A Comparison of Some Conditions for Non-Hamiltonicity of Graphs

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Abstract

Some sufficient conditions for non-hamiltonicity of graphs are compared.

Keywords: Graph, hamiltonian cycle, crossing

AMS Subject Classification (1991): 05C45

1 Introduction

We consider connected graphs and use the terminology of Bondy and Murty [1].

Most results on hamiltonian graphs concern sufficient condition for hamiltonicity, like the well-known conditions of Dirac, Ore, Pósa, Bondy and Chvátal, see [2]. These conditions are of numerical nature in the sense that some graph parameters are considered. Usually they can be calculated in polynomial time. If for the minimum degree δ we have $\delta \geq \frac{n}{2}$ $(n \geq 3)$, we have Dirac's original sufficient condition for hamiltonicity.

The reason for considering such conditions is, of course, the fact that the problem of determining whether a graph is hamiltonian is NP-complete, see Garey and Johnson [4].

A characterization of hamiltonian graphs that can be checked in polynomial time is not to be expected. There are some necessary and sufficient conditions for hamiltonian graphs, three of which will be mentioned. Vrba [14] gave a characterization by indicating a, very large, matrix, the determinant of which gives the number of Hamilton cycles. Hoede and Veldman [8] gave a necessary condition for 2-connected non-hamiltonian graphs, in terms of contractibility to two specific graphs, using a necessary and sufficient condition that enables the "coupling" of cycles. The generalization to k-connected graphs was given in [13]. These conditions are of structural nature and accept the fact that there is no way to check the conditions in polynomial time. This is also the case with the necessary and sufficient condition given by Hoede [6], in terms of crossing numbers and crossing lengths of path systems, that will be discussed in Section 3.

The remarkable thing about some of the conditions, that have appeared in the literature lately, is the fact that they tend to have a more structural nature and consider crossing path systems. Hendry [5] introduced the concept of path-toughness, in which paths are required to exist that cover the vertices of some subgraphs. Based on a result of Mader [11] on A-separators, recently Katona [10] introduced the concept of t-edge-toughness, in which a condition is required to hold, similar to that of t-toughness. That concept was introduced by Chvátal, who wrote a chapter on hamiltonian graphs, in the book The Traveling Salesman Problem [2], in which a sufficient condition for non-hamiltonicity is discussed as part of the theory of weakly hamiltonian graphs, where certain subgraphs called combs play a role. The result is related to the theory of factors of Tutte [12], in particular to that of 2-factors, and via that to the existence of perfect matchings. The form of the condition is somewhat similar to that of the condition of Mader. In the same book Grötschel and Padberg discuss polyhedral theory [2], in which inequalities based on Chvátal's combs are considered next to so-called 2-matching inequalities, due to Edmonds, that are special cases of the Chvátal comb inequalities.

All these conditions and concepts are interrelated. Some of these relationships will be discussed in Sections 2 and 3. The reason for this discussion is that the author formulated a simple sufficient condition for non-hamiltonicity, in terms of 2-matchings, and studied how it ties up with the various results mentioned in this introduction.

2 A-separators

In the appendix of the book of Garey and Johnson, a list of NP-complete problems, example [ND40] reads as follows.

DISJOINT CONNECTING PATHS

INSTANCE: Graph $G = (V, E)$, collection of disjoint vertex pairs $(s_1, t_1), \dots, (s_k, t_k)$.

QUESTION: Does G contain k mutually vertex-disjoint paths, one connecting s_i and t_i for each i, $1 \le i \le k$?

Let A be an independent set of k vertices in a graph G. A Hamilton cycle C in G is cut into k paths, with endvertices in A, that are mutually vertex-disjoint, but for the endvertices. On each path an edge can be chosen arbitrarily. Two of these k edges have at most a vertex of A in common. A *2-matching* (*perfect 2-matching*) in a graph is a set of edges such that every vertex is contained in at most (exactly) two edges. The k edges form therefore a special 2-matching, in which only vertices of A may be contained in two edges and that will be called an *A-2-matching*. Let Y be a set of edges of G. Y *disconnects* A if $G - Y$ has no path connecting two vertices of A. An *A-cut* is a set of edges that disconnects A. $\langle Y \rangle$ denotes the subgraph induced by Y.

