ON THE BIPARTITION NUMBERS OF RANDOM TREES, II

A. MEIR

York University
N. York, Ontario
M3J 1P3

J.W. Moon

University of Alberta
Edmonton, Alberta
T6G 2G1

Abstract. Let T_n denote any rooted tree with n nodes and let $p = p(T_n)$ and $q = q(T_n)$ denote the number of nodes at even and odd distance, respectively, from the root. We investigate the limiting distribution, expected value, and variance of the numbers $D(T_n) = |p - q|$ when the trees T_n belong to certain simply generated families of trees.

1. Introduction

Every non-trivial tree T_n is a bipartite graph, that is, its nodes can be partitioned into two subsets P and Q such that no two nodes of the same subset are joined by an edge. If T_n is a rooted tree, then we may take P and Q to consist of the nodes at even and odd distance from the root. We call $p = |P|$ and $q = |Q|$ the *bipartition numbers* of T_n and we let $D = D(T_n) = |p - q|$. Since $p + q = n$, the total number of nodes in the tree T_n, it follows that $D^2 = 2p^2 + 2q^2 - n^2$. This relation was exploited in [11] to show that the expected value of $D^2(T_n)$ over all trees T_n in certain simply generated families $\mathcal{F}$ of trees is asymptotic to $An/4$, where A is a constant whose value depends on $\mathcal{F}$.

Our main object here is to determine the limiting distribution of D over trees T_n in certain simply generated families $\mathcal{F}$ and, in particular, to show that the expected value of $D(T_n)$ is asymptotic to $(An/2\pi)^{1/2}$. In §2 we derive an expression for the number of trees T_n in $\mathcal{F}$ with given bipartition numbers in terms of coefficients of powers of a certain generating function. In §3 we present some results on the asymptotic behaviour of such coefficients. Then in §4 we apply these results to the expression found in §2 to obtain our main results on the distribution of D.

2. Simply Generated Trees with Given Bipartition Numbers

We recall that *ordered* trees are (finite) rooted trees with an ordering specified for the branches incident with each node as one proceeds away from the root (see [8; p. 306]). Given a sequence $\Gamma = \{c_0 = 1,\ c_1, \cdots\}$ of non-negative numbers, we define $\mathcal{F} = \mathcal{F}_\Gamma$ to be the set of *weighted* ordered trees such that each ordered tree T_n is assigned the *weight*

$$w(T_n) = \prod_i c_i^{N_i(T_n)},$$

where $N_i(T_n)$ denotes the number of nodes of T_n that are incident with i edges leading away from the root (see, e.g., [9] or [14]). We call such a family $\mathcal{F}$ a *simply generated* family of trees.

Let $y_n = \sum w(T_n)$ where the sum is over all trees T_n in $\mathcal{F}$ with n nodes; we refer to y_n as the (weighted) number of trees T_n of $\mathcal{F}$. It is not difficult to see that the generating function $y = \sum y_n x^n$ of the simply generated family $\mathcal{F}$ satisfies the relation

$$y = x\Phi(y), \tag{2.1}$$

where $\Phi(t) = 1 + \sum_1^\infty c_m t^m$. Two familiar examples of such families are the ordinary ordered trees, for which $\Phi(t) = (1-t)^{-1}$ and $y_n = n^{-1} \cdot \binom{2n-2}{n-1}$, and the rooted labelled trees, for which $\Phi(t) = e^t$ and $y_n = n^{n-1}/n!$. We assume henceforth that $\mathcal{F}$ is some particular simply generated family whose generating function satisfies relation (2.1).

Let $y_{n,p}$ denote the sum of the weights of all trees T_n of $\mathcal{F}$ such that T_n has p nodes at even distance from the root. In order to obtain a formula for these numbers we shall obtain, more generally, a formula for the numbers

$$y_{n,p}(k) = \sum y_{n_1,p_1} \cdots y_{n_k,p_k},$$

where the sum is over all solutions in positive integers to the equations $n_1 + \cdots + n_k = n$ and $p_1 + \cdots + p_k = p$, for $k = 1, 2, \cdots$. Note that $y_{n,p}(k)$ is the (weighted) number of forests consisting of an ordered collection of k trees from $\mathcal{F}$ such that there are n nodes altogether in these trees and p of these nodes are at even distance from the root of the corresponding tree.

In what follows we let $C_K\{G(t)\}$ or, more briefly, $C_K\{G\}$ denote the coefficient of t^K in the power series $G(t)$.

