Another equivalent of the graceful tree conjecture

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Abstract

Let $T = (V, E)$ be a tree on $|V| = n$ vertices. T is graceful if there exists a bijection $f : V \to \{0, 1, \ldots, n-1\}$ such that $\{|f(u) - f(v)| \mid uv \in E\} = \{1, 2, \ldots, n-1\}$. If, moreover, T contains a perfect matching M and f can be chosen in such a way that $f(u) + f(v) = n - 1$ for every edge $uv \in M$ (implying that $\{|f(u) - f(v)| \mid uv \in M\} = \{1, 3, \ldots, n-1\}$), then T is called strongly graceful. We show that the well-known conjecture that all trees are graceful is equivalent to the conjecture that all trees containing a perfect matching are strongly graceful. We also give some applications of this result.

Keywords: graceful, tree

AMS Subject Classifications (1991): 05C05, 05C99

1 Introduction

We use BONDY & MURTY [2] for terminology and notation not defined here and consider finite simple graphs only.

Let $T = (V, E)$ be a tree on $|V| = n$ vertices. A bijection $f : V \to \{0, 1, \ldots, n-1\}$ is called a *labeling* of (the vertices of) T. T is *graceful* if f is a *graceful labeling*, that is, if $\{|f(u) - f(v)| \mid uv \in E\} = \{1, 2, \ldots, n-1\}$.

If, moreover, T contains a perfect matching M and f can be chosen in such a way that $f(u) + f(v) = n - 1$ for every edge $uv \in M$ (implying that $\{|f(u) - f(v)| \mid uv \in M\} = \{1, 3, \ldots, n-1\}$), then T and f are called *strongly graceful* (See also [11]). Examples of trees with a (strongly) graceful labeling are given in Figure 1.c (1.a,1.b).

The conjecture that the edges of a complete graph on $2n+1$ vertices can be covered by $2n+1$ copies of an arbitrary tree on $n+1$ vertices is known as the Ringel Conjecture. The stronger Ringel-Kotzig Conjecture asserts that this decomposition can be done in a certain cyclic way. Rosa [12] pointed out that both conjectures would be settled if an arbitrary tree admits a certain valuation. One of these valuations, which he called a β-valuation is nowadays commonly referred to as a graceful labeling, as defined above. This term was introduced by Golomb [9]. We refer to [1], [4], [6], [8], and [9] for more information on the Ringel-Kotzig Conjecture and other surveys of results, conjectures and open problems concerning the labeling of graphs. For later reference we formulate the graceful tree conjecture explicitly.

Conjecture 1
Every tree is graceful.

In Section 2 we show that Conjecture 1 and the following conjecture are equivalent.

Conjecture 2
Every tree containing a perfect matching is strongly graceful.

In Section 3 we consider some transformations of strongly graceful graphs. In Section 4 we give some applications to show that the established equivalence could be useful in trying to prove Conjectures 1 and 2 or special cases of the conjectures. We also give a procedure to generate a large class of (strongly) graceful trees.

2 Conjectures 1 and 2 are equivalent

Before we prove that Conjecture 1 and Conjecture 2 are equivalent, we first need some definitions. The *spiketree of a tree* $T = (\{v_1, \ldots, v_n\}, E)$ is obtained from T by adding n distinct new vertices $u_1, \ldots, u_n$ and the edges $u_1v_1, \ldots, u_nv_n$. A *spiketree* is a tree which is a spiketree of some tree. The *contree of a tree* T with a perfect matching M is obtained from T by contracting the edges of M.

Theorem 3
Conjecture 1 and Conjecture 2 are equivalent.

