ON THE INEQUALITY $dk(G) \leq k(G) + 1$

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Abstract

Let D be an acyclic digraph. The competition graph of D has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. The competition-common enemy graph of D has the same set of vertices as D and an edge between vertices u and v if and only if there are vertices w and x in D such that (w, u), (w, v), (u, x), and (v, x) are arcs of D. The competition number (respectively, double competition number) of a graph G, denoted by $k(G)$ (respectively, $dk(G)$), is the smallest number k such that G together with k isolated vertices is a competition graph (respectively, competition-common enemy graph) of an acyclic digraph.

It is known that $dk(G) \leq k(G) + 1$ for any graph G. In this paper, we give a sufficient condition under which a graph G satisfies $dk(G) \leq k(G)$ and show that any connected triangle-free graph G with $k(G) \geq 2$ satisfies that condition. We also give an upper bound for the double competition number of a connected triangle-free graph. Finally we find an infinite family of graphs each member G of which satisfies $k(G) = 2$ and $dk(G) > k(G)$.

Key words. competition number, double competition number, triangle-free graphs, Harary graphs

1 Introduction

The *competition graph* of D has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. Since Cohen [2] introduced the notion of competition graph in 1968, various variations have been defined and studied by

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many authors. (See the survey articles by Kim [6] and Lundgren [10].) The notion of competition-common enemy graph was introduced by Scott [14] in 1987 as one of these variants. The *competition-common enemy graph* (*CCE graph*) of an acyclic digraph D has the same set of vertices as D and an edge between vertices u and v if and only if there are vertices w and x in D such that (w,u), (w,v), (u,x), (v,x) are arcs of D.

We suppose that $D = (V, A)$ is an acyclic digraph. (For all undefined graph theory terminology, see [1] or [12].) For each vertex of D, we define $S_D(v)$ and $P_D(v)$ as follows:

$$\begin{aligned} S_D(v) &= \{x \in V(D) \mid (x,v) \in A(D)\} \\ P_D(v) &= \{x \in V(D) \mid (v,x) \in A(D)\} \end{aligned}$$

The edge sets of the competition graph and the CCE graph of an acyclic digraph D can be described as follows: The vertices u and v are adjacent in the competition graph of D if and only if $P_D(u) \cap P_D(v) \neq \emptyset$. Also, vertices u and v are adjacent in the CCE graph of D if and only if $S_D(u) \cap S_D(v) \neq \emptyset$ and $P_D(u) \cap P_D(v) \neq \emptyset$.

The definition of the competition number $k(G)$ of a graph G was introduced by Roberts [13]. The *competition number* $k(G)$ of a graph G is the smallest number k such that $G \cup I_k$ is a competition graph of some acyclic digraph when $G \cup I_k$ is G together with k isolated vertices. Roberts [13] showed that the competition number $k(G)$ is well-defined. The literature of competition graphs is summarized in [5], [6], [10], and [16]. Analogously to the definition of $k(G)$ of a graph G, Scott [14] defined the *double competition number* $dk(G)$ of G to be the smallest number k such that $G \cup I_k$ is a CCE graph of some acyclic digraph and then showed that $dk(G)$ of a graph G is well-defined. The CCE graphs have been studied by Füredi [3], Jones *et al.* [4], Kim *et al.* [9], Scott [14], and Seager [15].

Scott [14] observed that $2 \leq dk(G) \leq k(G) + 1$ for any graph G without isolated vertices. If $k(G) = 1$ and G does not have isolated vertices, then those inequalities are immediately replaced by equalities. This observation led Kim *et al.* [9] to ask for conditions under which $dk(G) = k(G) + 1$ for the case $k(G) \geq 2$ or for interesting families of graphs for which this is true. In Section 2, we partially answer their question by giving a sufficient condition for a graph G satisfying $dk(G) \leq k(G)$. We also show that a large number of graphs including triangle-free graphs satisfy this condition. This suggests that it should not be easy to find a graph G with $k(G) \geq 2$ and $dk(G) = k(G)+1$. In Section 3, we present an infinite family of graphs with $k(G) = 2$ and $dk(G) = k(G)+1$. In Section 2, we also give an upper bound for the double competition number of a connected, triangle-free graph G with $k(G) \geq 2$. Finally, we pose some open questions in Section 4.