Theorem 1. *If* $1 \le k \le p \le n$, *then*

$$y_{n,p}(k) = \frac{k}{p}\, C_{p-k}\{\Phi^{n-p}\} \cdot C_{n-p}\{\Phi^{p}\}. \tag{2.2}$$

Proof. The proof will be by induction on n. If $n = p$ then $y_{n,n}(k)$ equals 1 or 0 according as $k = n$ or $k < n$, and it is easy to see that formula (2.2) holds in this case. So we may assume that $1 \le k \le p < n$ and that formula (2.2) holds for all admissible values of the parameters when the total number of nodes involved is less than n.

Let F denote any forest of the type counted by $y_{n,p}(k)$ where $n > p$. If we remove the k root nodes of F and all their incident edges, we obtain a forest F' consisting of an ordered collection of j trees from $\mathcal{F}$ for some integer j such that $1 \le j \le n-p$; we may regard the trees in F' as being rooted at the nodes originally joined to the k root nodes of F. There are $n-k$ nodes in F' and $n-p$ of these nodes are at even distance from the root of the corresponding tree of F'. Now there are $y_{n-k,n-p}(j)$ such forests F', by definition; and it is not difficult to see that the number of ways of attaching the k root nodes of F to the j root nodes of F', taking the weight factors associated with the k root nodes of F into account, is equal to $C_j\{\Phi^k\}$. Consequently, when $n > p$ the numbers $y_{n,p}(k)$ satisfy the recurrence relation

$$y_{n,p}(k) = \sum_{j=1}^{n-p} C_j\{\Phi^k\} \cdot y_{n-k,n-p}(j). \tag{2.3}$$

When we apply the induction hypothesis to the factor $y_{n-k,n-p}(j)$ in relation (2.3) and simplify, using the relation $mC_m\{G\} = C_{m-1}\{G'\}$ twice, we find that

$$\begin{aligned}
y_{n,p}(k) &= (n-p)^{-1} C_{p-k}\{\Phi^{n-p}\} \sum_j jC_j\{\Phi^k\} \cdot C_{n-p-j}\{\Phi^{p-k}\} \\
&= (n-p)^{-1} C_{p-k}\{\Phi^{n-p}\} \sum_j C_{j-1}\{k\Phi^{k-1}\Phi'\} \cdot C_{n-p-j}\{\Phi^{p-k}\} \\
&= kC_{p-k}\{\Phi^{n-p}\} \cdot (n-p)^{-1} C_{n-p-1}\{\Phi^{p-1}\Phi'\} \\
&= kp^{-1} C_{p-k}\{\Phi^{n-p}\} \cdot C_{n-p}\{\Phi^{p}\},
\end{aligned}$$

as required. This suffices to complete the proof of the theorem. (Expression (2.2) can also be deduced from Lagrange's inversion formula. The argument used above can be formulated so as to provide another combinatorial proof of Lagrange's formula.)

Since the numbers $y_{n,p}$ introduced earlier are the same as the numbers $y_{n,p}(1)$, it follows from formula (2.2) that

$$y_{n,p} = p^{-1} \cdot C_{p-1}\{\Phi^{n-p}\} \cdot C_{n-p}\{\Phi^{p}\}. \tag{2.4}$$

In order to investigate the asymptotic behaviour of $y_{n,p}$ we need information on the behaviour of the coefficients of high powers of functions. Such coefficients are closely related to the distribution of sums of suitably defined independent, identically distributed random variables. There is an extensive literature on such problems (cf. [6; Chap. 9] and [4; Chap. XVI]). To make this paper fairly self-contained, we shall present versions of certain results of this nature in the next section; these particular versions are useful in treating various problems of asymptotic enumeration.