Proof First assume every tree is graceful. Let T be a tree on n vertices (n even) containing a perfect matching M, and let L be the contree of T on $\frac{n}{2}$ vertices. By assumption L is graceful, hence there exists a bijection $f : V(L) \to \{0, 1, \ldots, |V(L)| - 1\}$ such that $\{|f(u) - f(v)| \mid uv \in E(L)\} = \{1, 2, \ldots, |V(L)| - 1\}$. Assign to each vertex $x \in V(L)$ the label $\lambda(x) = 2f(x)$. Then $\{|\lambda(u) - \lambda(v)| \mid uv \in E(L)\} = \{2, 4, \ldots, 2|V(L)| - 2\} = \{2, 4, \ldots, n - 2\}$. Every vertex $x \in V(L)$ corresponds to an edge $pq \in M$ of T. We will assign labels $\lambda(x)$ and $n - 1 - \lambda(x)$ to p and q in such a way that $\lambda : V(T) \to \{0, 1, \ldots, |V(T)| - 1\}$ is a strongly graceful labeling. Let p_iq_i be the edge of M in T corresponding to $x_i \in V(L)$ (i=1,...,$|V(L)|$) chosen in such a way that every edge x_ix_j in L corresponds to an edge p_ip_j or to an edge q_iq_j in T. This can be done by starting at an end vertex x_k of L and choosing either p_k or q_k corresponding to x_k. For all other vertices and edges of L and T the correspondence is easily determined using the connectedness of L and T. Define $\lambda(p_i) = \lambda(x_i)$ and $\lambda(q_i) = n - 1 - \lambda(x_i)$. Then clearly $\lambda(p_i) + \lambda(q_i) = n - 1$ for every $p_iq_i \in M$. Hence the edges of M in T yield all odd (absolute) differences $1, 3, \ldots, n - 1$. Moreover, by the choice of p_i, q_i, all other edges p_ip_j (or q_iq_j) of T correspond to edges x_ix_j of L, so that $|\lambda(p_i) - \lambda(p_j)| = |\lambda(x_i) - \lambda(x_j)|$ (or $|\lambda(q_i) - \lambda(q_j)| = |n - 1 - \lambda(x_i) - (n - 1 - \lambda(x_j))| = |\lambda(x_i) - \lambda(x_j)|$). Hence the edges of $E(T) \setminus M$ yield all even (absolute) differences 2,...,$n-2$. This proves that $\lambda : |V(T)| \to \{0, 1, \ldots, n - 1\}$ is a strongly graceful labeling of T.

Conversely, assume every tree containing a perfect matching is strongly graceful. Let $T = (\{v_1, \ldots, v_n\}, E)$ be a tree on n vertices and let $G = (\{u_1, \ldots, u_n, v_1, \ldots, v_n\}, E \cup \{u_1v_1, \ldots, u_nv_n\})$ be the spiketree of T. By assumption G is strongly graceful, hence there exists a bijection $f : V(G) \to \{0, 1, \ldots, 2n - 1\}$ such that $f(u_i) + f(v_i) = 2n - 1$ for i=1,2,...,n. Since the edges of E yield all even differences $\{2, \ldots, 2n - 2\}$ in G, without loss of generality we may assume that all $f(v_i)$ are even and all $f(u_i)$ are odd (otherwise we use labels $2n - 1 - f(x)$ for all $x \in V(G)$). Define $\lambda(v_i) = \frac{1}{2}f(v_i)$ (i=1,...,n). Then $\{\lambda(v_i) \mid i = 1, \ldots, n\} = \{0, 1, \ldots, n - 1\}$ and the edges of E yield the differences $\{1, 2, \ldots, n-1\}$. Clearly $\lambda : V(T) \to \{0, 1, \ldots, n - 1\}$ is a graceful labeling of T. ∎

To prove Conjecture 2 it would be sufficient to prove that every spiketree is strongly graceful. This is shown in the next section. Since there are far less spiketrees (and trees with a perfect matching) on n vertices than trees, one might hope that Conjecture 2 is easier to prove than Conjecture 1. On the other hand, of course, the conclusion of Conjecture 2 is much stronger, and might therefore be more difficult to reach.

3 Transformations of strongly graceful trees

We consider some transformations of strongly graceful trees that might be useful in a possible proof of Conjecture 2, and its applications to generate large classes of (strongly) graceful graphs. We first need one more definition.

If f is a labeling of the tree T on n vertices, then the *complementary labeling* g of T is defined by: $g(v) = n - 1 - f(v)$ for all $v \in V(T)$.

Lemma 4
Let T be a tree on n vertices containing a perfect matching M and having a strongly graceful labeling f. Let uv and xy be edges of M and let $ux \in E(T)$. Then the following two types of transformations of T give another strongly graceful tree.

Type 1. Delete ux and add vy.

Type 2. Delete ux and add uy or vx.

Proof Consider the labels of u, v, x, and y. By the definition of f, $f(u) + f(v) = n - 1 = f(x) + f(y)$. Hence $|f(v) - f(y)| = |n - 1 - f(u) - (n - 1 - f(x))| = |f(u) - f(x)|$. So after a transformation of Type 1 the resulting tree has exactly the same strongly graceful labeling.