Before preceding, we need the following results:

Theorem 1 (Roberts [13]) *If G is a triangulated graph, then $k(G) \leq 1$.*

Theorem 2 (Roberts [13]) *If a graph G is connected, $|V(G)| > 1$, and G has no triangles, then $k(G) = |E(G)| - |V(G)| + 2$.*

He also showed that for any triangle-free graph G with $|V(G)| > 1$, $k(G) \geq |E(G)| - |V(G)| + 2$. Kim [7] gave the following formula for computing the competition number of triangle-free graphs without isolated vertices. For a triangle-free graph G, we define $f(G)$ by

$$f(G) = \max\{0, -|E(G)| + |V(G)| - 1\}.$$

Theorem 3 (Kim [7]) *If G has no isolated vertices and no triangles, then*

$$k(G) = |E(G)| - |V(G)| + 2 + f(G).$$

2 A condition under which a graph G satisfies $dk(G) < k(G) + 1$

Given a graph G with n vertices and without isolated vertices, we take a vertex v of G. We denote by $H_G(v)$ the graph induced by the vertices in the nontrivial components of $G - v$. We denote by $i_G(v)$ the number of isolated vertices in $G - v$. We denote by $D_G(v)$ an acyclic digraph whose competition graph is $H_G(v)$ plus $k(H_G(v))$ isolated vertices. From the digraph $D_G(v)$, we construct a digraph denoted by $\mathcal{D}_G(v)$ as follows. We let $m = n - i_G(v) - 1$. We label v as v_n and the isolated vertices of $G - v_n$ as $v_{m+1}, \ldots, v_{n-1}$. We note that $v_{m+1}, \ldots, v_{n-1}$ are pendant vertices that are adjacent to v_n. Since $D_G(v)$ is acyclic, the vertices of $H_G(v)$ can be labeled $v_1, \ldots, v_m$ so that if (v_i, v_j) is in $A(D_G(v))$, then $i > j$. Now we define $\mathcal{D}_G(v)$ as follows:

$$\begin{aligned} V(\mathcal{D}_G(v)) &= V(D_G(v)) \cup \{v_{m+1}, \ldots, v_n\} \cup \{b\}; \\ A(\mathcal{D}_G(v)) &= A(D_G(v)) \cup \{(v_n, w) \mid w \in V(D_G(v))\} \\ &\cup \{(v_n, v_i) \mid i = m+1, \ldots, n-2\} \\ &\cup \{(b, v_n)\} \cup \{(b, w) \mid v_n w \in E(G)\} \\ &\cup \{(v_{i+1}, v_i) \mid i = m, \ldots, n-2\} \end{aligned}$$

It is obvious that $\mathcal{D}_G(v)$ is acyclic. We call $\mathcal{D}_G(v)$ a *v-dominating digraph* obtained from $D_G(v)$. Now we may claim as follows:

Lemma 4 *Suppose that a graph G has no isolated vertices and $G \neq K_{1,l}$ for any integer $l \geq 1$. For a vertex v^* of G, we take an acyclic digraph*

$D_G(v^*)$ whose competition graph is $H_G(v^*)$ together with $k(H_G(v^*))$ isolated vertices. If $\mathcal{D}_G(v^*)$ is a v^*-dominating digraph obtained from $D_G(v^*)$, then the CCE graph of $\mathcal{D}_G(v^*)$ is G together with $k(H_G(v^*)) + 1$ isolated vertices.

Proof. We let G' be the CCE graph of $\mathcal{D}_G(v^*)$. Then v^* is labeled v_n in $\mathcal{D}_G(v^*)$. We let $H = H_G(v^*)$, $D = D_G(v^*)$, and $\mathcal{D} = \mathcal{D}_G(v^*)$. Since there is no vertex incoming toward b, b is isolated in G'. It is enough to show that $V(G') = V(G) \cup I_{k(H)} \cup \{b\}$ and $E(G') = E(G)$. Since $V(\mathcal{D}) = V(D) \cup \{v_{m+1}, \ldots, v_n\} \cup \{b\}$ and $V(D) = V(G) - \{v_{m+1}, \ldots, v_n\} \cup I_{k(H)}$, $V(\mathcal{D}) = V(G) \cup I_{k(H)} \cup \{b\}$.

