$ and $P_2$ be shortest paths from $u$ to $v$ and from $v$ to $u$ respectively. Then $W(u,v) + P_2$, $P_1 + W(v,u)$, and $P_1 + P_2$ are all closed walks and are all shorter than $W$. Thus,

$$|W(u,v)| + |P_2| \equiv |P_1| + |W(v,u)| \equiv |P_1| + |P_2| \equiv 0 \pmod{m}.$$

But this implies $|W(u,v)| + |W(v,u)| \equiv 0 \pmod{m}$, a contradiction. □

In Lemma 2.8 we find, given a vertex $u \in V(G)$, an integer $m \geq 2$, and a closed walk $W$ in $G$ whose length is not divisible by $m$, that not all paths from $u$ to $W$ are congruent modulo $m$.

**Lemma 2.8.** *Let $G$ be a primitive directed graph and let $m$ be a positive integer. Let $W$, with $|W| \geq 2$, be a closed walk in $G$ whose lengh is not divisible by $m$. Let $u \in V(G)$. Then there is an arc $(r,s) \in W$ such that shortest paths $R$ from $r$ to $u$ and $S$ from $s$ to $u$ have the property $|R| \not\equiv |S| + 1 \pmod{m}$. As well, there is an arc $(a,b) \in W$ such that shortest paths $P$ and $Q$ from $u$ to $a$ and $b$ respectively have the property $|P| + 1 \not\equiv |Q| \pmod{m}$.*

**Proof:** This is a proof by contradiction. We note $W$ exists since $G$ is primitive. Suppose, contrary to the statement of the lemma, the arc $(r,s)$ does not exist. Let the vertex sequence of $W$ be: $a_0, a_1, \ldots, a_{|W|}$ with $a_0 = a_{|W|}$. Let $P_i$ be a shortest path from $a_i$ to $u$. (There may of course be more than one shortest path from $a_i$ to $u$, but they all have length equal to $d(a_i, u)$.) Then $|P_i| + 1 \equiv |P_{i-1}| \pmod{m}$. Summing we obtain

$$\sum_{i=1}^{|W|} |P_i| + |W| \equiv \sum_{i=1}^{|W|} |P_{i-1}| \pmod{m}$$

which implies $|W| \equiv 0 \pmod{m}$, a contradiction. Similarly the arc $(a,b)$ also exists. □

We are now ready to characterize those primitive symmetric graphs with exponent $2d$.

**Theorem 2.9.** *The exponent of a primitive graph $G$ with diameter $d$ is $2d$ if and only if there is a vertex $u \in V(G)$ such that among all closed walks of odd length containing $u$ a shortest one has length $2d+1$.*

**Proof:** Let $u \in V(G)$ and among all closed walks of odd length containing $u$ let $W_u$ be a shortest one. Let $M = \max\{|W_u|: u \in V(G)\}$. We note by Lemma 2.7 that $M \leq 2d+1$.

Suppose there is a vertex $u \in V(G)$ such that $|W_u| = 2d+1$. Since there is no closed walk containing $u$ of length $2d-1$, $\exp(G; u) \geq 2d$. Thus, by Theorem 2.3. $\exp(G) = 2d$.

Suppose $\exp(G) = 2d$ and suppose $M \leq 2d-1$. Let $u, v$ be any vertices of $G$. We wish to show there are $u \to v$ walks of different parity and each of length at most $2d$. Then, by Lemma 2.6, $\exp(G; u, v) < 2d$ and since $u$ and $v$ are arbitrary vertices of $G$ we would have $\exp(G) < 2d$, a contradiction.

By Lemma 2.8 there is an arc $(a, b)$ of $W_u$ such that if $P$ and $Q$ are shortest $a \to v$ and $b \to v$ paths, then $|P| \not\equiv |Q| + 1 \pmod 2$. Thus, the walks $W_u(u, a) + P$ and $W_u(u, b) + Q$ have different parity.

If $|W_u(u, b)| \leq d$, then there are $u \to v$ walks of lengths $|W_u(u, b)| + |Q| \leq 2d$ and $|W_u(u, a)| + |P| \leq 2d-1$. If $|W_u(u, b)| \geq d+1$, then, since $|W_u| \leq M \leq 2d-1$, we have $|W_u'(u, a)| \leq d-1$ and $|W_u'(u, b)| \leq d-2$. Now the $u \to v$ walks $W_u'(u, a) + P$ and $W_u'(u, b) + Q$ have different parity and $|W_u'(u, a)| + |P| \leq 2d-1$ and $|W_u'(u, b)| + |Q| \leq 2d-2$.

