the Extremal Question for Cycles with Chords

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ABSTRACT. Minimum degree two implies the existance of a cycle. Minimum degree 3 implies the existence of a cycle with a chord. We investigate minimum degree conditions to force the existence of a cycle with k chords.

1 Introduction

The first theorem of nearly every graph theory course is the statement that the sum of the degrees is twice the number of edges. In many of those courses the second theorem is the extremal theorem for cycles:

Theorem 1. *If G is a graph with $n \geq 2$ vertices and G has either*

i) *minimum degree ≥ 2 or*

ii) *at least n edges*

then G contains a cycle.

Posa [4] proposed, and Czipszer [3] published a solution of, a variation of this problem as an exercise in a Hungarian journal.

Theorem 2. *If G is a graph with $n \geq 4$ vertices and G has either*

i) *minimum degree ≥ 3 or*

ii) *at least $2n - 3$ edges*

then G contains a cycle with a chord.

Our purpose here is to investigate the extremal question for cycles with k chords. For fixed k we find best possible minimum degree conditions which forces the existence of a cycle with k chords. Additionally, we consider the edge version of the extremal question for cycles with chords, and we investigate the effect of connectivity assumptions.

Theorem 3. *Let G be a graph with minimum degree $\delta \geq 2$. Then*

a) *G has a cycle with at least $\left\lceil \frac{\delta^2-2\delta}{2} \right\rceil$ chords.*

b) *If G contains no 3-cycle and no 5-cycle, then G has a cycle with at least $\delta^2 - 2\delta$ chords.*

Proof: If $P\colon V_1, V_2, \ldots, V_K$ is a longest path in G, note that all neighbors of V_1, are vertices of P. Denote by $\ell(P)$ the largest index of a neighbor of V_1 and assume that among all longest paths, P has been chosen so that $\ell = \ell(P)$ is maximum. The vertices $V_1, V_2, \ldots, V_\ell$ form a cycle C, and it will be shown that C has the required number of chords. To see this, let V_i be a neighbor of V_1 and note that the path $Q\colon V_{i-1}, V_{i-2}, \ldots V_1, V_i, V_{i+1}, \ldots V_K$ is a longest path since it contains all vertices of P. Hence $\ell(Q) \leq \ell$ and it follows that all neighbors of V_{i-1} are among $V_1, V_2, \ldots, V_\ell$, that is, all neighbors of V_{i-1} are on C. Since V_1 has at least δ neighbors, there are at least δ vertices on C with all their neighbors on C. Each of these δ vertices is incident with 2 edges of C, and the remaining $\delta - 2$ edges must be chords of C. Allowing for the possibility that these chords will now be counted twice, once at each end, we conclude that C has at least $\left\lceil \frac{\delta^2-2\delta}{2} \right\rceil$ chords.

If G contains no 3-cycle and no 5-cycle, the above scheme for counting chords encounters no duplications. For if V_i and V_j are neighbors of V_1 with $i < j$, then the lack of triangles means $j - 1 \neq i$ and $i \neq 3$. Hence, if V_{i-1} were adjacent to V_{j-1}, the vertices V_1, V_i, V_{i-1}, V_{j-1}, and V_j would be distinct and would induce a 5-cycle. It follows that the cycle C has at least $\delta(\delta - 2)$ chords. □

Part a) of Theorem 3 is best possible in the loose sense that a complete graph on δ vertices has minimum degree $\delta - 1$ and contains no cycle with as many as $\left\lceil \frac{\delta^2-2\delta}{2} \right\rceil$ chords. Part b) is sharp in a stronger sense. The complete bipartite graph $K_{\delta,\delta}$, has minimum degree δ and has no cycle with more than $\delta^2 - 2\delta$ chords.

Note that when $\delta = 3$, Theorem 3 improves Theorem 2 since it guarantees a cycle with 2 chords. A graph in which every block is K_4 shows that this is best possible. Later we will see that a connectivity assumption changes the situation.

In order to focus on the desired number of chords, we reformulate Theorem 3, expressing the necessary minimum degree in terms of the guaranteed number of chords.

Corollary 4. *If k is a nonnegative integer and G is a graph with minimum degree δ, then*

a) *If* $\delta \geq 1 + \sqrt{2k+1}$ *then G has a cycle with at least k chords.*

b) *If G contains no 3-cycle and no 5-cycle and* $\delta \geq 1 + \sqrt{k+1}$*, then G has a cycle with at least k chords.*

Proof: a) $\frac{\delta^2 - 2\delta}{2} \geq k$ b) $\delta^2 - 2\delta \geq k$

We now add a connectivity assumption to the interesting case of $\delta = 3$.

