A Note on Hamiltonian Cycles in $K_{1,r}$–Free Graphs

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ABSTRACT. A graph is called $K_{1,r}$–free if it does not contain $K_{1,r}$ as an induced subgraph. In this paper we generalize a theorem of Markus for Hamiltonicity of 2–connected $K_{1,r}$–free ($r \geq 5$) graphs and present a sufficient condition for 1–tough $K_{1,r}$–free ($r \geq 4$) graphs to be Hamiltonian.

1 Introduction

We consider only finite undirected graphs without loops and multiple edges. For notation and terminology not defined here we refer to [2]. We use n to denote the order of a graph. A graph G is called $K_{1,r}$–free if it does not contain $K_{1,r}$ as an induced subgraph. A graph G is called 1–tough if $t(G-S) \leq |S|$ for every subset S of $V(G)$ with $t(G-S) > 1$, where $t(G-S)$ denotes the number of components of $G-S$. We use $\sigma_k(G)$ to denote the minimum value of the degree sum of any k pairwise nonadjacent vertices if $k \leq \alpha$; if $k > \alpha$, we set $\sigma_k(G) = k(n-1)$. Here α denotes the independence number of G. Let A, B be two disjoint subsets of $V(G)$, we define

$$E(A, B) = \{ab : a \in A, b \in B; ab \in E(G)\}.$$

The following two theorems are due to Markus.

Theorem 1 *[7] Let G be a 2–connected $K_{1,4}$–free graph of order n with $\delta \geq (n+2)/3$. Then G is Hamiltonian.*

Theorem 2 *[7] Let G be a 2–connected $K_{1,r}$–free ($r \geq 5$) graph of order n with $\delta \geq (n+r-3)/3$. Then G is Hamiltonian unless $G - E(G-T)$ is $K_{r-1,r-2}$, where T is any largest independent set of G.*

The aim of this paper is to present the following results, the first of which is a generalization of Theorem 2.

Theorem 3 *Let G be a 2-connected $K_{1,r}$-free ($r \geq 5$) graph of order n with $\sigma_3 \geq n+r-3$. Then G is Hamiltonian unless $G - E(G-T)$ is $K_{r-1,r-2}$, where T is any largest independent set of G.*

Theorem 4 *Let G be a 1-tough $K_{1,r}$-free ($r \geq 5$) graph of order n with $\sigma_3 \geq n+r-5$. Then G is Hamiltonian.*

Notice that $K_{1,4}$-free graphs are also $K_{1,5}$-free graphs, so we have the following corollary.

Corollary 1 *Let G be a 1-tough $K_{1,4}$-free graph of order n with $\sigma_3 \geq n$. Then G is Hamiltonian.*

Remark 1. Theorem 21 in [3] implies that every 2-connected $K_{1,r}$-free graph of order $n \geq 9r-13$ and $\sigma_3 \geq n+2$ is Hamiltonian. So Theorem 21 in [3] is stronger than Theorem 3 if the order n of the graph is at least $9r-13$. Since any 1-tough graph is 2-connected, Theorem 21 in [3] is also stronger than Theorem 4 if $r \geq 7$ and the order n of the graph is at least $9r-13$.

Remark 2. For $n = 3s+1 \geq 10$, construct the graph H from $3K_s + K_1$ by choosing one vertex from each copy of K_s, say u, v and w, and adding the edges uv, vw and wu. Then the graph H is a 1-tough $K_{1,r}$-free (where $r = 4$ and 5) graph of order n with $\sigma_3(H) \geq n-1$, but H is not Hamiltonian. So Theorem 4 for $r=5$ and Corollary 1 are best possible.

2 Lemmas

Lemma 1 *(Theorem 10 in [1]) Let G be a 2-connected graph on n vertices such that $\sigma_3 \geq s \geq n+2$. Then G contains a cycle of length at least $min\{n, n+s/3-\alpha\}$.*

Lemma 2 *[6] Let G be a balanced bipartite graph of order n with $\delta \geq (n+2)/4$. Then G is Hamiltonian.*

Lemma 3 *[5] Let G be a 1-tough graph of order n with $\sigma_3 \geq s \geq n$. Let C be a longest cycle in G . Then $|V(C)| \geq min\ \{n, n+s/3-\alpha+1\}$.*

Lemma 4 *[4] Let G be graph of order at least 3 with $\alpha \leq k$, where k is the connectivity of G. Then G is Hamiltonian.*

3 Proofs

Proof of Theorem 3. Suppose G is a graph satisfying the conditions in Theorem 3 and it is nonhamiltonian. By Lemma 4, we can assume that $\alpha \geq 3$. By Lemma 1, we have

$n + (n + r - 3)/3 - \alpha \leq n - 1$, so that

$\alpha \geq (n + r)/3$.