Therefore, we conclude if $\exp(G) = 2d$, then there is a vertex $u \in V(G)$ such that among all closed walks of odd length containing $u$ a shortest one has length $2d+1$. □

## 2.2 A Characterization of Primitive Symmetric Graphs on $n$ Vertices with Exponent at least $n$

In this section we characterize those primitive symmetric graphs on $n$ vertices which have exponent at least $n$. Recall that if $W(u, v)$ is a portion of a walk $W$ in $G$, then $W'(v, u)$ is the $v \to u$ walk whose vertex sequence consists of the vertices of $W(u, v)$ listed in reverse order. Note $|W(u, v)| = |W'(v, u)|$.

**Lemma 2.10.** *Let $G$ be a primitive symmetric graph on $n$ vertices and let $u \in V(G)$. Among all the closed walks of odd length containing $u$, let $W$ be a shortest one. Then the number of distinct vertices in $W$ is at least $(|W|+1)/2$. Also, if the number of distinct vertices in $W$ is exactly $(|W|+1)/2$, then $W$ contains a loop and is unique (up to isomorphism.).*

**Proof:** We first show that no vertex of $G$ can occur more than twice in $W$. This implies that the number of distinct vertices in $W$ is at least $(|W|+1)/2$ since $W$ is of odd length.

Suppose a vertex $u \in V(G)$ occurs three times in $W$. Let $v_1$, $v_2$, and $v_3$ be the first, second, and third occurrences of $v$ in $W$. Let $W = W_1 + W_2 + W_3 + W_4$ where $W_1 = W(u, v_1)$, $W_2 = W(v_1, v_2)$, $W_3 = W(v_2, v_3)$, and $W_4 = W(v_3, u)$. The closed walk $W_1 + W_4$ containing $u$ must be of even length since it is shorter than $W$. This implies exactly one of $W_2$ or $W_3$ is of odd length; suppose $|W_2|$ is odd. But now the closed walk $W_1 + W_2 + W_4$ containing $u$ is of odd length and shorter than $W$ contradicting the minimality of $W$. Thus, no vertex of $G$ can occur more than twice in $W$.

Suppose the number of distinct vertices in $W$ is exactly $(|W| + 1)/2$. Then, every vertex in $W$ occurs exactly twice. Let $v_a$ and $v_b$ denote the first and second occurrences of vertex $v$ in $W$. We note $W(v_a, v_b)$ is of odd length; otherwise $W(u, v_a) + W(v_b, u)$ is of odd length and shorter than $W$, a contradiction. Let $(u, x)$ and $(y, u)$ be arcs of $W$. We show first that $x = y$.

Suppose to the contrary that $x \neq y$. We note both $|W(x_a, x_b)|$ and $|W(y_a, y_b)|$ are odd and hence $|W'(y_b, y_a)|$ is odd. Also, $W(x_b, y_b)$ is of even length or else the closed walk $(u, x_a) + W(x_b, y_b) + (y_b, u)$ is of odd length and shorter than $W$, a contradiction. But now the closed walk $(u, x_a) + W(x_b, y_b) + W'(y_b, y_a) + (y_b, u)$ is of odd length and shorter than $W$, a, contradiction. Therefore, we conclude $x = y$.

We now observe that vertex $x$ is contained in a closed walk $W_1 = W(x_a, x_b)$ of odd length $|W_1| = |W| - 2$, that the number of distinct vertices in $W_1$ is exactly $(|W_1| + 1)/2$, and that $W_1 \subseteq W$. Thus, by repeating this observation we obtain a sequence of walks $W_1, W_2, \ldots, W_k$ where the number of distinct vertices in $W_i$ is $(|W_i| + 1)/2$ for each $i$, $1 \leq i \leq k$ and $W \supseteq W_1 \supseteq W_2 \ldots \supseteq W_k$ and $|W_1| = |W| - 2, |W_2| = |W_1| - 2, \ldots, |W_k| = |W_{k-1}| - 2 = 1$.