We take an edge vw of G. If neither of v, w is v_n, then vw is an edge of H and there is a vertex x in $V(D)$ so that arcs (v, x) and (w, x) are in $A(D)$. Since arcs (v_n, v) and (v_n, w) are added to $A(D)$ to obtain $A(\mathcal{D})$, v and w are adjacent in G'. If one of v and w, say v, is v_n, then either w is a vertex of H or $w = v_j$ for some $j \in \{m+1, \ldots, n-1\}$. We assume the former. Since H does not have isolated vertices, w is adjacent to some vertex in H and therefore there is a vertex x in $V(D)$ such that arc (w, x) is in $A(D)$. Since arcs (v_n, x), (b, v_n), (b, w) are added to $A(D)$ to obtain $A(\mathcal{D})$, v_n is adjacent to v in G'. Now we suppose that $w = v_j$ for some $j \in \{m+1, \ldots, n-1\}$. Then there are arcs (b, v_n), (b, v_j), (v_n, v_{j-1}), (v_j, v_{j-1}) and therefore $v_n v_j$ is in $E(G')$. We have just shown that $E(G) \subseteq E(G')$.

We take two nonadjacent vertices v, w of G. If v and w are both vertices of H, then they are not adjacent in H and therefore $P_D(v) \cap P_D(w) = \emptyset$ since $H \cup I_{k(H)}$ is the competition graph of D. Since no arcs outgoing from v or w were added to obtain $\mathcal{D}$, it is also true that $P_{\mathcal{D}}(v) \cap P_{\mathcal{D}}(w) = \emptyset$ and therefore they are not joined in G'. Now we assume that v or w, say v, is v_n. Since all the arcs go from a higher index to a lower one, $S_{\mathcal{D}}(v_n) = \{b\}$. However, by the definition of $\mathcal{D}_G(v^*)$, b is not in $S_{\mathcal{D}}(w)$ since v_n and w are not adjacent in G. Therefore $S_{\mathcal{D}}(v_n) \cap S_{\mathcal{D}}(w) = \emptyset$ and v_n and w are not adjacent in G'. Finally, we suppose that v or w, say v, is v_j for some $j \in \{m+1, \ldots, n-1\}$. Then w cannot be v_n. It is obvious from the definition of $\mathcal{D}_G(v^*)$ that $P_{\mathcal{D}}(v_j) = \{v_{j-1}\}$ and $S_{\mathcal{D}}(v_{j-1}) = \{b, v_j, v_n\}$. The latter implies that v_{j-1} is not in $P_{\mathcal{D}}(w)$. Therefore, $P_{\mathcal{D}}(v_j) \cap P_{\mathcal{D}}(w) = \emptyset$ and v_j is not adjacent to w in G'. Thus $E(G) \supseteq E(G')$. Hence $E(G) = E(G')$. □

The following result gives a large class of graphs G with $dk(G) \leq k(G)$.

Theorem 5 *Suppose that a graph G has no isolated vertices and $G \neq K_{1,l}$ for $l \geq 1$. If there exists a vertex v^* in $V(G)$ such that $k(H_G(v^*)) < k(G)$, then $dk(G) \leq k(G)$*

Proof. We take an acyclic digraph $D_G(v^*)$ such that the competition graph of $D_G(v^*)$ is $H_G(v^*) \cup I_{k(H_G(v^*))}$. If $\mathcal{D}_G(v^*)$ is a v^*-dominating digraph obtained from $D_G(v^*)$, then the CCE graph of $\mathcal{D}_G(v^*)$ is $G \cup I_{k(H_G(v^*))}$ together with one more isolated vertex by Lemma 4. Therefore,

$$dk(G) \leq k(H_G(v^*)) + 1 \leq k(G) - 1 + 1 = k(G).$$

□

A large class of graphs satisfy the hypothesis of Theorem 5. We present the following two interesting families of graphs which are included in this class. We denote by $d_G(v)$ the degree of a vertex v of a graph G.