Let $T = \{ v_1, v_2, ..., v_\alpha \}$ be any largest independent set in G. Without loss of generality, we assume that $d(v_1) \leq d(v_2) \leq ... \leq d(v_\alpha)$. By assumption, we have

$d(v_i) + d(v_{i+1}) + d(v_{i+2}) \geq n + r - 3$.

where $1 \leq i \leq \alpha$, $v_{\alpha+1} = v_1$ and $v_{\alpha+2} = v_2$.

Since G is $K_{1,r}$–free, we can obtain the following inequality:

$$(\star) \qquad \alpha(n + r - 3) \leq 3(d(v_1) + d(v_2) + ... + d(v_\alpha)) = 3|E(T, G - T)| \leq 3(r - 1)(n - \alpha),$$

Hence, using $\alpha \geq (n + r)/3$, we have $2r - 3 \leq n \leq 2r$.

If $n = 2r - 1$ or $2r - 2$, then by $\alpha \geq (n + r)/3$, we have $\alpha \geq r$. Using inequality $(\star)$, we can easily derive a contradition. So we may assume that $n = 2r$ or $2r - 3$.

If $n = 2r$, then $\alpha \geq (n + r)/3 = r$ and

$$r(3r - 3) \leq \alpha(n + r - 3) \leq 3(d(v_1) + d(v_2) + ... + d(v_\alpha)) = 3|E(T, G - T)| \leq 3(r - 1)(n - \alpha) \leq r(3r - 3)$$

Hence $\alpha = r$; $d(v_i) = r - 1$, $1 \leq i \leq \alpha$; $d_T(u) = r - 1$, for each $u \in G - T$. Let H be the graph $G - E(G - T)$. Then H is a balanced bipartite graph with $\delta = r - 1 \geq (n + 2)/4$. So by Lemma 2, H is Hamiltonian and so is G, a contradiction.

If $n = 2r - 3$, then $\alpha \geq (n + r)/3 = r - 1$ and

$3(r-1)(r-2) \le \alpha(n+r-3) \le 3(d(v_1)+d(v_2)+...+d(v_\alpha)) = 3|E(T,G-T)|$
$\le 3(r-1)(n-\alpha) \le 3(r-1)(r-2)$.

Hence $\alpha = r-1$; $d(v_i) = r-2$, $1 \le i \le \alpha$; $d_T(u) = r-1$ for each $u \in G-T$. Thus $G - E(G-T)$ is isomorphic to $K_{r-1,r-2}$.

Proof of Theorem 4. Suppose G is a graph satisfying the conditions in Theorem 4 and it is nonhamiltonian. Since any 1–tough graph is 2–connected, by Lemma 4, we can assume that $\alpha \ge 3$. By Lemma 3, we have

$n + (n+r-5)/3 - \alpha + 1 \le n-1$, so that

$\alpha \ge (n+r+1)/3$.

Let $T = \{v_1, v_2, ..., v_\alpha\}$ be any largest independent set in G. Without loss of generality, we can assume that $d(v_1) \le d(v_2) \le ... \le d(v_\alpha)$. Then by assumption, we have

$d(v_i) + d(v_{i+1}) + d(v_{i+2}) \ge n+r-5$.

where $1 \le i \le \alpha$, $v_{\alpha+1} = v_1$ and $v_{\alpha+2} = v_2$.

Since G is $K_{1,r}$–free, we can obtain the following inequality:

$\alpha(n+r-5) \le 3(d(v_1) + d(v_2) + ... + d(v_\alpha)) = 3|E(T,G-T)| \le 3(n-\alpha)(r-1)$,

Hence, using $\alpha \ge (n+r+1)/3$, we have $2r-4 \le n \le 2r+2$.

Since G is 1–tough, $\alpha \le \lfloor n/2 \rfloor$. By $\alpha \ge (n+r+1)/3$, we have $n = 2r+2$. Thus $\alpha \ge (n+r+1)/3 = r+1$ and

$3(r-1)(r+1) \le \alpha(n+r-5) \le 3(d(v_1) + d(v_2) + ... + d(v_\alpha)) = 3|E(T,G-T)| \le 3(n-\alpha)(r-1) \le 3(r-1)(r+1)$.

Hence $\alpha = r+1$; $d(v_i) = (n+r-5)/3 = r-1$, $1 \le i \le \alpha$; $d_T(u) = r-1$ for each $u \in G-T$. Let H be the graph $G - E(G-T)$. Then H is a balanced bipartite graph with $\delta = r-1 \ge (n+2)/4$. Thus by Lemma 2, H is Hamiltonian and so is G, a contradiction.

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