In case of a transformation of Type 2, let T_1 and T_2 denote the trees of $T - ux$ containing the vertices u and x, respectively. Maintain the labels of T_1 and replace the labels of T_2 by the complementary labels. It is easy to check that this gives a strongly graceful labeling for the tree $(T - ux) + uy$. Similarly, it is easy to give a strongly graceful labeling for the tree $(T - ux) + vx$. ■

Lemma 4 can be used to construct classes of graceful trees, starting from a strongly graceful labeling of a tree containing a perfect matching. Since the transformations can be applied in the "inverse" direction as well, any graph in such a generated class could be taken as a starting point. All graphs in such a class have the same contree.

Lemma 5
Let T be a tree containing a perfect matching, and let T^c be the contree of T. Then T is strongly graceful if and only if the spiketree T^* with contree T^c is strongly graceful.

Proof Let T be a strongly graceful tree containing a perfect matching, and let T^c be the contree of T. If T is a spiketree, there is nothing to prove. Suppose T is not a spiketree. Then T contains an edge $uv \in M$ such that the degree of both u and v is at least 2. By repeated transformations of Type 2 all edges incident with u can be made incident with v, yielding a strongly graceful tree T' in which u has degree 1, and with the same contree as T. Repeating this procedure as long as there are edges of M which are not incident with an end vertex, we obtain a strongly graceful spiketree T^* with the same contree as T.

For the proof of the converse, note that we can use the "inverse" transformations to find a strongly graceful labeling of T starting from a strongly graceful labeling of T^*. ■

Lemma 5 shows that to prove Conjecture 2 it would be sufficient to prove that every spiketree is strongly graceful. However, since a strongly graceful labeling of a spiketree immediately yields a graceful labeling of its contree (as in the proof of Theorem 3), this hardly improves the situation. As remarked by Van den Heuvel [10], interesting trees with a perfect matching might be those trees for which all nonmatching edges induce stars. It is easily seen that any spiketree can be transformed into such a type of tree using transformations of Type 1 or 2.

Attempts to prove Conjecture 1 for trees or Conjecture 2 for trees with a perfect matching, in any particular form, by induction, are frustrated essentially by the fact that the vertex label 0 cannot be assumed to occur at an arbitrary vertex. Trees in which the label 0 can be assigned to each vertex in some graceful labeling are called *0-rotatable* and were studied by Chung & Hwang [7].

If we can keep control over the location of the label 0, some other transformations, involving the growing of trees, can be mentioned. Again let M be a perfect matching of a tree T. An edge $uv \in M$ is a *pendent edge* of T if at most one of the vertices of $V \setminus \{u, v\}$ is adjacent to u or v. (Note that u or v has degree one.)

Suppose T contains a pendent edge uv and suppose $vw \in E$ for a vertex $x \in V \setminus \{u\}$ with $xy \in M$.

Lemma 6
If xy is a pendent edge of $T' = T - \{u,v\}$ and T' admits a strongly graceful labeling λ such that $\{\lambda(x), \lambda(y)\} = \{0, n-3\}$, then T admits a strongly graceful labeling μ such that $\{\mu(u), \mu(v)\} = \{0, n-1\}$.

Proof Define $\mu(w) = \lambda(w)+1$ for all $w \in V(T')$, and choose $\{\mu(u), \mu(v)\} = \{0, n-1\}$ in such a way that the edge vx yields the difference $n-2$. (If $\mu(x) = 1$, take $\mu(v) = n-1$; if $\mu(x) = n-2$, take $\mu(v) = 0$.) ■

Lemma 7
If there is a pendent edge $st \in M \setminus \{uv\}$ such that $tx \in E$, and $T' = T - \{u, v\}$ admits a strongly graceful labeling λ such that $\{\lambda(s), \lambda(t)\} = \{0, n-3\}$, then T admits a strongly graceful labeling μ such that $\{\mu(u), \mu(v)\} = \{0, n-1\}$.

Proof Without loss of generality assume $\lambda(s) = 0$ and $\lambda(t) = n-3$. First observe that $\lambda(x) = 1$ and $\lambda(y) = n-4$ (the difference $n-4$ can only be yielded by edges with vertex label pairs $(0, n-4)$ or $(1, n-3)$). Define $\mu(w) = \lambda(w)+2$ for all $w \in V(T')$ with an even label $\lambda(w)$, $\mu(w) = \lambda(w)$ for all other $w \in V(T')$, $\mu(u) = 0$, and $\mu(v) = n-1$. Since all even differences in T' are yielded by edges with odd label pairs, it is clear that in T all even differences $\{2, \ldots, n-2\}$ are yielded. The odd difference $n-1$ is yielded by uv. It remains to show that all other odd differences are yielded by the edges of $M \setminus \{uv\}$.

