Double Youden rectangles of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$

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ABSTRACT. A $k \times v$ double Youden rectangle (DYR) is a type of balanced Graeco-Latin design where each Roman letter occurs exactly once in each of the k rows, where each Greek letter occurs exactly once in each of the v columns, and where each Roman letter is paired exactly once with each Greek letter. The other properties of a DYR are of balance, and indeed the structure of a DYR incorporates that of a symmetric balanced incomplete block design (SBIBD). Few general methods of construction of DYRs are known, and these cover only some of the sizes $k \times v$ with $k = p$ (odd) or $p+1$, and $v = 2p+1$. Computer searches have however produced DYRs for those such sizes, $p \leq 11$, for which the existence of a DYR was previously in doubt. The new DYRs have cyclic structures. A consolidated table of DYRs of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$ is provided for $p \leq 11$; for each of several of the sizes, DYRs are given for different inherent SBIBDs.

1 Introduction

Bailey [1] defined a double Youden rectangle (DYR) of size $k \times v$, with $k < v$, as a $k \times v$ rectangular arrangement of the kv distinct ordered pairs

x, y formed when x is drawn from a set X of v elements, and y from a set Y of k elements, with

(i) each element of X appearing exactly once on each row,

(ii) each element of Y appearing exactly once in each column,

(iii) each element of X appearing at most once in each column, the sets of elements of X in the columns being the blocks of a symmetric balanced incomplete block design (SBIBD, or symmetric 2-design),

(iv) each element of Y appearing either n or $n+1$ times in each row, where n is the integral part of v/k, the remainder being $m = v - nk$, and where either $m = 1$ or, if n occurrences of each element from Y are removed from each row, the remaining sets of elements of Y in the rows are the blocks of an SBIBD.

The elements of X and Y could be chosen to be letters from, respectively, the Roman and Greek alphabets, thus justifying the description of a DYR as a special type of Graeco-Latin design; however, we do not use this notation in this paper.

DYRs of a given size can be classified into 'species' [12] akin to the 'main classes' (alias 'species') of Latin squares.

An example of a 4×7 DYR is the following:

^{*0}	$c2$	$a3$	$b1$	$B2$	$C3$	$A1$
$a1$	$b3$	$C2$	$B0$	^{*3}	$A2$	$c0$
$b2$	$C0$	$c1$	$A3$	$a0$	^{*1}	$B3$
$c3$	$B1$	$A0$	$a2$	$C1$	$b0$	^{*2}

Here $X = \{^*, A, B, C, a, b, c\}$ and $Y = \{0, 1, 2, 3\}$. The horizontal and vertical lines have been inserted to illustrate the DYR's p-cyclic structure with $p = 3$:

(a) within the 1×3 and 3×1 subrectangles, each successive entry is obtained from the previous one by use of the cyclic permutations (ABC) (abc) and (123);

(b) within each of the two 3×3 subsquares, these permutations are similarly used on the main diagonal and on broken diagonals parallel to it, the elements * and 0 being invariant;

(c) the isolated entry in the top left-hand corner contains the invariant elements.

Preece [12, 13] reviewed knowledge of DYRs. Special attention has been given to DYRs of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$ where p is odd. For these sizes, the parameter m from condition (iv) takes the values 1 and p respectively.

We define a $p \times (2p+1)$ DYR to be 'perfect' if, within each of two disjoint sets of p columns, the symbols from Y are disposed in a Latin square. We define a $(p+1) \times (2p+1)$ DYR to be 'perfect' if, within each of two sets of $p+1$ columns, these sets being disjoint save for a single common member, the symbols from Y are disposed in a Latin square. The 4×7 DYR given above is readily seen to be perfect.

For DYRs of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$, the inherent SBIBDs from condition (iii) must come from the well known mutually complementary series of SBIBDs with

$$(v, k, \lambda) = (2p+1, p, (p-1)/2) \text{ and } (2p+1, p+1, (p+1)/2).$$

No 3×7 DYR exists [7], but two species of 4×7 DYRs have been reported [4, 9, 14]. Two species of size 5×11 have been reported [12], and many species of size 6×11 [9, 12, 14]. Many 7×15 DYRs have been reported [8, 10, 15] and an 8×15 DYR [11]. No examples of 9×19 or 10×19 DYRs have hitherto been published. Some DYRs of size 11×23 have been reported [8, 15], but none of size 12×23.