Corollary 6 *For any connected, triangle-free graph G with $k(G) \geq 2$,*

$$dk(G) \leq k(G).$$

Proof. Since $k(G) \geq 2$, there exists a cycle C in G. We take a vertex v' on C. Since the two vertices on C that are adjacent to v' are in the same component in $G-v'$, $d_G(v') - i_G(v') \geq 2$. We let $H = H_G(v')$. If $f(H) = 0$, then

$$\begin{aligned} k(H) &= |E(H)| - |V(H)| + 2 && \text{(Theorem 3)} \\ &= (|E(G)| - d_G(v')) - (|V(G)| - i_G(v') - 1) + 2 \\ &= |E(G)| - |V(G)| + i_G(v') - d_G(v') + 3 \\ &\leq |E(G)| - |V(G)| + 1 < k(G). \end{aligned}$$

If $f(H) > 0$, then $f(H) = -|E(H)| + |V(H)| - 1$ by the definition of $f(H)$. Then

$$k(H) = |E(H)| - |V(H)| + 2 + f(H) = 1 < k(G).$$

Therefore, by Theorem 5, $dk(G) \leq k(G)$. □

Kim [8] showed that if a graph G has exactly one cycle of length at least four as an induced subgraph, then $k(G) \leq 2$.

Corollary 7 *Suppose that a graph G without isolated vertices has exactly one cycle of length at least four as an induced subgraph and satisfies equality $k(G) = 2$. Then*

$$dk(G) \leq k(G).$$

Proof. We take a vertex v' on the cycle of length at least four. Then $H_G(v')$ is triangulated and therefore $k(H_G(v')) \leq 1$ by Theorem 1. Hence $dk(G) \leq k(G)$ by Theorem 5. □

Kim *et al.* [9] asked for infinite families of graphs for which $k(G) - dk(G)$ remains bounded, though $dk(G) < k(G) + 1$. Corollary 7 answers their question. There are infinitely many graphs satisfying the hypothesis of

Corollary 7 and therefore the equality $k(G) - dk(G) = 0$. For example, each of wheels W_n has exactly one cycle of length at least four as a generated subgraph. Kim [8] showed that $k(W_n) = 2$ for wheels W_n and therefore wheels W_n satisfy the hypothesis of Corollary 7. We also note that the hypothesis of Theorem 5 can be satisfied by graphs with many triangle such as wheels.

For a nontrivial graph G that is not isomorphic to $K_{1,l}$ for any integer $l \geq 1$, we let $\Delta^*(G) = \max_{v \in V(G)}\{d_G(v) - i_G(v) - f(H_G(v))\}$ and $\Delta^*(K_{1,l}) = 1$ for any integer $l \geq 1$. The following theorem gives an upper bound for the double competition number of a triangle-free graph:

Theorem 8 *Suppose that a nontrivial graph G is connected and triangle-free. Then*

$$dk(G) \leq |E(G)| - |V(G)| + 4 - \Delta^*(G).$$

Proof. If $G = K_{1,l}$ for some $l \geq 1$ then $|E(G)| - |V(G)| + 4 = 3$ and $\Delta^*(K_{1,l}) = 1$. Since $dk(K_{1,l}) = 2$, the theorem follows. Hence we may assume that $G \neq K_{1,l}$. Let $|V(G)| = n$ and $|E(G)| = e$. We take a vertex v' such that $d_G(v') - i_G(v') - f(H_G(v')) = \Delta^*(G)$. Then $H_G(v')$ is not an empty graph and is still triangle-free. We let $H_G(v') = H$ and $i = i_G(v')$. Then by Theorem 3,

$$k(H) = (e - d_G(v')) - (n - 1 - i) + 2 + f(H) = e - n + 3 - d_G(v') + i + f(H).$$

We let $D_G(v')$ be an acyclic digraph such that the competition graph of $D_G(v')$ is H plus $k(H)$ isolated vertices. Then the CCE-graph of a v'-dominating digraph $\mathcal{D}_G(v')$ obtained from $D_G(v')$ is G together with $k(H)+1$ isolated vertices by Lemma 4. Thus

$$dk(G) \leq k(H) + 1 \leq e - n + 3 - d_G(v') + i + f(H) + 1.$$

Since $d_G(v') - i - f(H) = \Delta^*(G)$, the theorem follows. □

The upper bound given in the above theorem is sharp and it can be achieved by C_4. In fact, if G has no cut vertices, the upper bound in the above theorem can be simplified as follows:

Corollary 9 *If G is connected, triangle-free and has no cut vertices, then*

$$dk(G) \leq k(G) - \Delta(G) + 2$$

where $\Delta(G)$ denotes the maximum degree of G.

