HAMILTON DECOMPOSITIONS OF BLOCK-INTERSECTION GRAPHS OF STEINER TRIPLE SYSTEMS

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Abstract

Block-intersection graphs of Steiner triple systems are considered. We prove that the block-intersection graphs of non-isomorphic Steiner triple systems are themselves non-isomorphic. We also prove that each Steiner triple system of order at most 15 has a Hamilton decomposable block-intersection graph.

1 Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. By $V(G)$ and $E(G)$ we denote the vertex set and edge set, respectively, of the graph G.

A *cycle* is a 2-regular connected graph. A *Hamilton cycle* in a graph G is a 2-regular connected spanning subgraph of G.

A *Hamilton decomposition* of a regular graph G consists of a set of Hamilton cycles (plus a 1-factor if $\Delta(G)$ is odd) of G such that these cycles (and the 1-factor when $\Delta(G)$ is odd) partition the edges of G. If G has a Hamilton decomposition, it is said to be *Hamilton decomposable*.

A *Steiner triple system* (S, B) of order n consists of a set B of blocks, each being a 3-subset of $S = \{1, \ldots, n\}$, such that each pair of elements from the set S occurs in exactly one block.

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The *block-intersection graph* for the Steiner triple system (S, B) is the graph $G(S, B)$ for which $V(G(S, B)) = B$, and where two vertices of $G(S, B)$ are joined by an edge if and only if the corresponding blocks in (S, B) contain a common element.

Definitions omitted in this paper can be found in [6] or [13].

In this paper we concern ourselves with two questions: whether non-isomorphic Steiner triple systems have non-isomorphic block-intersection graphs, and whether these graphs are Hamilton decomposable.

We prove that any pair of non-isomorphic Steiner triple systems do indeed have non-isomorphic block-intersection graphs.

Concerning Hamilton decompositions, block-intersection graphs for all Steiner triple systems of order $n \leq 15$ are found to be Hamilton decomposable.

2 The Isomorphism Question

Theorem 1 *If (S_1, B_1) and (S_2, B_2) are non-isomorphic Steiner triple systems, then their block-intersection graphs are non-isomorphic.*

Proof. Clearly if the two Steiner triple systems are not of the same order, then their block-intersection graphs will not have the same number of vertices and hence will not be isomorphic. We thus need only consider pairs of non-isomorphic Steiner triple systems having the same order.

It is known that for $n \in \{3, 7, 9\}$ there is precisely one (up to isomorphism) Steiner triple system of order n. So we need only consider Steiner triple systems having larger order.

For $n = 13$ there are exactly two non-isomorphic Steiner triple systems. Their block-intersection graphs were tested by computer and determined to be non-isomorphic. Regarding the algorithm used, it consists of testing permutations of mappings from the vertex set of one graph to the vertex set of the other; complete details appear in [18] as well as on the author's web site.

For $n = 15$ there are precisely 80 non-isomorphic Steiner triple systems. We refer to each system by its number given in [9, 15] since this ordering has become somewhat entrenched in the literature.

In considering the 80 corresponding block-intersection graphs, we note that if a Steiner triple system has a subsystem of order 7, then its block-intersection graph will contain a corresponding clique of size 7. We also note that each of the 15 points in these Steiner triple systems gives rise to a clique of size 7 in each block-intersection graph, since each point is in exactly 7 blocks.

Noting that graphs having differing numbers of cliques of size 7 will clearly not be isomorphic, we may reduce the problem at hand by dividing the 80 systems into groups based on the number of subsystems of order 7 that each has.

Steiner triple system numbers 1 and 2 are the only systems to have 15 and 2, respectively, subsystems of order 7 [15]. Hence their block-intersection graphs cannot be isomorphic to any other block-intersection graphs.

Steiner triple systems $4, 5, 6$, and 7 each have three subsystems of order 7, while Steiner triple systems $7, \ldots, 22$, and 61 each have a single subsystem of order 7. All of the remaining Steiner triple systems contain no subsystems of order 7 [15].

Steiner triple systems 4 through 7 were tested by computer and found to have mutually non-isomorphic block-intersection graphs. Likewise, Steiner triple systems $7, \ldots, 22$, and 61 were tested by computer and found to have mutually non-isomorphic block-intersection graphs.

In considering the Steiner triple systems having no subsystems, we note that there is a natural isomorphism between each Steiner triple system and its block-intersection graph. The mapping from a Steiner triple system (S, B) to its block-intersection graph $G(S, B)$ is well-defined, so we need only show that the inverse mapping is equally well-defined. This is done by observing that each clique of size 7 in a block-intersection graph can only arise from one of the 15 points of the set S. We may arbitrarily assign unique points from S to each of these 15 cliques. Three cliques are then mapped to a block of the design if and only if they intersect in a vertex; the Steiner triple system obtained from these blocks is isomorphic to the original one used to construct the graph.

We thus conclude that non-isomorphic Steiner triple systems of order 15 have non-isomorphic block-intersection graphs.