Most of the above-mentioned published DYRs of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$ are p-cyclic in the sense illustrated above. But only those of sizes 7×15 and 11×23 were obtained from general methods of construction [15], which, till now, have been available only for sizes $p \times (2p+1)$ where p is a prime power congruent to 3 (modulo 4). Otherwise, the published DYRs that have been referred to have been obtained by trial-and-error matching methodology, as described by Preece [14].

2 The new table of DYRs

DYRs have now been found for all of the sizes 5×11, 7×15, 9×19 and 11×23, and for all of the complementary sizes 6×11, 8×15, 10×19 and 12×23 (as well, of course, as 4×7). This progress has been achieved primarily by complete computerisation of the search process, and mostly by restricting the search so as to produce only outcomes that are p-cyclic in the sense illustrated above. Newly found DYRs have been gathered together with previously published ones to give the consolidated Table 1 below, for $p \leq 11$. Some of the DYRs in the Table are perfect, but no computer search was made specifically for such DYRs.

Within the mutually complementary series of SBIBDs with $(v, k, \lambda) = (2p+1, p, (p-1)/2)$ and $(2p+1, p+1, (p+1)/2)$, the number Nd of mutually non-isomorphic SBIBDs for a particular value of p is as follows:

p	3	5	7	9	11
Nd	1	1	5	6	1102

Thus, for $p = 7$, 9 or 11 there is a choice of inherent SBIBD for a DYR of size $p \times (2p+1)$ or $(p+1) \times (2p+1)$. Table 1 gives a DYR for each of these SBIBDs for which a DYR has now been found. For each such SBIBD, only one DYR with a particular cyclic structure is given (even though others, perhaps from other species, may be known), except that a perfect and a non-perfect DYR are given, if known. For size 7×15, the labellings (e.g. $C5$) of the inherent SBIBDs are those of Bhat and Shrikhande [3]; for size 9×19, the labellings (e.g. $D1$) are those of Bhat [2]; for size 11×23, the labelling $aC5$ (a = analogue) refers to an SBIBD analogous to the SBIBD $C5$ for size 7×15, etc. For size 7×15, the inherent SBIBDs $C5$ and $C2$ are each self-dual, whereas $C3$ and $C1$ are the duals of one another; for size 9×19, $D1$ and $D2$ are each self-dual. For sizes 8×15, 10×19 and 12×23, the inherent SBIBDs are complements of inherent SBIBDs for sizes 7×15, 9×19 and 11×23; the symbol $\sim$ is used to denote complement. For further identification of each of the inherent SBIBDs, the order $|A|$ of its automorphism group is given in Table 1.

For the p-cyclic $p \times (2p+1)$ DYRs, the $2p+1$ symbols in the set X are taken to be the p letters $A, B, C, \ldots$, the p letters $a, b, c, \ldots$, and the asterisk *; the p symbols in the set Y are taken to be $1, 2, \ldots, p$ except that, when $p = 11$, they are written $1, 2, \ldots, 9, t, u$. For the p-cyclic $(p+1) \times (2p+1)$ DYRs, the symbols in Y are taken to be $0, 1, 2, \ldots$. For both types, the permutations for the cyclic generation of the DYRs are

$$(ABC\ldots)(abc\ldots) \text{ and } (123\ldots p).$$

Amongst the new DYRs in Table 1 is a 7×15 DYR whose inherent SBIBD is $C2$. This DYR has

$$X = \{{}^*, A, B, \ldots, F, a, b, \ldots, f, G, g\} \text{ and } Y = \{0, 1, 2, \ldots, 6\},$$

and is 3-cyclic with permutations

$$(ABC)(DEF)(abc)(def) \text{ and } (123)(456),$$

the symbols *, G and g from X being invariant, as is the symbol 0 from Y. Inspection of the DYR is sufficient to show that the role of symbol * in the design differs from that of G or g.

Also amongst the new DYRs in Table 1 is a perfect 9×19 DYR whose inherent SBIBD is $D3$ and whose structure is bicyclic with permutations

$$(ABC)(DEF)(GHI)(abc)(def)(ghi) \text{ and } (123)(456)(789)$$

and

$$(ADG)(BEH)(CFI)(adg)(beh)(cfi) \text{ and } (147)(258)(369).$$

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Table 1

$k \times v$ double Youden rectangles (DYRs)
with $v = 2p+1$ and $k = p$ or $p+1$, where $p \leq 11$.

For each DYR, $|A|$ = the order of the automorphism group of the inherent SBIBD.
Within the DYRs, horizontal and vertical lines are used to show the cyclic structure (see text).
For sizes 11×23 and 12×23, the symbols $\ldots, 8, 9, t, u$ are used for Y, to avoid confusion with symbols from X.