Lemma 2.1. Let A be an independent set of a hamiltonian graph $G(V, E)$. If Y is an A-cut, then $\langle Y \rangle$ contains an A-2-matching with $|A|$ edges.

Proof. The $|A|$ paths of the Hamilton cycle, between pairs of vertices of A, each contain at least one edge of Y. □

It is sufficient for non-hamiltonicity to find an A-cut that induces a subgraph not containing an A-2-matching with $|A|$ edges. This is the simple sufficient condition meant in the introduction.

As an example we consider three classes of non-hamiltonian graphs, considered often in hamiltonian graph theory, see e.g. Jackson [9]. In Figure 1 we give these classes consisting of three cliques K_p, K_q and K_r, $p \geq 3$, $q \geq 3$, $r \geq 3$ and respectively

(a) two adjacent vertices x_1 and x_2 connected to all other vertices,

(b) one vertex x adjacent to all other vertices and three vertices v_1 v_2 and v_3, one in each clique, forming a triangle,

(c) two times three vertices v_1, v_2 and v_3 respectively w_1, w_2 and w_3, one in each clique, forming a triangle.

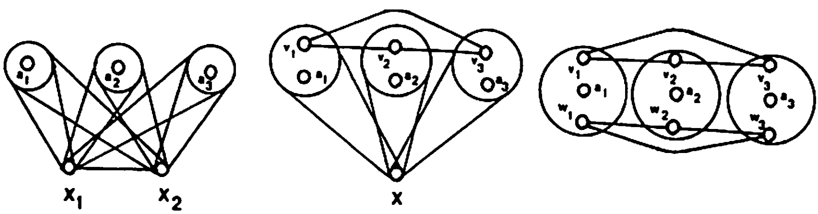

Figure 1
Three classes of non-hamiltonian graphs

The A-set consists of the three independent vertices a_1, a_2 and a_3. The A-cuts we consider consist of

(a) the set Y_1 of edges incident with x_1 or x_2,

(b) the set Y_2 of edges incident with x and the three edges of the triangle,

(c) the set Y_3 of six edges that form the two triangles.

Y_1, Y_2 and Y_3 have at most two edges in an A-2-matching. In case (a) all edges are incident with x_1 or x_2, but two edges cannot meet in x_1 or x_2, as $x_1, x_2 \notin A$. In case (b) only one edge can be incident with x and only one edge of the triangle can be chosen as $x, v_1, v_2, v_3 \notin A$. In case (c) each triangle can contribute at most one edge to the A-2-matching, as all six vertices v_1, v_2, v_3, w_1, w_2 and w_3 do not belong to A. All graphs are non-hamiltonian according to Lemma 2.1.

Let us now start the comparison with some other results and problems. The concept of an A-2-matching may seem somewhat unusual. However, we may replace each vertex a_i in A by two non-adjacent vertices $a_{i,1}$ and $a_{i,2}$ that both are made adjacent to all vertices of G that were adjacent to a_i. Consider the set A^* of these new vertices in the resulting graph G^*. An edge cut Y^* that disconnects, in G^*, the vertices $a \in A^*$ that belong to different pairs of a's induces a graph $\langle Y^* \rangle$, that contains a matching of $|A|$ edges if G is hamiltonian. The paths into which the Hamilton cycle is cut by the vertices of A can be set in correspondence with paths in G^* which do not share an endvertex in A^*. Due to this transformation we are in the context of the NP-complete problem DISJOINT CONNECTING PATHS mentioned in the beginning of this section. A Hamilton cycle in G determines an ordering of the vertices of A. The paths in G^* between elements of A^* determine the disjoint vertex pairs (s_j, t_j) in the formulation of the problem, s_j and t_j belonging to different pairs of a's. If, conversely, no ordering of the vertices of A gives a positive answer to the question, for the graph G^*, then the graph G is non-hamiltonian.