Proof. For any vertex v in G, $f(H_G(v)) = 0$ and $i_G(v) = 0$ since $H_G(v) = G - v$ is connected. Hence, $\Delta^*(G) = \max_{v \in V(G)}\{d_G(v)\} = \Delta(G)$ and the corollary immediately follows. □

3 A family of graphs G with $k(G) = 2$ and $dk(G) = 3 = k(G) + 1$

We let $|i-j| \ (mod\ n) = \min\{|i-j|, n-|i-j|\}$. A graph G of n vertices is called a *Harary graph of the form* $H(2m,n)$ for some positive integers m, n with $m \leq n$ if there is a vertex labeling $v_1, \ldots, v_n$ such that v_i is adjacent to v_j if and only if

$$|i-j| \ (mod\ n) \leq m-1.$$

Harary graph $H(6,6)$ is given in Figure 1.

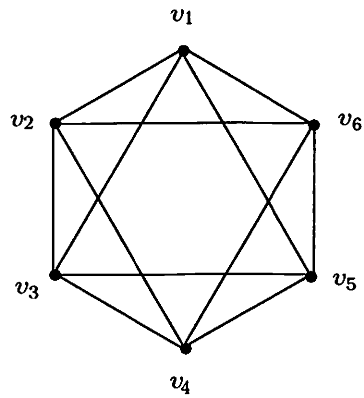

Figure 1: $H(6,6)$.

We denote the edge clique covering number of a graph G by $\theta(G)$.

Theorem 10 *For any positive integer* m *and* n, $n \geq 6$, $\frac{n}{3} + 1 \leq m \leq \frac{n}{2}$, $k(H(2m,n)) = 2$ *and* $dk(H(2m,n)) = 3$.

Proof. It is known that a Harary graph $H(2m,n)$ is a proper circular arc graph. (See Wang [16].) Wang [16] showed that the competition number of a proper circular arc graph is less than or equal to two. Therefore $k(H(2m,n)) \leq 2$. Opsut [11] showed that $k(G) \geq \min_v \theta(N(v))$ for any graph G, where $N(v)$ is the open neighborhood of v. Since $\theta(N(v)) = 2$ for any v in $H(2m,n)$, $k(H(2m,n)) \geq 2$. Therefore, $k(H(2m,n)) = 2$.

Now we suppose that $dk(H(2m,n)) = 2$. We will reach a contradiction. We let D be a minimal digraph among the acyclic digraphs whose CCE graphs are $H(2m,n)$ plus two isolated vertices a_1 and a_2. Since D is acyclic, there exists a labeling $w_0, w_1, \ldots, w_n, w_{n+1}$ of the vertices of D so that if $(w_i, w_j) \in A(D)$, then $i > j$. Since w_0 and w_{n+1} have only incoming arcs and only outgoing arcs, respectively, they are isolated in the CCE graph of D and therefore they should be a_1 and a_2. Since the arcs outgoing from w_1 can go toward only w_0 (i.e. $P_D(w_1) = \{w_0\}$), $S_D(w_0) \supseteq N[w_1]$ in $H(2m, n)$ where $N[w_1]$ is the closed neighborhood of w_1 in $H(2m, n)$. Since $H(2m, n)$ is point-symmetric, we may assume that $v_1 = w_1$. Then there are arcs from $v_1, \ldots, v_m, v_{n-m+2}, \ldots, v_n$ to w_0 in D since they belong to $N[v_1]$ in $H(2m, n)$. Now we consider the following cases:

Case 1 $w_n = v_j$ *for some* $2 \le j \le m-1$.

Since $w_n = v_j$, $S_D(v_j) = \{w_{n+1}\}$. We take vertices v_{j+1} and $v_{n-m+j+1}$. By the definition of $H(2m, n)$, v_{j+1} and $v_{n-m+j+1}$ both are adjacent to v_j. Then since $S_D(v_j) = \{w_{n+1}\}$, $(w_{n+1}, v_{n-m+j+1})$, (w_{n+1}, v_{j+1}), $(w_{n+1}, v_j) \in A(D)$. Since $j \le m-1$, v_{j+1} is adjacent to v_1. Since $1 \le n-[(n-m+j+1)-1] = m-j \le m-2$, $v_{n-m+j+1}$ is adjacent to v_1. Therefore, there are arcs (v_1, w_0), (v_{j+1}, w_0), $(v_{n-m+j+1}, w_0)$ in D since $P_D(v_1) = \{w_0\}$. Hence we can conclude that v_{j+1} and $v_{n-m+j+1}$ are adjacent in $H(2m, n) \cup \{a_1, a_2\}$ which is the CCE of D. However, since $(n-m+j+1)-(j+1) = n-m \ge 2m-m = m$ and $n - [(n-m+j+1)-(j+1)] = m$, v_{j+1} and $v_{n-m+j+1}$ are not adjacent in $H(2m, n)$ and therefore we reach a contradiction.