$k \times v$ SBIBD $\|A\|$	DYR
4×7 sole 168	Perfect:

*0	c2	a3	b1	B2	C3	A1
a1	b3	C2	B0	*3	A2	c0
b2	C0	c1	A3	a0	*1	B3
c3	B1	A0	a2	C1	b0	*2

Ditto Non-perfect:

*0	b3	c1	a2	B1	C2	A3
c3	C1	a3	A0	*2	b0	B2
a1	B0	A2	b1	C3	*3	c0
b2	c2	C0	B3	a0	A1	*1

5 × 11 sole 660 — 3-cyclic with permutations (ABC) (DEF) (GHI) and (123):

E3	I5	J3	K2	C2	H4	G1	F4	B5	A1	D1
J1	F1	G5	I4	K3	A3	C5	H2	D4	B2	E2
H5	J2	D2	B1	G4	K1	E4	A5	I3	C3	F3
C4	A4	B4	H3	I1	G2	F2	D3	E1	J5	K5
I2	G3	H1	F5	D5	E5	B3	C1	A2	K4	J4

6 × 11 sole 660 — 3-cyclic with permutations (ABC) (DEF) (GHI) and (123) (456):

K6	H2	E2	A3	J5	F1	I4	B3	G4	D5	C6
F3	K4	I3	D2	B1	J6	H5	G5	C1	E6	A4
G1	D1	K5	J4	E3	C2	A2	I6	H6	F4	B5
B2	E5	C4	G6	A6	D3	J1	K1	F5	I2	H3
A5	C3	F6	E1	H4	B4	D6	J2	K2	G3	I1
D4	B6	A1	C5	F2	I5	K3	E4	J3	H1	G2

Ditto p-cyclic with $p = 5$:

*0	e1	a2	b3	c4	d5	B2	C3	D4	E5	A1
b2	a0	c5	E4	D1	C4	A0	*2	B5	e3	d3
c3	D5	b0	d1	A5	E2	e4	B0	*3	C1	a4
d4	A3	E1	c0	e2	B1	b5	a5	C0	*4	D2
e5	C2	B4	A2	d0	a3	E3	c1	b1	D0	*5
a1	b4	D3	C5	B3	e0	*1	A4	d2	c2	E0

7 × 15 C5 20160 — Perfect:

A1	*1	c7	e6	F5	b4	G3	D2	a1	C7	E6	g5	B4	d3	f2
B2	E3	*2	d1	f7	G6	c5	A4	g3	b2	D1	F7	a6	C5	e4
etc.				etc.							etc.			

Ditto Non-perfect:

A1	*1	c5	e2	F5	b3	G3	D2	a1	C7	E6	g6	B4	d7	f4
B2	E3	*2	d6	f3	G6	c4	A4	g5	b2	D1	F7	a7	C5	e1
etc.				etc.							etc.			

7 × 15 C3 168 — Perfect:

A1	*1	c7	e6	F5	b4	G3	D2	a1	E7	B6	f5	C4	g3	d2
B2	E3	*2	d1	f7	G6	c5	A4	e3	b2	F1	C7	g6	D5	a4
etc.				etc.							etc.			

Ditto Non-perfect:

A1	*1	c5	e2	F5	b3	G3	D2	a1	d7	g6	B6	f4	E7	C4
B2	E3	*2	d6	f3	G6	c4	A4	D5	b2	e1	a7	C7	g5	F1
etc.				etc.							etc.			

7 × 15 Perfect:
C1
168

A1	*1	c6	e4	G2	b7	D5	F3	a1	E4	B7	g2	C6	d5	f3
B2	G4	*2	d7	f5	A3	c1	E6	g4	b2	F5	C1	a3	D7	e6
etc.				etc.							etc.			

Ditto Non-perfect:

A1	*1	d3	g5	G2	f2	D5	F3	a1	E4	B7	c7	C6	b4	e6
B2	G4	*2	e4	a6	A3	g3	E6	f7	b2	F5	C1	d1	D7	c5
etc.				etc.							etc.			