Thus, we conclude there is a vertex of $W$ which has a loop. As well, the decomposition of $W$ shows that it is unique (up to isomorphism). □

Lemma 2.10 tells us that if $W$ contains no loops, then the number of distinct vertices in $W$ is at least $(|W|+3)/2$. If $W$ contains no loops and the number of vertices in $W$ is exactly $(|W|+3)/2$, then by an argument similar to that in Lemma 2.10 we again obtain a decomposition of $W$ showing uniqueness (up to isomorphism).

**Theorem 2.11.** *Let $G$ be a primitive symmetric graph on $n$ vertices. Let $u \in V(G)$ and among all closed walks of odd length containing $u$, let $W_u$ be a shortest one. Let $M = \max\{|W_u| : u \in V(G)\}$.*

(a) *Suppose $G$ admits loops. If $M \geq n$, then $\exp(G) = M - 1$ and if $M \leq n$, then $\exp(G) \leq n - 1$.*

(b) *Suppose $G$ admits no loops. If $M \geq n - 2$, then $\exp(G) = M - 1$ and if $M \leq n - 2$, then $\exp(G) \leq n - 3$.*

**Proof:** Let $u, v \in V(G)$. By Lemma 2.6, to prove $\exp(G; u, v) < \gamma$ it suffices to show there are two $u \to v$ walks of opposite parity and each of length at most $\gamma$. We first assume $G$ admits loops.

**Case 1:** Suppose the closed walks $W_u$ and $W_v$ intersect. Let $x$ be a vertex contained in both $W_u$ and $W_v$. Assume with no loss of generality $|W_u(u,x)| < |W_u(x,u)|$ and $|W_v(x,v)| < |W_v(v,x)|$. We note the $u \to v$ walk $U = W_u(u,x) + W_v(x,v)$ is of length less than $\max\{|W_u|, |W_v|\}$. Now either $|W_u(u,x)| < |W_v(x,v)|$ in which case the $u \to v$ walk $W_u(u,x) + W_v'(x,v)$, which has parity opposite $U$, is of length less than $|W_v|$, or $|W_v(x,v)| \leq |W_u(u,x)|$ in which case the $u \to v$ walk $W_u'(u,x) + W_v(x,v)$, which also has parity opposite $U$, is of length at most $|W_u|$. Thus, $\exp(G; u, v) \leq \max\{|W_u|, |W_v|\} - 1$.

**Case 2:** Suppose the closed walks $W_u$ and $W_v$ do not intersect. We wish to show $\exp(G; u, v) \leq \max\{n, M\} - 1$. Let $P$ be a shortest $u \to v$ path. Let $x$ be the final vertex of $P$ which is also a vertex of $W_u$ and let $y$ be the first vertex of $P$ after $x$ which is also a vertex of $W_v$. With no loss of generality suppose $|W_u(u,x)| < |W_u(x,u)|$ and $|W_v(y,v)| < |W_v(v,y)|$. We note $|P(u,x)| \leq |W_u(u,x)|$ since $P$ is a shortest path. In fact $|P(u,x)| = |W_u(u,x)|$ for if $|P(u,x)| < |W_u(u,x)|$, then one of the closed walks $W_u(u,x) + P'(x,u)$ or $W_u(u,x) + P'(x,u)$ containing $u$ would be shorter than $W_u$ and of odd length since $W_u(u,x)$ and $W_u'(u,x)$ have different parity. Similarly, $|P(y,v)| = |W_v(y,v)|$. Thus, the $u \to v$ walk $W_u(u,x) + P(x,y) + W_v(y,v)$ has length $|P| \leq n - 1$.

We may suppose

$$\max\{n, M\} < |W_u'(u,x)| + |P(x,y)| + |W_v(y,v)| \tag{1}$$

or else this $u \to v$ walk, which has parity opposite $P$, is of length at most $\max\{u, M\}$ in which cases we are done. Similarly, we may suppose

$$\max\{n, M\} < |W_u(u,x)| + |P(x,y)| + |W_v'(y,v)| \tag{2}$$

Adding equations (1) and (2) yields $2\max\{n, M\} < |W_u| + 2|P(x,y)| + |W_v|$ which implies $|P(x,y)| > \max\{n, M\} - (|W_u| + |W_v|)/2$. Now, by Lemma 2.10, the number of distinct vertices in $W_u$, $W_v$, and $P$ total at least $(|W_u| + 1)/2 + (|W_v| + 1)/2 + |P(x,y)| - 1 > |W_u|/2 + |W_v|/2 + n - (|W_u| + |W_v|)/2 = n$ which contradicts the assumed number of vertices in $G$. Thus, $\exp(G; u, v) \leq \max\{n, M\} - 1$.