A-cuts in G have been extensively studied for $|A| = 2$, in which case the well-known max-flow min-cut theorem of Ford and Fulkerson has a central place. In trying to prove that a graph G is non-hamiltonian one looks for A-cuts without an A-2-matching of $|A|$ edges. The natural question is then that for A-cuts that have minimum cardinality. For $|A| \geq 3$ this is an interesting but difficult problem. The related problem, analogous to the relation of cut and flow in flow theory, is to determine the maximum number of vertex-disjoint paths with endvertices in A. We shall come back to this natural concept in our discussion later. It is essentially what the concept of 1-edge-toughness of Katona is about too.

Let $X \subseteq V(G-A)$ and $Y \subseteq E(G-A-X)$ be a pair (X, Y) that disconnects A. If $X = \emptyset$ we are back to our A-cut, if $Y = \emptyset$ we are back to the well-known situation wherein $\omega(G-X) \geq |A|$ and $|A| > |X|$ leads to $\omega(G-X) > |X|$, a sufficient condition for non-hamiltonicity. The components Q_i, $i = 1, \ldots, p$, of $\langle Y \rangle$ are now considered with respect to the number of vertex-disjoint paths that may "cross" them. Katona defines for a subgraph B of a graph A the *boundary* and the *inner vertices of B with respect to A* by

$$\begin{aligned} bd_A(B) &= \{v \in V(B) \mid v \text{ has a neighbour outside } B\} \qquad \text{and} \\ in_A(B) &= V(B) - bd_A(B). \end{aligned}$$

The cardinality of the boundary of Q_i with respect to $G - X$ determines how many vertex-disjoint paths may cross Q_i. Paths go in and out of Q_i via boundary vertices, so at most $\lfloor \frac{|bd_{G-X}(Q_i)|}{2} \rfloor$ can pass Q_i. Only one path can cross a vertex of X. Referring to Mader, Katona states

Lemma 2.2. [Mader] If there exists a pair (X, Y) disconnecting A such that

$$|A| > |X| + \sum_{i=1}^{p} \lfloor \frac{|bd_{G-X}(Q_i)|}{2} \rfloor,$$

then there is no cycle containing all vertices of A.

A pair (X, Y) satisfying the condition in Lemma 2.2 is called an *A-separator*. The right hand side of the inequality gives an upper bound on the number of vertex disjoint paths that have their endvertices in A. As an example from Katona's paper, consider the graph in Figure 2.

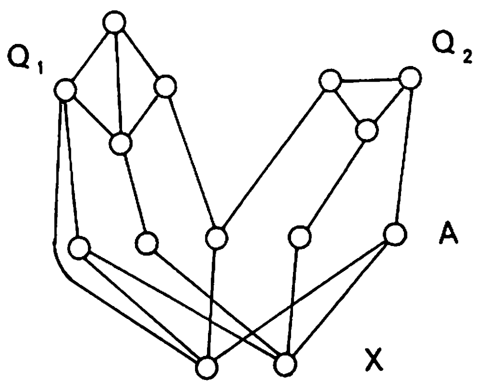

Figure 2

$|A| = 5$ and $|X| = 2$. Q_1 and Q_2 should allow three paths to cross. However, both have $|bd_{G-X}(Q_i)| = 3$ and allow only one path to cross. Q_1 has an inner vertex. As $5 > 2 + \sum_{i=1}^{2} \lfloor \frac{3}{2} \rfloor$, the graph does not contain a cycle containing the vertices of A. The condition is sufficient for non-hamiltonicity and can be checked easily.

A graph is *t-tough* if for every vertex set X with $\omega(G - X) > 1$ we have

$$\omega(G - X) \leq \frac{|X|}{t}.$$

Katona defines the *t-edge-toughness* analogously. A graph is *t-edge-tough* if

$$\omega(G - X - Y - \bigcup_{i=1}^{p} in_{G-X}(Q_i)) \leq \frac{|X| + \sum_{i=1}^{p} \lfloor \frac{|bd_{G-X}(Q_i)|}{2} \rfloor}{t},$$

for every $X \subseteq V(G)$ and $Y \subseteq E(G-X)$ with $\omega(G-X-Y-\bigcup_{i=1}^{p} in_{G-X}(Q_i)) > 1$.