7 × 15 3-cyclic, with permutations (ABC) (DEF) (abc) (def)
C2 and (123) (456):
96

*0	D2	E3	F1	f5	d6	e4	B2	C3	A1	b4	c5	a6	G0	g0
F6	*1	G5	C0	A2	e2	b5	D3	f0	c4	B6	g1	E4	a1	d3
D4	A0	*2	G6	c6	B3	f3	a5	E1	d0	F5	C4	g2	b2	e1
E5	G4	B0	*3	d1	a4	C1	e0	b6	F2	g3	D6	A5	c3	f2
A3	b3	g6	d5	*4	E0	c2	F4	G2	e5	D1	a0	f1	B4	C6
B1	e6	c1	g4	a3	*5	F0	f6	D5	G3	d2	E2	b0	C5	A4
C2	g5	f4	a2	D0	b1	*6	G1	d4	E6	c0	e3	F3	A6	B5

8 × 15
C5 ~
20160

*0	F6	G7	A1	B2	C3	D4	E5	c2	d3	e4	f5	g6	a7	b1
e5	D3	g1	b4	C0	E2	f2	c7	F7	A6	d0	a6	*5	B4	G3
f6	d1	E4	a2	c5	D0	F3	g3	A4	G1	B7	e0	b7	*6	C5
etc.				etc.							etc.			

8 × 15
C3 ~
168

*0	F5	G6	A7	B1	C2	D3	E4	f5	g6	a7	b1	c2	d3	e4
d4	D7	g5	B3	c0	e6	b4	G2	*1	a5	f2	F6	E3	C7	A0
e5	A3	E1	a6	C4	d0	f7	c5	B0	*2	b6	g3	G7	F4	D1
etc.				etc.							etc.			

8 × 15
C1 ~
168

*0	C7	D1	E2	F3	G4	A5	B6	c7	d1	e2	f3	g4	a5	b6
a1	g0	C3	f7	A7	e3	d6	G2	b5	F4	D0	*4	E6	c2	B5
b2	A3	a0	D4	g1	B1	f4	e7	C6	c6	G5	E0	*5	F7	d3
etc.				etc.							etc.			

9 × 19
D1
171

A1	*1	a6	B9	I5	i7	G4	d1	C4	g8	F9	E7	D3	f8	b6	H3	e5	h2	c2
B2	h9	*2	b7	C1	A6	a8	H5	e2	D5	d3	G1	F8	E4	g9	c7	I4	f6	i3
etc.					etc.									etc.				

9 × 19
D2
9

B1	*1	D7	G5	A3	c2	e7	i1	I6	a4	g3	f6	d8	F2	C4	E9	H8	h5	b9
C2	b5	*2	E8	H6	B4	d3	f8	a2	A7	c1	h4	g7	e9	G3	D5	F1	I9	i6
etc.					etc.									etc.				

9 × 19 Perfect and dicyclic (see text):
D3
72

A1	*1	g4	d7	E8	G2	i5	I6	e9	D3	a1	f8	h6	C5	H7	c2	B9	b3	F4
B2	e8	*2	h5	g6	F9	H3	E1	G4	f7	i4	b2	d9	a3	A6	I8	D5	C7	c1
C3	i6	f9	*3	I1	h4	D7	d8	F2	H5	e7	g5	c3	G9	b1	B4	a2	E6	A8
D4	C9	h3	G6	*4	a7	g1	H2	A5	c8	E3	e6	I7	d4	i2	b9	F8	B1	f5
E5	H4	A7	i1	h2	*5	b8	a9	I3	B6	G8	F1	f4	c7	e5	g3	d6	D9	C2
F6	g2	I5	B8	c9	i3	*6	C4	b7	G1	d5	H9	D2	h1	a8	f6	A3	e4	E7
G7	B5	D8	f2	F3	b6	A9	*7	d1	a4	I2	E4	i8	H6	h9	C1	g7	c5	e3
H8	d3	C6	E9	B7	D1	c4	b5	*8	e2	g9	G3	F5	A2	I4	i7	f1	h8	a6
I9	F7	e1	A4	a5	C8	E2	f3	c6	*9	D6	h7	H1	g8	B3	G5	b4	d2	i9

10 × 19
D1 ~
171

*0	C9	D1	E2	F3	G4	H5	I6	A7	B8	b9	c1	d2	e3	f4	g5	h6	i7	a8
d6	b7	I2	h5	D9	B3	i0	g2	a9	C3	*1	f6	G0	A5	E8	e8	c4	F4	H7
e7	D4	c8	A3	i6	E1	C4	a0	h3	b1	I8	*2	g7	H0	B6	F9	f9	d5	G5
etc.					etc.									etc.				

10 × 19
$D2 \sim$
9

*0	D9	E1	F2	G3	H4	I5	A6	B7	C8	h5	i6	a7	b8	c9	d1	e2	f3	g4
a4	g2	e7	h6	C6	A2	D4	b0	H3	i9	*1	F3	I8	G0	E7	B9	f5	d5	c8
b5	a1	h3	f8	i7	D7	B3	E5	c0	I4	d9	*2	G4	A9	H0	F8	C1	g6	e6
etc.					etc.									etc.				