Case 2 $w_n = v_m$.

We take two vertices v_{m-1} and v_{2m-1}. It is obvious that v_{m-1} and v_{2m-1} both are adjacent to v_m. Therefore, there are arcs (w_{n+1}, v_{m-1}), (w_{n+1}, v_{2m-1}), $(w_{n+1}, v_m) \in A(D)$. Since $1 < n - [(2m-1)-1] \le 3(m-1) - 2m + 2 = m-1$, v_{2m-1} is adjacent v_1. Clearly v_{m-1} is also adjacent to v_1. Therefore, there are arcs (v_{m-1}, w_0), (v_{2m-1}, w_0), (w_{n+1}, v_{m-1}), (w_{n+1}, v_{2m-1}). Hence v_{m-1} and v_{2m-1} are adjacent in $H(2m, n) \cup \{a_1, a_2\}$. However, since $(2m-1)-(m-1) = m$ and $n - [(2m-1)-(m-1)] = n-m \ge m$, they are not adjacent in $H(2m, n)$ and therefore we reach a contradiction.

Case 3 $w_n = v_j$ *for some* $m+1 \le j \le n-m+1$.

In this case, we take two vertices $v_{j-n+2m-1}$ and v_{m+j-1}. Since $1 < (j-n+2m-1)-1 \le (n-m+1)-n+2m-2 = m-1$ and $1 \le j-(j-n+2m-1) = n-2m+1 \le (3m-3)-2m+1 = m-2$, $v_{j-n+2m-1}$ is adjacent to both w_1 and w_n. Since $1 \le n-[(m+j-1)-1] = n-m-j+2 \le (3m-3)-m-(m+1)+2 = m-2$ and $(m+j-1)-j = m-1$, v_{m+j-1} is adjacent to both w_1 and w_n. Therefore, by applying a similar arguments as in the previous cases, we can show that $v_{j-n+2m-1}$ and v_{m+j-1} are adjacent in $H(2m, n) \cup \{a_1, a_2\}$. However, since $(m+j-1)-(j-n+2m-1) = n-m \ge 2m-m = m$ and $n - [(m+j-1)-(j-n+2m-1)] = m$, $v_{j-n+2m-1}$ and v_{m+j-1} are not adjacent in $H(2m, n)$. Hence we reach a contradiction.

Case 4 $w_n = v_{n-m+2}$.

This case is symmetric to Case 2. We take v_{n-m+3} and v_{n-2m+3} and then apply a similar argument.

Case 5 $w_n = v_j$ *for some* $n - m + 3 \leq j \leq n$.

This case is symmetric to Case 1. We take v_{j-1} and $v_{j-1-n+m}$ and then apply a similar argument.

In each case, we obtain a contradiction. Thus $dk(H(2m,n)) \geq 3$. Since $dk(G) \leq k(G) + 1$ for any graph G and $k(H(2m,n)) = 2$, it follows that $dk(H(2m,n)) = 3$. □

4 Further questions

Theorem 5 partially answers the problem of characterizing graphs G with $dk(G) = k(G)+1$. We note that the family of graphs given in Theorem 10 do not satisfy the hypothesis of Theorem 5. To see why, for $n \geq 6$ and $m \leq n$, we take $G = H(2m,n)$ and a vertex v of G. Then $H_G(v) = H(2m,n) - v$. It can be checked that $\theta(N(w)) \geq 2$ for any vertex w of $H_G(v)$ where $N(w)$ is the open neighborhood of w in $H_G(v)$. Therefore $k(H_G(v)) \geq 2 = k(G)$. From the above observation, it seems natural to ask whether or not the converse of Theorem 5 holds. In fact, it will be an interesting problem to characterize graphs G satisfying $k(H_G(v)) < k(G)$ for some vertex v of G.

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