Among his results are

Lemma 2.3. [Katona] If G is t-edge-tough then G is t-tough.

Lemma 2.4. [Katona] If G is hamiltonian then G is 1-edge-tough.

Lemma 2.5. [Katona] If G is $2t$-tough then G is t-edge-tough.

Comparing Lemma 2.1 with Lemma 2.2 we note that Lemma 2.1 does not show the graph in Figure 2 to be non-hamiltonian and Lemma 2.2 does not show the graph in Figure 1(c), with $p = q = r = 3$, to be non-hamiltonian. Both lemmata focus on the fact that, given an independent set A in G, for G to be hamiltonian there should be enough paths with endvertices in A. In the next section we discuss a necessary and sufficient condition consisting of four parts, of which this fact is the first.

3 Crossing theory

Let A be an independent set of vertices of G and let $\langle R\rangle$ be the subgraph induced by $R = V(G) - A$ with components R_i, $i = 1, \ldots, p$. A component R_i and its neighbours N_i in A determine a *fragment* $F_i = \langle V(R_i) \cup N_i\rangle$. $\mathcal{P}_i$ is a set of vertex-disjoint paths of this fragment with distinct endvertices in N_i. We exclude the uninteresting case in which $|A| = 1$. $\mathcal{P}_i$ is called a *crossing* of F_i. The *crossing number* $CN(R_i, \mathcal{P}_i)$ is the number $|\mathcal{P}_i|$of paths in a crossing of R_i. The maximum value of this number over all crossings is denoted by $CN(R_i)$. The *crossing length* $CL(R_i, \mathcal{P}_i)$ of a crossing is the sum of the lengths of the paths in $\mathcal{P}_i$.

Theorem 3.1. [Hoede] A graph G is hamiltonian if and only if for every independent set $A(|A| \geq 2)$ there exist crossings $\mathcal{P}_i$, such that for each of the p components R_i of $G - A$

(i) $\sum_{i=1}^{p} CN(R_i, \mathcal{P}_i) \geq |A|$;

(ii) $\forall_{i=1,\ldots,p} : CL(R_i, \mathcal{P}_i) = |V(R_i)| + |\mathcal{P}_i|$ and $\sum_{i=1}^{p} CL(R_i, \mathcal{P}_i) = |V(G)|$;

(iii) Each vertex of A occurs twice as endvertex of a crossing path.

(iv) The graph $H(A, P)$, with vertex set A and $a_i a_j \in P$ if and only if a_i and a_j are the endvertices of a crossing path, is a cycle of length $|A|$.

The simple content of this theorem is the following. When the components R_i offer enough crossing possibilities (i), the crossing paths pass all vertices of the components (ii), the crossing paths form a 2-factor (iii) and this 2-factor is one cycle (iv), then G is hamiltonian. The converse is obvious. The Petersen graph only fails to satisfy condition (iv), see [6].

The conditions (i), (ii) and (iii) are necessary and sufficient for a graph to have a 2-factor with at least two vertices of A on each cycle. This latter statement is superfluous in Theorem 3.1 as there we consider a 2-factor consisting of one cycle and the vertices of A are automatically on it.

Although the theorem is very simple, it gives a clear picture of the road to hamiltonicity. The comparison with other results will focus on conditions (i) and (ii). When these are satisfied condition (iii) leads to the existence of a 2-factor. From there to the existence of a Hamilton cycle is a step that the author [7] recently found the following condition for.

Theorem 3.2. [7] If a connected graph G has a 2-factor then G is hamiltonian if G does not contain one of the eleven graphs in Figure 3 as an induced subgraph.