Theorem 5. *Let G be 2-connected with* $n \geq 5$ *vertices and* $\delta \geq 3$*. Then G has a cycle with at least 3 chords.*

Proof: As in the proof of Theorem 1, we let $P\colon V_1, V_2, \ldots, V_K$ be a longest path maximizing ℓ. Consider these cases:

a) If $\ell = 4$, then V_1 must be adjacent to all of V_2, V_3, and V_4. Hence, since V_3 and V_4 are neighbors of V_1, both V_2 and V_3 have all their neighbors on C. Hence V_1, V_2, V_3, and V_4 induce K_4 and, since $n \geq 5$, V_4 is a cut vertex, ruled out by hypothesis.

b) Now, if $\ell = 5$, then, if $n = 5$, G has at least 8 edges, so C has at least 3 chords. If $n > 5$ and V_1 has degree larger than 3, then every vertex of C has all its neighbors on C, and so C has at least 3 chords. If V_1 has degree exactly 3, then V_1 is adjacent to V_2, V_5, and either V_3, or V_4. If V_1 is adjacent to V_4, then V_3 has all neighbors on C, so V_3 is adjacent to V_5 and $V_2, V_1, V_4, V_3, V_5, V_6, \ldots, V_K$ is a longest path. It follows that V_2 has all its neighbors on C. Since V_1 is adjacent to V_2, V_4, and V_5, it follows that V_1, V_3, and V_4 have all neighbors on C. Hence V_5 is a cut vertex. If the neighbors of V_1 are V_2, V_3, and V_5, then immediately we know that V_1, V_2, and V_4 have all neighbors on C. But this forces V_4 to be adjacent to V_2, and the path $V_3, V_1, V_2, V_4, V_5, V_6, \ldots, V_K$ is a longest path. Hence V_3 has all its neighbors on C and again V_5 is a cut vertex.

c) Finally if $\ell \geq 6$, we will show C has at least 3 chords. Suppose V_1 is adjacent to V_j in addition to V_2 and V_ℓ. If $j \neq \ell - 1$ then, since both $V_{\ell-1}$ and V_{j-1} have all neighbors on C, the chord $V_1 - V_j$ along with the chords incident with $V_{\ell-1}$ and V_{j-1} total 3 unless $V_{\ell-1}$ is adjacent to V_{j-1}. If V_{j-1} is adjacent to $V_{\ell-1}$, then the path $V_{\ell-2}, V_{\ell-3}, \ldots V_j, V_1, V_2, \ldots V_{j-1}, V_{\ell-1}, V_\ell, \ldots V_K$ is a longest path and it follows that $V_{\ell-2}$ has all neighbors on C. This produces a third chord. If $j = \ell - 1$, we have the chord $V_1 - V_j$ and another chord incident with V_{j-1}, say $V_{j-1} - V_i$ with $i < j - 1$. In this case $V_{i+1}, \ldots V_{j-1}, V_i, V_{i-1}, \ldots V_1, V_j, V_\ell, \ldots V_K$ is a longest path and there is a third chord incident with V_{i+1}. If in this case $j = \ell - 1$ we have V_{j-1} incident with V_ℓ, then the path $V_{j-2}, V_{j-3}, \ldots V_1, V_j, V_{j-1}, V_\ell, \ldots V_K$ is a longest path and again there must be a third chord. □

The graph $K_{3,n-3}$ shows that the hypotheses of Theorem 5 do not force more than 3 chords.

Theorem 6. *If G has n vertices, $n \geq 4$ and at least $2n-2$ edges, then G contains a cycle with at least 2 chords.*

Proof: For $n = 4$ the assertion is true, the only such graph being K_4. Assume the assertion for graphs with fewer than n vertices and consider G with n vertices and at least $2n-2$ edges. If G has minimum degree at least 3, G has a cycle with at least 2 chords by Theorem 5. If not, deletion of a vertex V of degree 2 or less leaves a subgraph to which the induction assumption applies. That subgraph, and hence G, has a cycle with at least 2 chords. □

Theorem 7. *If G is 2-connected with n vertices, $n \geq 5$, and at least $2n-2$ edges, then G contains a cycle with at least 3 chords.*

Proof: If $n = 5$, G is missing at most 2 edges from K_5, and it is easily vertified that G has a 5-cycle with 3 chords. Making the appropriate induction assumption, we consider the two cases. If G has minimum degree at least 3, the result follows from Theorem 5. If not, we delete a vertex V of degree less than 3. If $G - V$ is 2-connected, then the induction assumption can be invoked. If not, then there is a vertex W such that $G - \{V, W\}$ is not connected. Say C_1 is a component and C_2 is the union of the remaining components of $G - \{V, W\}$. Note that V must have a neighbor V_1 in C and a neighbor V_2 in C_2 or else W would be a cut vertex of G. Now consider the graph H obtained from G by removing V and introducing an edge joining V_1 and V_2. G has $n-1$ vertices, at least $2n-3$ edges, and is 2-connected, so, by the induction hypothesis, H has a cycle C with at least 3 chords. The edge V_1V_2 is not a chord of that cycle since the removal of edge V_1V_2 leaves a subgraph of H in which V_1 and V_2 are in distinct blocks. If C does not contain the edge V_1V_2, then C is a cycle of G with 3 chords. If V_1V_2 is an edge of C, replace it by the path $V_1\ V\ V_2$ to obtain a cycle of G with at least 3 chords. □

Some further questions we find interesting are discussed in [1] and [2]. Among these we mention especially the following, due to Peter Hamburger. What minimum degree forces the existence of a cycle with as many chords as vertices?

References

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