If $n = 4k+2$ for some positive integer k, then the edges of $M \setminus \{uv\}$ in T' yield the label pairs $\{(0, 4k-1), (1, 4k-2), \ldots, (2k-2, 2k+1), (2k-1, 2k)\}$. By adding 2 to all even labels, in T the same edges yield the pairs $\{(2, 4k-1), (1, 4k), \ldots, (2k, 2k+1), (2k-1, 2k+2)\}$ corresponding to differences $\{4k-3, 4k-1, \ldots, 1, 3\} = \{1, 3, \ldots, n-3\}$.

If $n = 4k$, then similarly for T' we obtain the pairs $\{(0, 4k-3), (1, 4k-4), \ldots, (2k-4, 2k+1), (2k-3, 2k), (2k-2, 2k-1)\}$, and for T the pairs $\{(2, 4k-3), (1, 4k-2), \ldots, (2k-2, 2k+1), (2k-3, 2k+2), (2k, 2k-1)\}$ corresponding to differences $\{4k-5, 4k-3, \ldots, 3, 5, 1\} = \{1, 3, \ldots, n-3\}$.

This completes the proof. ■

Note that Lemma 6 and Lemma 7 can be applied whenever a specific suitable labeling of a smaller tree is known. We will come back to this in the next section.

4 Applications

Theorem 3, Lemma 6, and Lemma 7 can be used to "grow" (strongly) graceful trees, what BLOOM [1] calls "horticulture", whenever a specific suitable labeling of a smaller tree is known.

The transformations of Type 1 and 2 give the possibility of changing the structure of a strongly graceful tree. By "spiking" (putting a pendent edge at every vertex of) any graceful tree, as considered in the proof of Theorem 3, another (strongly) graceful tree, on twice the number of vertices, is obtained. This spiking procedure can be continued leading to a growth of each vertex of the original tree into a subtree on 2,4,8,16, etc. vertices.

In the next examples Lemma 6 and 7 can be applied to yield a strongly graceful labeling.

It is not difficult to see that every spiketree (or in fact every tree containing a perfect matching) can be reduced to a path on two vertices by successively deleting the vertices of a pendent edge at each stage. This is not enough for our purpose.

We say that two vertex disjoint pendent edges uv and xy of T with $d(u) = d(x) = 1$ are *close* if either vy is an edge of T or v and y have a common neighbor in T. We say that T is *linear* if we can reduce T to a path on two vertices by successively deleting the vertices of a pendent edge at each stage in such a way that every two successive pendent edges in the series of deleted pendent edges are close.

Theorem 8
Every linear spiketree is (strongly) graceful.

Proof It is easy to prove this result by induction on the number of vertices, using Lemma 6 and 7. We leave the details to the reader. ∎

A *caterpillar* is a tree for which the deletion of all vertices of degree one results in a path.

The following result is also easy to prove. We omit the proof.

Lemma 9

(a) The contree of a linear spiketree is a caterpillar.

(b) The spiketree of a caterpillar is linear.

Corollary 10 ([6],[12])
All caterpillars are graceful.

Proof The result is a direct consequence of Lemma 9 (b), Theorem 8, and the construction used in the second part of the proof of Theorem 3. ■

Corollary 11
Every tree containing a perfect matching and having a caterpillar as its contree is (strongly) graceful.

Proof The result is a direct consequence of Lemma 5, Lemma 9 (b), Theorem 8, and the transformation method of Lemma 4 (resulting in a spiketree of a caterpillar). ■

Note that we can use the trees satisfying the hypothesis of Corollary 11 again to generate new (strongly) graceful trees with the former trees as their contrees, and so on, by spiking. This procedure generates (strongly) graceful "long-legged caterpillars".

As an example we consider the tree T_1 of Figure 1.a, in which a perfect matching is indicated by the heavy lines, its contree T_2 in Figure 1.b together with a perfect matching, and the contree T_3 of T_2 in Figure 1.c. A graceful labeling of the caterpillar T_3 is indicated in Figure 1.c; the strongly graceful labelings of T_2 and T_1 obtained from this labeling are indicated in Figures 1.b and 1.a, respectively.