Since $u$ and $v$ are arbitrary vertices of $G$, we have from Case 1 and Case 2 that $\exp(G) \leq \max\{n, M\} - 1$. Thus, if $M \leq n$, then $\exp(G) \leq n - 1$ and if $M \geq n$, then $\exp(G) \leq M - 1$. We observe vertex $u$ is contained in no closed walk of length $|W_u| - 2$ which implies $\exp(G) = \max_u\{\exp(G; u)\} \geq \max_u\{|W_u| - 1\} = M - 1$. Thus, we conclude if $M \geq n$, then $\exp(G) = M - 1$. This proves part (a).

We now assume $G$ admits no loops. By the argument in Case 1 we again have the result $\exp(G; u, v) \leq \max\{|W_u|, |W_v|\} - 1 \leq M - 1$. To apply the argument in Case 2, we first observe that if $G$ contains no loops, then the diameter of $G$ is at most $n-2$ and so $|P| \leq n-2$. Also, by Lemma 2.10, for each $u \in V(G)$ the walk $W_u$ contains at least $(|W_u|+3)/2$ distinct vertices. Then, by an argument similar to that in Case 2, we have $\exp(G) \leq \max\{n-2, M\} - 1$. Thus. if $M \leq n-2$, then $\exp(G) \leq n-3$. Now observing that $\exp(G) \geq M-1$, we have $\exp(G) = M-1$ provided $M \geq n-2$. We also observe that if $\exp(G) \geq n-2$, then $M \geq n-1$. This proves part (b). □

Theorem 2.11 tells us that if $G$ admits loops, then for each integer $m$ in the set $\{n, n+1, \ldots, 2n-2\}$, $\exp(G) = m$ if and only if $m+1 = \max\{|W_u| : u \in V(G)\}$. Also, if $G$ admits no loops, then Theorem 2.11 tells us that for each integer $m$ in the set $\{n-2, n-1, n \ldots, 2n-4\}$, $\exp(G) = m$ if and only if $m+1 = \max\{|W_u| : u \in V(G)\}$.

In the case $G$ admits loops and $M = n$, we obtain, by Theorem 2.11, $\exp(G) = M-1 = n-1$. However, if $\exp(G) = n-1$ we do not necessarily have $M = n$. For example. consider the primitive symmetric graph with diameter $n-1$ and a loop at every vertex.

## 2.3 The Exponent Set of Symmetric Graphs

We call now apply Theorems 2.9 and 2.11 to determine the exponent set of primitive symmetric graphs. This exponent set was determined by J. Shao [9] but can be proved much more expediently using the two theorems mentioned.

**Theorem 2.12.** *The exponent set of primitive symmetric graghs on $n$ vertices, $E_n^s$, is the set $\{1, 2, \ldots, 2n-2\}/S$ where $S$ is the set of odd numbers in $\{n, n+1, \ldots, 2n-3\}$. Moreover, if $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $E = \{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(n, n)\}$. then $G$ is the unique (up to isomorphisrn) symmetric graph with exponent $2n-2$.*

**Proof:** By Theorem 2.11 there is no primitive symmetric graph of order $n$ which has an exponent in $S$ since if $\exp(G) \geq n$, then it must be even. It remains to be shown that all other integers in $\{1, 2, \ldots, 2n-2\}$ can be attained as exponents of primitive symmetric graphs on $n$ vertices.

We note first of all the complete graph on $n$ vertices with $n$ loops has exponent 1 and the complete graph with $k$ loops where $1 \leq k < n$ has exponent 2. Figure 1 shows graphs $G$ on $n$ vertices and diameter $d$, $2 \leq d \leq n-1$, which have the property that for vertex $u_0 \in V(G)$, the shortest closed walk of odd length containing $u_0$ has length $2d+1$. Thus, by Theorem 2.9, $\exp(G) = 2d$ and hence for each integer $k$ in $\{2, 4, \ldots, 2n-2\}$ there is a graph on $n$ vertices with exponent $k$.