3. On the Coefficients of Powers of Functions

We assume from now on that the function

$$\Phi(t) = 1 + \sum_{1}^{\infty} c_m t^m$$

satisfies the following conditions:

$$c_m \geq 0 \quad \text{for} \quad m \geq 1 \quad \text{and} \quad c_j > 0 \quad \text{for some} \quad j \geq 1; \tag{3.1}$$

$$\gcd\{m : \ m \geq 1 \quad \text{and} \quad c_m > 0\} = 1; \tag{3.2}$$

$$\Phi(t) \quad \text{is analytic when} \quad |t| < R, \quad \text{where} \quad 0 < R \leq \infty. \tag{3.3}$$

The functions $g(t)$ and $A(t)$ are defined as follows for $0 < t < R$;

$$g(t) = t\,\frac{d}{dt} \log\, \Phi(t) = t\Phi'(t)/\Phi(t); \tag{3.4}$$

$$A(t) = tg'(t) = t^2\Phi''(t)/\Phi(t) + g(t) - g^2(t). \tag{3.5}$$

It is not difficult to see that $A(t) > 0$ for $0 < t < R$, so

$$g(t) \quad \text{is strictly increasing for} \quad 0 < t < R. \tag{3.6}$$

This implies that the inverse function $g^{-1}(s)$ exists for $s \in (0, S)$, where $S = \sup\,\{g(t) : \ 0 < t < R\}$.

Theorem 2. *Let* $[s_1, s_2]$ *be any fixed closed subinterval of* $(0, S)$; *let* K *and* N *be integers tending to infinity in such a way that*

$$s_1 \le K/N \le s_2 \tag{3.7}$$

for all sufficiently large values of N; *and let* $r := g^{-1}(K/N)$. *Then*

$$C_K\{\Phi^N\} = \left(2\pi A(r)N\right)^{-1/2} \cdot \Phi^N(r) r^{-K}\{1 + O(N^{-1})\} \tag{3.8}$$

holds uniformly as $K, N \to \infty$ *and the constant implicit in the* O*-term depends only on* s_1 *and* s_2.

This can be proved by applying the saddlepoint method. See, for example, [3; p. 646], [7; p. 868], [1; p. 1115], [12; p. 290] or [5; p. 193] for results that are either essentially equivalent or closely related to this; in particular, refinements of the error term are given in [7] and [1].

We now impose more restrictive conditions on the relative sizes of K and N; this permits us to obtain more explicit estimates for the individual coefficients $C_K\{\Phi^N\}$ and for certain sums involving these coefficients.

Theorem 3. *Let* α *be a constant such that* $0 < \alpha < S$ *and let* $\eta := g^{-1}(\alpha)$. *Suppose the integers* K *and* N *tend to infinity in such a way that*

$$\Delta := K - \alpha N = O(N^{2/3}) \tag{3.9}$$

as $N \to \infty$. *Then*

$$\begin{aligned} C_K\{\Phi^N\} &= \left(2\pi A(\eta)N\right)^{-1/2} \cdot \Phi^N(\eta)\eta^{-K} e^{-\Delta^2/2A(\eta)N} \\ &\quad \times \{1 + O(1/N) + O(\Delta/N) + O(\Delta^3/N^2)\} \end{aligned} \tag{3.10}$$

holds uniformly as $N \to \infty$. *Furthermore,*

$$\sum{}' \eta^v C_v\{\Phi^N\} \le \Phi^N(\eta) e^{-\Delta^2/2A(\eta)N} \cdot \{1 + O(\Delta^3/N^2)\}, \tag{3.11}$$

where the sum is over all v *such that* $v \ge K$ *if* $\Delta > 0$ *and over all* v *such that* $v \le K$ *if* $\Delta < 0$.

Remark. Relation (3.10) is, in effect, a version of the classical local limit theorem for sums of suitably defined independent, identically distributed random variables (cf. [6; p. 243], [4; p. 533], or [13; p. 208]). The error terms in the right-hand sides of relations (3.10) and (3.11) appear as multiplicative factors of the leading terms; and there are no isolated additional error terms. This fact will be useful in estimating sums involving the coefficients $C_K\{\Phi^N\}$.

Proof of (3.10). Since $K/N = \alpha + \Delta/N = \alpha + o(1)$ as $N \to \infty$ and $0 < \alpha < S$, hypothesis (3.7) of Theorem 2 will be satisfied with suitably chosen s_1 and s_2 for all sufficiently large values of N. We let $r = g^{-1}(K/N)$, as before.