11 × 23 Perfect:
aC5
660

A1	*1	Ct	f7	G6	I4	K2	e8	bu	j3	H5	d9	a1	E3	i5	B7	F9	Ju	c2	g4	k6	D8	ht
B2	et	*2	Du	g8	H7	J5	A3	f9	c1	k4	I6	iu	b2	F4	j6	C8	Gt	K1	d3	h5	a7	E9
etc.						etc.											etc.					

11 × 23 Non-perfect:
aC5
660

A1	*1	Ct	f4	G6	I4	K2	et	b6	j2	H5	d5	a1	E3	iu	B7	F9	Ju	c9	g3	k8	D8	h7
B2	e6	*2	Du	g5	H7	J5	A3	fu	c7	k3	I6	i8	b2	F4	j1	C8	Gt	K1	dt	hA	a9	E9
etc.						etc.											etc.					

11 × 23 Perfect:
aC3
55

A1	*1	Ct	f7	G6	I4	K2	e8	bu	j3	H5	d9	a1	Ju	g2	F9	D8	B7	hA	kt	c5	E3	i6
B2	et	*2	Du	g8	H7	J5	A3	f9	c1	k4	I6	j7	b2	K1	h3	Gt	E9	C8	i5	au	d6	F4
etc.						etc.											etc.					

11 × 23 Non-perfect:
aC3
55

A1	*1	Ct	f4	G6	I4	K2	et	b6	j2	H5	d5	a1	Ju	g9	F9	D8	B7	h3	k7	cu	E3	i8
B2	e6	*2	Du	g5	H7	J5	A3	fu	c7	k3	I6	j9	b2	K1	ht	Gt	E9	C8	i4	a8	d1	F4
etc.						etc.											etc.					

11 × 23 Perfect:
aC1
55

A1	*1	f7	Bt	e8	j3	d9	D6	J5	E4	bu	F2	a1	H6	iu	K5	Gt	C4	c9	g3	k8	I2	h7
B2	G3	*2	g8	Cu	f9	k4	et	E7	K6	F5	c1	i8	b2	I7	j1	A6	Hu	D5	dt	hA	a9	J3
etc.						etc.											etc.					

11 × 23 Non-perfect:
aC1
55

A1	•1	e6	Bt	b5	ft	j4	D6	J5	E4	d2	F2	a1	Hu	iu	K9	G8	C7	c9	g3	k8	I3	h7
B2	G3	•2	f7	Cu	c6	gu	k5	E7	K6	F5	e3	i8	b2	I1	j1	At	H9	D8	dt	h4	a9	J4
etc.						etc.											etc.					

12 × 23
aC5 ~
660

•0	A1	B2	C3	D4	E5	F6	G7	H8	I9	Jt	Ku	j6	k7	a8	b9	ct	du	e1	f2	g3	h4	i5
j5	i2	D0	et	K8	d4	au	B5	ht	F4	I2	g8	•1	A6	C9	Ju	f3	H7	k9	E7	c6	b0	G3
k6	h9	j3	E0	fu	A9	e5	b1	C6	iu	G5	J3	H4	•2	B7	Dt	K1	g4	I8	at	F8	d7	c0
etc.						etc.											etc.					

12 × 23
aC3 ~
55

•0	G4	H5	I6	J7	K8	A9	Bt	Cu	D1	E2	F3	h9	it	ju	k1	a2	b3	c4	d5	e6	f7	g8
c3	g0	It	C9	A8	D6	b8	f5	e7	K9	a4	k7	•1	Fu	H0	B5	E4	Gu	Jt	j3	d2	i6	h2
d4	a8	h0	Ju	Dt	B9	E7	c9	g6	f8	At	b5	i3	•2	G1	I0	C6	F5	H1	Ku	k4	e3	j7
etc.						etc.											etc.					

12 × 23
aC1 ~
55

•0	F9	Gt	Hu	I1	J2	K3	A4	B5	C6	D7	E8	e2	f3	g4	h5	i6	j7	k8	a9	bt	cu	d1
a1	D8	d5	C9	E6	e4	bu	fu	c0	Bt	gt	I9	J4	F8	•3	i2	H7	G5	A0	k2	j6	h3	K7
b2	Jt	E9	e6	Dt	F7	f5	c1	g1	d0	Cu	hu	A8	K5	G9	•4	j3	I8	H6	B0	a3	k7	i4
etc.						etc.											etc.					