Suppose in Figure 1 that $d$ is odd and a loop is added to each vertex. The diameter of the resulting graph $G'$ is then $d$ and so $\exp(G') \geq d$. Also, since $G'$ has a loop at every vertex, $\exp(G') \leq d$ and so $\exp(G') = d$. Since $3 \leq d \leq n-1$ and $1 \in E_n^s$ we have that for each odd integer $k$ in the set $\{1, 2, \ldots, n-1\}$ there is a symmetric graph on $n$ vertices with exponent $k$. This proves the first part of the theorem.

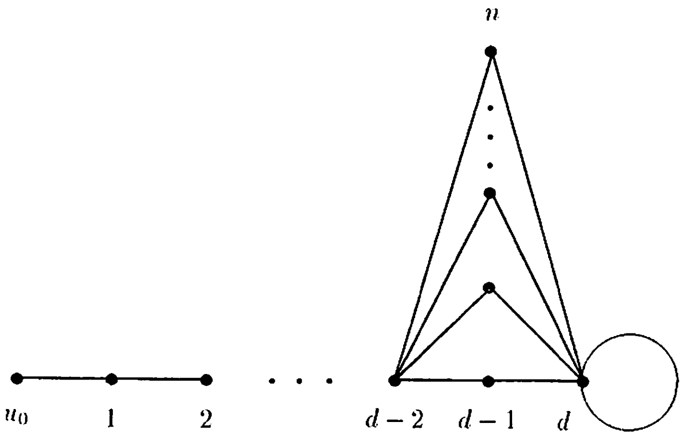


**Figure 1.** Graphs with even exponent

Suppose $\exp(G) = 2n-2$. Then by Theorem 2.3, the diameter of $G$ is $n-1$ and by Theorem 2.9 there is a vertex $u_0$ in $G$ such that among all closed walks of odd length containing $u_0$, a shortest one $W$ is of length $|W| = 2d+1 = 2n-1$. Now by Lemma 2.10, the number of distinct vertices in $W$ is $(|W|+1)/2 = n$ and thus, also by Lemma 2.10, the extremal graph attaining the exponent $2n-2$ is unique (up to isomorphism). This graph is given in Figure 1 when $d = n-1$. This proves the second part of the theorem. □

B. Liu *et al* [8] obtained the exponent set for primitive symmetric graphs $G$ with no loops. We show that Theorems 2.9 and 2.11 can be applied here as well to obtain the exponent set of symmetric primitive graphs with no loops and, in addition, characterize those symmetric primitive graphs with exponent $2n-4$.

**Theorem 2.13.** *The exponent set of primitive symmetric graphs on $n$ vertices with no loops, $E_n^{s'}$, is the set $\{2, \ldots, 2n-4\}/S$ where $S$ is the set of odd integers in $\{n-2, n-1, \ldots, 2n-5\}$. Moreover, if $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $E = \{(i, i+1) : 1 \leq i \leq n-2\} \cup \{(n-2, n), (n-1, n)\}$, then $G$ is the unique (up to isomorphism) symmetric graph with no loops and exponent $2n-4$.*

**Proof:** Clearly $1 \notin E_n^{s'}$. Also, the largest achievable exponent is $2n-4$ since if $\exp(G) = 2n-2$, then $G$ has a loop and, by Theorem 2.12, the exponent $2n-3$ is not permitted.

By Theorem 2.11 there is no graph $G$ on $n$ vertices and odd exponent $k$ where $k$ is in the set $\{n-2, n-1, n, \ldots, 2n-5\}$. It remains to be shown that all other integers in $\{2, 3, \ldots, 2n-4\}$ can be attained as exponents of symmetric primitive graphs on $n$ vertices with no loops.

If the diameter $d = 1$ consider the complete symmetric graph on $n \geq 3$ vertices. This graph has exponent 2. For $2 \leq d \leq n-2$ consider the graghs shown in Figure 2. Here a shortest closed walk $W$ of odd length containing vertex $u_0$ is of length $|W| = 2d+1$ and $5 \leq |W| = 2d+1 \leq 2n-3$. Thus, by Theorem 2.9, $\exp(G) = 2d = |W|-1$ and hence $\{2, 4, \ldots, 2n-4\} \subseteq E_n^{s'}$. Let $T$ be the set of odd integers in $\{3, 5, \ldots, n-3\}$. We now show $T \subseteq E_n^{s'}$.