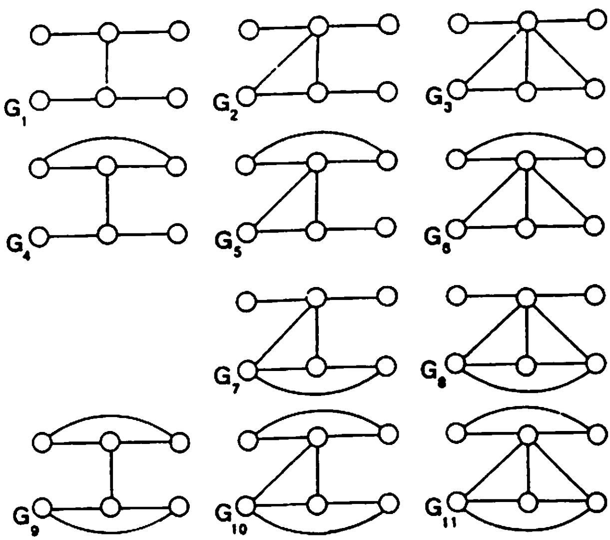

Figure 3

The non-existence of a 2-factor is the central point of Chvátal's [2] discussion of non-hamiltonian graphs. We quote (page 207): "In fact, the problem of finding a 2-factor in G can be reduced to the problem of finding a perfect matching in another graph easily constructible from G [Tutte]. Combined with the efficient algorithm for finding a largest matching in a graph [Edmonds], this trick (*replacing vertices of degree d by bipartite graphs* $K_{d,d-2}$) yields an efficient algorithm, which, given any graph, finds either a 2-factor in G or a partition (*of the vertices of G into disjoint sets R, S and T*) satisfying

$$2|S| + m(T) + |R| + \sum_{Q} \lfloor \frac{m(Q,T)}{2} \rfloor < n, \text{“} \qquad \text{(italics by the author).}$$

In this inequality $|S|$ and $|R|$ are cardinalities, as is $n = |V(G)|$. $m(T)$ is the number of edges of G with both endvertices in T, $m(Q,T)$ is the number of edges in G with one endvertex in Q and one in T. The summation is over all components Q of $\langle R \rangle$.

The existence of R, S and T such that this condition holds is necessary and sufficient for the non-existence of a 2-factor. Subtracting $n = |R| + |S| + |T|$ on both sides we have

$$|S| + m(T) - |T| + \sum_{Q} \lfloor \frac{m(Q,T)}{2} \rfloor < 0.$$

$m(T)$ is an upper bound for the number of edges of a hamiltonian cycle with both endvertices in T. As this latter number is at most $|T| - k(T)$, where $k(T)$ is the number of components of $\langle T \rangle$, we find

Theorem 3.3. [Chvátal] G is non-hamiltonian if, for $T \neq V(G)$, $V(G)$ can be partitioned into sets R, S and T such that

$$|S| - k(T) + \sum_{Q} \lfloor \frac{m(Q,T)}{2} \rfloor < 0.$$

For $R = \emptyset$, $m(Q,T) = 0$ and we find $|S| < k(T)$. For a hamiltonian graph G we must then have $|S| \geq k(T)$, meaning that $k(T)$, the number of components of $G - S$, is at most $|S|$ or that G is 1-tough.

For the condition for non-hamiltonicity of Theorem 3.3 there is a direct interpretation in terms of crossings. A Hamilton cycle C has to cross the components of $\langle T \rangle$ at least $k(T)$ times. On the other hand C has to cross the vertices of S and the components Q of $\langle R \rangle$. The number of these crossings has upper bound $|S| + \sum_{Q} \lfloor \frac{m(Q,T)}{2} \rfloor$. Therefore

$$|S| + \sum_{Q} \lfloor \frac{m(Q,T)}{2} \rfloor \geq k(T)$$

is necessary for hamiltonicity. Sofar the comparison with Chvátal's theory.

The link of crossing theory with path-toughness and edge-toughness is the following. Given a set X of vertices on a Hamilton cycle C of a graph G, the components of $G - X$ are covered by at most $|X|$ vertex-disjoint paths belonging to C, possibly of length 0. In the case of an independent set A there are exactly $|A|$ paths covering $G - A$. Denote by $\mu(G - X)$ the minimum number of vertex-disjoint paths that are needed to cover all vertices of $G - X$. A graph is called *path-tough* if and only if $\mu(G - X) \leq |X|$ for all non-empty sets $X \subseteq V(G)$. Clearly a graph G is path-tough if G is hamiltonian.