It follows from Taylor's theorem that

$$\begin{aligned}\Delta/N = K/N - \alpha = g(r) - g(\eta) &= g'(\eta)(r-\eta) + \mathcal{O}((r-\eta)^2)\\ &= A(r/\eta - 1) + \mathcal{O}((r/\eta-1)^2),\end{aligned}$$

where we write A for $A(\eta) = \eta g'(\eta)$. This implies that

$$r/\eta - 1 = \Delta/AN + \mathcal{O}(\Delta^2/N^2). \tag{3.12}$$

Next we expand the function $Q(t) = N\ \log\Phi(t) - K\ \log t$ about $t = \eta$. We find, taking (3.4) and (3.5) into account, that

$$Q'(\eta) = (N\alpha - K)\eta^{-1} \quad \text{and} \quad Q''(\eta) = (NA - N\alpha + K)\eta^{-2}$$

and, since $K - N\alpha = \Delta$, that

$$Q(r) = Q(\eta) - \Delta(r/\eta - 1) + \tfrac{1}{2}\,(AN+\Delta)(r/\eta-1)^2 + \mathcal{O}(N(r/\eta-1)^3). \tag{3.13}$$

Relations (3.12) and (3.13) imply that

$$Q(r) = Q(\eta) - \Delta^2/2AN + \mathcal{O}(\Delta^3/N^2)$$

or, equivalently, that

$$\Phi^N(r)r^{-K} = \Phi^N(\eta)\eta^{-K}\cdot e^{-\Delta^2/2AN}\{1 + \mathcal{O}(\Delta^3/N^2)\}. \tag{3.14}$$

Finally,

$$A(r) = A(\eta) + \mathcal{O}(r-\eta) = A + \mathcal{O}(\Delta/N), \tag{3.15}$$

appealing to (3.12) again. Conclusion (3.10) now follows from relations (3.8), (3.14), and (3.15).

Proof of (3.11). If $\Delta < 0$, then $K/N < \alpha$ and $r = g^{-1}(K/N) < g^{-1}(\alpha) = \eta$, provided that N is sufficiently large to ensure that (3.7) is satisfied. But then

$$\begin{aligned}\sum_{v\le K}\eta^v C_v\{\Phi^N\} &\le \sum_{v\le K}\eta^K\cdot r^{v-K}C_v\{\Phi^N\}\\ &\le (\eta/r)^K\cdot\Phi^N(r),\end{aligned} \tag{3.16}$$

and the required conclusion now follows from relation (3.14). An analogous argument applies when $\Delta > 0$.

Remark. Inequality (3.16) and its derivation are similar in spirit to the statement and derivation of Chernoff's inequality (cf. [2] or [1]). For examples of stronger and more complicated results on large deviations, see, e.g. [1; p. 1117], [12; p. 290], or [4; p. 552].

4. Main Results

We recall that $\mathcal{F}$ denotes some simply generated family of trees whose generating function $y = \sum y_n x^n$ satisfies the relation $y = x\Phi(y)$ where $\Phi(t) = 1 + \sum_1^\infty c_m t^m$. We shall assume henceforth that the function $\Phi(t)$ not only satisfies conditions (3.1) - (3.3) but that, in addition, there exists a number τ, where $0 < \tau < R$, such that

$$\tau\Phi'(\tau) = \Phi(\tau). \tag{4.1}$$

This implies that we may apply Theorem 3 with $\alpha = 1$ and $\eta = \tau$ in this case. Moreover, $A(\tau) = \tau^2\Phi''(\tau)/\Phi(\tau)$, in view of (4.1) and definitions (3.4) and (3.5). From now on we write A for $A(\tau)$. We remark that

$$y_n = \tau(2\pi A n^3)^{-1/2}(\Phi(\tau)/\tau)^n \cdot \{1 + O(n^{-1})\} \tag{4.2}$$

as $n \to \infty$ (cf. [9; p. 1000] or [14; p. 32]). More generally, it follows from Lagrange's inversion formula and (3.10) that if $k = O(n^{2/3})$, then

$$\begin{aligned} C_n\{y^k\} &= \frac{k}{n}\, C_{n-k}\{\Phi^{n-k}\} \\ &= (2\pi A n^3)^{-1/2} k \Phi^n(\tau)\tau^{k-n} e^{-k^2/2An} \\ &\quad \times \{1 + O(1/n) + O(k/n) + O(k^3/n^2)\}. \end{aligned}$$

Other versions of this last relation, valid for more restricted values of k, have been given in [10, p. 581] and [14; p. 41].

We now use the results of the preceding sections to determine the limiting behaviour of the fraction $y_{n,p}/y_n$ of trees T_n in $\mathcal{F}$ that have bipartition numbers p and $n - p$ when $|2p - n|$ is relatively small; and we give a simple upper bound for this fraction when $|2p - n|$ is larger. We then use these results to determine the limiting behaviour of the expected value, the variance, and the distribution of $D(T_n)$.