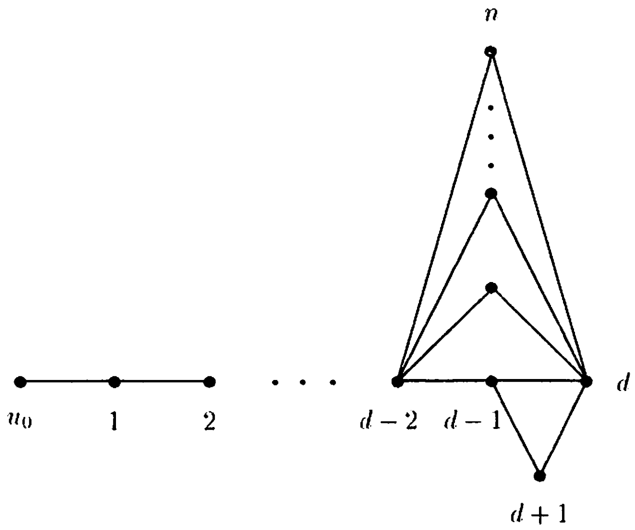


**Figure 2.** Graphs with no loops and even exponent

The graphs in Figure 3 consist of a path of odd length $d$, where $3 \leq d \leq n-3$, and 3-cycles attached as shown. The diameter of $G$ is $d$ and so $\exp(G) \geq d$. Let $P$ be a shortest $u \to v$ path. If $P$ intersects a 3-cycle, then there is a $u \to v$ walk of length at most $d+1$ and parity opposite $P$ so $\exp(G; u, v) \leq d$ by Lemma 2.6. Thus. suppose $u$ and $v \in \{2, 3, \ldots, d-2\}$. Because of the symmetrically placed 3-cycles $(u_0, 1, d+1, u_0)$ and $(d-1, d, d+2, d-1)$ we may suppose with no loss of generality that $u \leq v$ and $d-1-v \leq u-1$ which implies $u+v \geq d$. Then the $u \to v$ walk $u, u+1, \ldots, d, d+2, d-1, \ldots, v$ has parity opposite $P$ and has length $(d-1-u)+3+(d-1-v) = 2d+1-(u+v) \leq d+1$. Thus, by Lemma 2.6, $\exp(G; u, v) \leq d$. Thus, for all choices of $u$ and $v$ we have $\exp(G; u, v) \leq d$ and so $\exp(G) \leq d$. We conclude then $\exp(G) = d$ and hence $T \subseteq E_n^{s'}$. This completes the proof of the first part of the theorem.

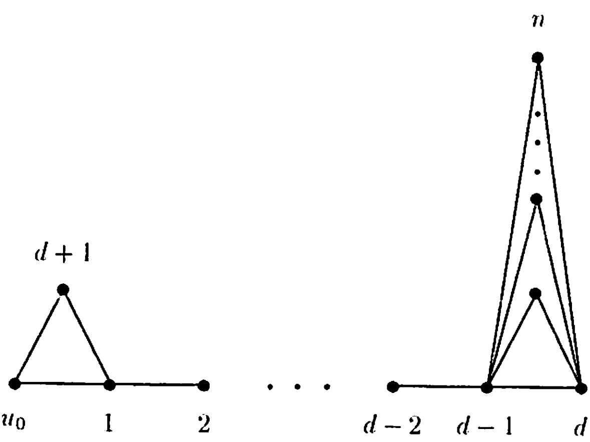


**Figure 3.** Graphs with no loops and odd exponent

We note that if $G$ has no loops, then the diameter of $G$ is at most $n-2$. Thus, if $\exp(G) = 2n-4$, then $d = n-2$ by Theorem 2.3, and by Theorem 2.9 there must be a vertex $u_0 \in V(G)$ such that a shortest closed walk $W$ of odd length containing $u_0$ is of length $2d+1 = 2n-3$. Since $G$ has no loops, by Lemma 2.10, the number of distinct vertices in $W$ is at least $(|W|+3)/2 = n$. Thus, also by Lemma 2.10, the extremal graph shown in Figure 2, with $d = n-2$, is the unique (up to isomorphism) symmetric graph on $n$ vertices and no loops with exponent $2n-4$. □

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