For $n \geq 19$, we use the same technique as for Steiner triple systems of order 15 having no subsystems, except that we now identify each point of S with a clique of size $(n-1)/2$. Noting that it is impossible for a clique of size larger than 7 to arise from blocks that do not all have an element in common, this technique provides the basis for establishing a natural isomorphism between Steiner triple systems and their block-intersection graphs, no two graphs being isomorphic unless their corresponding Steiner triple systems are also isomorphic.

□

3 The Hamilton Decomposition Question

Block-intersection graphs for Steiner triple systems are regular, and have been shown to have vertex-connectivity equal to their degree [8, 10]. Given these properties it is natural to inquire if these graphs might possess Hamilton decompositions.

It has been shown that block-intersection graphs for a variety of block designs, including Steiner triple systems, are Hamiltonian [4, 11]. Moreover, block-intersection graphs for Steiner triple systems have been shown to be edge-pancyclic [3]. We prove the following:

Theorem 2 *If* (S, B) *is a Steiner triple system of order* $n \leq 15$ *then its block-intersection graph is Hamilton decomposable.*

Proof. The Steiner triple systems of order 3 and 7 have block-intersection graphs which are complete graphs on 1 and 7 vertices, respectively. It is a classical result that complete graphs are Hamilton decomposable [14].

For Steiner triple systems of orders 9, 13, and 15, Hamilton decompositions of the corresponding block-intersection graphs were obtained by means of an exhaustive deterministic algorithm implemented on a computer. We omit a lengthy presentation of these Hamilton decompositions from this paper. Rather, the Hamilton cycles comprising each Hamilton decomposition, as well as specific details of the algorithm used to find the decompositions, appear in [18] and are also available on the author's web site.

□

4 Discussion

We note the following conjecture, due to Nash-Williams and extended by Jackson [2, 12, 16]:

Conjecture 1 *Every* k*-regular graph on at most* $2k+1$ *vertices is Hamilton decomposable.*

While this conjecture remains open, we notice that block-intersection graphs for Steiner triple systems of order 15 or less satisfy the hypothesis. And while the results which we have presented do not settle this conjecture, they do tend to support it.

Block-intersection graphs for Steiner triple systems of orders greater than 15 do not meet Nash-William's hypothesis. So, based on the empirical evidence presented, we now make the following conjecture:

Conjecture 2 *If* (S, B) *is a Steiner triple system then its block-intersection graph is Hamilton decomposable.*

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References

[1] I.Š.o. Aliev and E. Seiden, Steiner Triple Systems and Strongly Regular Graphs, *J. Combinatorial Theory* 6 (1969) 33–39.

[2] B. Alspach, J.-C. Bermond, and D. Sotteau, Decomposition into Cycles I: Hamilton Decompositions. *Cycles and Rays* (eds. G. Hahn, et al.) Kluwer Academic Publishers, Dordrecht (1990) 9–18.

[3] B. Alspach and D. Hare, Edge-Pancyclic Block-Intersection Graphs, *Discrete Math.* 97 (1991) 17–24.

[4] B. Alspach, K. Heinrich, and B. Mohar, A Note on Hamilton Cycles in Block-Intersection Graphs, *Finite Geometries and Combinatorial Designs - Contemporary Mathematics* 111 (1990) 1–4.

[5] J.-C. Bermond, Hamiltonian Decompositions of Graphs, Directed Graphs and Hypergraphs, *Advances in Graph Theory, Annals of Discrete Mathematics* 3 (1978) 21–28.

[6] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North-Holland Publishing Company, New York (1979).

[7] R.C. Bose, Strongly Regular Graphs, Partial Geometries and Partially Balanced Designs, *Pacific J. Math.* 13 (1963) 389–419.

[8] A.E. Brouwer and D.M. Mesner, The Connectivity of Strongly Regular Graphs, *European Journal of Combinatorics* 6 (1985) 215–216.

[9] P.B. Gibbons, *Computing Techniques for the Construction and Analysis of Block Designs*, PhD thesis, University of Toronto (1976).

[10] D.R. Hare and W. McCuaig, The Connectivity of the Block-Intersection Graphs of Designs, *Designs, Codes and Cryptography* 3 (1992) 5–8.

[11] P. Horák and A. Rosa, Decomposing Steiner Triple Systems into Small Configurations, *Ars Combinatoria* 26 (1988) 91–105.

[12] B. Jackson, Hamilton Cycles in Regular Graphs, *J. Graph Theory* 2 (1978) 363–365.

[13] C.C. Lindner and C.A. Rodger, *Design Theory*, draft edition.

[14] D.E. Lucas, *Recréations Mathématiques*, Vol. II, Gauthier Villars, Paris (1892) 162.

[15] R.A. Mathon, K.T. Phelps, and A. Rosa, Small Steiner Triple Systems and Their Properties, *Ars Combinatoria* 15 (1983) 3–110.

[16] C.St.J.A. Nash-Williams, Hamiltonian Arcs and Circuits, in *Recent Trends in Graph Theory, Lecture Notes in Mathematics* 186, Springer-Verlag (1971) 197–210.

[17] K.T. Phelps, Parallelism in Designs and Strongly Regular Graphs, *Congressus Numerantium* 27 (1979) 365–384.

[18] D.A. Pike, *Hamilton Decompositions of Graphs*, Ph.D. thesis, Auburn University (1996).