Theorem 4. *Let n and p be integers such that $1 \le p \le n-1$; and for any fixed positive constant h, let $H(n) = \lceil (2Ahn \log n)^{1/2} \rceil$. If*

$$|2p-n| \le H(n),$$

then

(4.3)

$$y_{n,p}/y_n = 4(2\pi An)^{-1/2} e^{-2\Delta^2/An} \cdot \{1 + \mathcal{O}(1/n) + \mathcal{O}(\Delta/n) + \mathcal{O}(\Delta^3/n^2)\}$$

holds uniformly as $n \to \infty$ where $\Delta = 2p - n$. If

$$|2p-n| \ge H(n),$$

then

$$y_{n,p}/y_n = \mathcal{O}(n^{3/2-h}) \tag{4.4}$$

as $n \to \infty$.

Proof of (4.3). We showed earlier that

$$y_{n,p} = p^{-1} \cdot \mathcal{C}_{p-1}\{\Phi^{n-p}\} \cdot \mathcal{C}_{n-p}\{\Phi^p\}. \tag{4.5}$$

It is not difficult to see that the relation $(p-1)-(n-p) = \mathcal{O}\big((n-p)^{2/3}\big)$ certainly holds here, so we may apply (3.10) with $K = p-1$, $N = n-p$, $\alpha = 1$, and $\eta = \tau$ to estimate the factor $\mathcal{C}_{p-1}\{\Phi^{n-p}\}$; and then, since $(n-p)-p = \mathcal{O}(p^{2/3})$, we may apply (3.10) with $K = n-p$, $N = p$, $\alpha = 1$, and $\eta = \tau$ to estimate the other factor $\mathcal{C}_{n-p}\{\Phi^p\}$. When we combine these estimates in (4.5) and take into account that $p^{-1} = 2n^{-1} \cdot \big(1 + \mathcal{O}(\Delta/n)\big)$, we find that

$$y_{n,p} = (2\tau/\pi An^2)\big(\Phi(\tau)/\tau\big)^n \cdot e^{-2\Delta^2/An} \cdot \{1+\mathcal{O}(1/n)+\mathcal{O}(\Delta/n)+\mathcal{O}(\Delta^3/n^2)\}.$$

This and relation (4.2) imply conclusion (4.3). We remark that (4.3) remains valid when $|2p-n| = \mathcal{O}(n^{2/3})$.

Proof of (4.4). We shall carry out the proof when $2p - n \ge H = H(n)$. In this case $n - p \le p - H$ and $H = \mathcal{O}(p^{2/3})$. If we appeal to (3.11) with $K = p - H$, $N = p$, $\alpha = 1$, and $\eta = \tau$, we find that

$$\tau^{n-p}\mathcal{C}_{n-p}\{\Phi^p\} \le \sum_{v \le p-H} \tau^v \mathcal{C}_v\{\Phi^p\}$$
$$= \mathcal{O}\big(\Phi^p(\tau) n^{-hn/p}\big) = \mathcal{O}\big(\Phi^p(\tau) n^{-h}\big),$$

so

$$(4.6) \qquad C_{n-p}\{\Phi^p\} = \mathcal{O}\big(\Phi^p(\tau)\tau^{p-n}n^{-h}\big).$$

And, clearly,

$$(4.7) \qquad C_{p-1}\{\Phi^{n-p}\} \leq \tau^{-(p-1)}\Phi^{n-p}(\tau).$$

Hence,

$$y_{n,p} = \mathcal{O}\big(\Phi^n(\tau)\tau^{-n}n^{-h}\big) = \mathcal{O}(n^{3/2-h}y_n),$$

as required, by relations (4.5), (4.6), (4.7), and (4.2). The proof when $2p - n \leq -H$ is similar except that now the argument involving (3.11) is applied to $C_{p-1}\{\Phi^{n-p}\}$.

Theorem 5. *Let $E(n)$ and $V(n)$ denote the expected value and the variance of $D(T_n)$ over all trees T_n in $\mathcal{F}$. Then*

$$E(n) = (An/2\pi)^{1/2} + \mathcal{O}(1) \quad \textit{and} \quad V(n) = A(\pi-2)n/4\pi + \mathcal{O}(n^{1/2})$$

as $n \to \infty$.