The spiking procedure can be generalized. Instead of putting a pendent edge at every vertex, we may put a $K_{1,p}$ at every vertex for some integer p. By doing this the number of vertices is multiplied by $p+1$. We illustrate how to adjust the labeling by the following example. Suppose we start with a path $P_3 = uvw$ with graceful labeling f given by $f(u) = 0$, $f(v) = 2$, and $f(w) = 1$. Suppose we carry out a generalized spiking by putting a $K_{1,3}$ at every vertex. We obtain a tree with 12 vertices, in which u has new neighbors u_1, u_2, u_3, v has new neighbors v_1, v_2, v_3, and w has new neighbors w_1, w_2, w_3. We replace the labels at u, v, w by four times their value, and label the other vertices in such a way with the remaining labels from $0, 1, \ldots, 11$ that in all copies of the added $K_{1,3}$'s the sums of the labels of the three adjacent pairs are 9, 10, and 11. This leads to a graceful labeling g of the new tree given by $g(u) = 0, g(u_1) = 9, g(u_2) = 10, g(u_3) = 11, g(v) = 8, g(v_1) = 1, g(v_2) = 2, g(v_3) = 3, g(w) = 4, g(w_1) = 5, g(w_2) = 6, g(w_3) = 7$. We leave it to the reader to check the generality of this procedure.

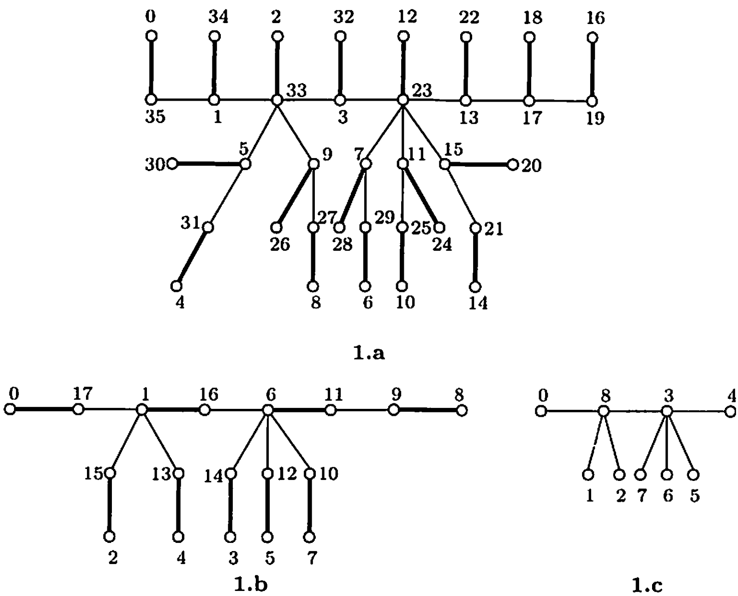

Figure 1: The trees T_1 (1.a), T_2 (1.b), and T_3 (1.c)

As our final application we refer to the interesting problem posed in CAHIT [4] to prove Conjecture 1 for trees with a fixed diameter, with diameter 5 as the first open case. We cannot prove this special case by our techniques, but we can generate a large infinite class of graceful trees with diameter 5. Take as a starting point any tree with diameter 3. This tree is a caterpillar and hence admits a graceful labeling. Now carry out a generalized spiking by putting a $K_{1,p}$ at every vertex for some integer p. The new tree is graceful and has diameter 5.

References

[1] G. S. Bloom, *A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful.* Topics in Graph Theory. Annals of the New York Acad. Sci. **328** (1979) 32–51.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications.* Macmillan, London and Elsevier, New York (1976).

[3] I. Cahit, *Cordial graphs: a weaker version of graceful and harmonious*

graphs. Ars Combinatoria **23** (1987) 201–208.

[4] I. Cahit, *Status of graceful tree conjecture in 1989*, in: Topics in Combinatorics and Graph Theory (R. Bodendiek and R. Henn, Eds.). Physica-Verlag, Heidelberg (1990) 175–184.

[5] I. Cahit, *On graceful trees*. Bulletin of the ICA **12** (1994) 15–18.

[6] I. Cahit and R. Cahit, *On the graceful numbering of spanning trees*. Information Processing Letters **3** (1974/75) 115–118.

[7] F. R. K. Chung and F. K. Hwang, *Rotatable graceful graphs*. Ars Combinatoria **11** (1981) 239–250.

[8] J. A. Gallian, *A survey: recent results, conjectures, and open problems in labeling graphs*. J. Graph Theory **13** (1984) 491–504.

[9] S. W. Golomb, *How to number a graph*. Graph Theory and Computing. Academic Press, New York (1972) 23–37.

[10] J. van den Heuvel, private communication.

[11] M. Maheo, *Strongly graceful graphs*. Discrete Math. **29** (1980) 39–46.

[12] A. Rosa, *Theory of Graphs* (International Symposium, Rome, 1966). Gordon and Breach, New York (1967) 339–345.