Lemma 3.4. [Hendry] If G is non-path-tough then G is non-hamiltonian.

The concept of path-toughness clearly ties up with the second condition of Theorem 3.1, where crossing paths are required to cover all vertices of $G - A$. Differences are that in Theorem 3.1 paths are required to have endvertices in A and A is required to be an independent set.

More directly related to crossing theory is the concept of edge-toughness.

Lemma 3.5. If a graph G satisfies condition (i) of Theorem 3.1 then G is 1-edge-tough.

Proof. Consider vertex set X and edge set Y in G such that $\omega(G - X - Y - \bigcup_{i=1}^{p} in_{G-X}(Q_i)) > 1$. Choose arbitrarily one vertex in each of the ω components, but not in $\bigcup_{i=1}^{p} bd_{G-X}(Q_i)$. By the definition of boundary vertices such vertices exist. These ω vertices constitute an independent set A. The vertices of X and the vertices of the components Q_i, of the graph induced by Y, belong to $G - A$, that has components R_j, $j = 1, \ldots, n$. In the definition of 1-edge-toughness the right hand side of the inequality reads

$$|X| + \sum_{i=1}^{p} \lfloor \frac{|bd_{G-X}(Q_i)|}{2} \rfloor .$$

By condition (i) we have

$$\sum_{j=1}^{n} CN(R_j, \mathcal{P}_j) \geq |A|.$$

As we have chosen $|A| = \omega(G - X - Y - \bigcup_{i=1}^{p} in_{G-X}(Q_i))$ we only have to compare the right hand side with the sum of the crossing numbers. $\sum_{j=1}^{n} CN(R_j)$ is the maximum number of vertex-disjoint paths with distinct endvertices in A. These paths have to pass the vertices of X or go in and out boundary vertices of the components Q_i as (X, Y) is an A-separator. Therefore

$$|X| + \sum_{i=1}^{p} \lfloor \frac{|bd_{G-X}(Q_i)|}{2} \rfloor \geq \sum_{j=1}^{n} CN(R_j),$$

and the result follows. □

4 Discussion

Unlike the sufficient conditions for hamiltonicity, the known sufficient conditions for non-hamiltonicity are not easily checked and are less clearly interrelated. The necessary and sufficient condition given by Theorem 3.1 in terms of crossings gives some hold in comparing them.

Condition (i) concerns crossing numbers. Both Lemma 2.1 and Lemma 2.2 involve conditions on the crossing number. If condition (i) is satisfied then G is 1-edge-tough by Lemma 3.5. Lemma 2.4 stated that G is 1-edge-tough under the much stronger condition that G is hamiltonian, i.e. satisfies conditions (i) up to (iv).

Condition (ii) concerns crossing lengths and the covering of vertices, like in the concept of path-toughness. If condition (ii) does not hold for a graph G it implies, like non-path-toughness by Lemma 3.4, that G is non-hamiltonian.

The link with crossing theory is most clear in the interpretation of Theorem 3.3, that stems from a necessary and sufficient condition for the existence of a 2-factor. Conditions (i), (ii) and (iii) also imply the existence of a 2-factor, but in reference to the independent set A.

The step from the existence of a 2-factor to the existence of a Hamilton cycle is made by posing condition (iv). For this we would have to have a survey of all systems of crossings paths satisfying the first three conditions of Theorem 3.1. An attempt to make the step from 2-factor to Hamilton cycle without reference to crossings is given in Theorem 3.2.

The interesting conjecture that 2-tough graphs are hamiltonian does not really tie up with the conditions that we discussed. By Lemma 2.5 a 2-tough graph is 1-edge-tough and, by Lemma 2.3, also 1-tough. These results were derived, however, in the discussion of non-hamiltonicity. Enomoto *et al* [3] showed that 2-tough graphs have a 2-factor. The existence of a 2-factor does not directly imply that the first three conditions of Theorem 3.1 hold, as a 2-factor may fail to have two vertices from A on each of its cycles. To tie up with Theorem 3.1 we would have to prove that 2-tough graphs imply a 2-factor consisting of paths with distinct endvertices in A for which condition (iv) holds.

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