Proof. Let $H = \lceil (7An \ \log n)^{1/2} \rceil$. Then

$$E(n) = \sum_{p=1}^{n-1} |2p-n| \cdot (y_{n,p}/y_n) = \Sigma_1 + \Sigma_2$$

where Σ_1 and Σ_2 denote the contributions from the terms for which $|2p-n| \leq H$ and $|2p-n| > H$, respectively. It follows from inequality (4.4) with $h = 7/2$ that

$$\Sigma_2 \leq n^2 \cdot \mathcal{O}(n^{-2}) = \mathcal{O}(1),$$

so it remains to estimate Σ_1.

If we let $x_p := (2p-n) \cdot 2(An)^{-1/2}$ and appeal to relation (4.3), we find that

$$\Sigma_1 = (2/\pi)^{1/2} \sum |x_p| e^{-x_p^2/2} \cdot \big\{1 + \mathcal{O}(n^{-1}) + \mathcal{O}\big((|x_p| + |x_p|^3) \cdot n^{-1/2}\big)\big\}$$

where the sum, as before, is over p such that $|2p-n| \leq H$. Now let

$$f(x) := (2/\pi)^{1/2} x e^{-x^2/2} \cdot \big\{1 + \mathcal{O}(n^{-1}) + \mathcal{O}\big((x+x^3)n^{-1/2}\big)\big\}.$$

Then the expression for Σ_1 can be rewritten as

$$\Sigma_1 = 2\sum f(x_p)$$

where now the sum is over p such that $0 \leq 2p - n \leq H$, i.e., over p such that $a := \lceil n/2 \rceil \leq p \leq \lfloor (n+H)/2 \rfloor =: b$. We observe that $4(An)^{-1/2}\Sigma_1$ equals a Riemann sum, with uniform subdivision size $4(An)^{-1/2}$, of the function $2f(x)$ over the interval $[x_a, x_b]$, where x_a equals 0 or $2(An)^{-1/2}$ according as n is even or odd, and $x_b = 2(7\log n)^{1/2} + O(n^{-1/2})$. Hence, by the Trapezoidal Rule,

$$\begin{aligned} 4(An)^{-1/2}\Sigma_1 &= 2\int_{x_a}^{x_b} f(x)dx + 4(An)^{-1/2} \\ &\qquad \times \{f(x_a) + f(x_b)\} + O\left((\log n)^{1/2} \cdot n^{-1}\right) \\ &= 2(2/\pi)^{1/2}\int_{x_a}^{x_b} xe^{-x^2/2}dx + O(n^{-1/2}) \\ &= 2(2/\pi)^{1/2} + O(n^{-1/2}), \end{aligned}$$

so

$$\Sigma_1 = (An/2\pi)^{1/2} + O(1).$$

This suffices to prove the formula for $E(n)$.

Similarly, it can be shown that the expected value of $D^2(T_n)$ equals $An/4 + O(n^{1/2})$; this and the formula for $E(n)$ imply the formula for $V(n)$.

Our final result follows readily by the same type of argument as was just used to estimate Σ_1.

Theorem 6. *Let λ be any positive constant. Then*

$$Pr\{D(T_n) \leq \lambda(An/4)^{1/2}\} = (2/\pi)^{1/2}\int_0^{\lambda} e^{-x^2/2}dx + O(n^{-1/2}).$$

The preparation of this paper was assisted by grants from the Natural Sciences and Engineering Research Council of Canada.

References

1. D. Blackwell and J.L. Hodges, Jr., *The probability in the extreme tail of a convolution*, Ann. Math. Stat. 30 (1959), 1113-1120.

2. H. Chernoff, *A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations*, Ann. Math. Stat. 23 (1952), 493-507.

3. H.E. Daniels, *Saddlepoint approximations in statistics*, Ann. Math. Stat. 25 (1954), 631-650.

4. W. Feller, *An Introduction to Probability and its Applications*, Vol. II, 2nd ed., Wiley, N.Y., 1971.

5. D. Gardy, *Some results on the asymptotic behaviour of coefficients of large powers of functions*, Disc. Math. 139 (1995), 189-217.

6. B.V. Gnedenko and A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, 1954.

7. I.J. Good, *Saddle-point methods for the multinomial distribution*, Ann. Math. Stat. 28 (1957), 861-881.

8. D.E. Knuth, *The Art of Computer Programming*, Vol. I, Addison-Wesley, Reading, 1973.

9. A. Meir and J.W. Moon, *On the altitudes of nodes in random trees*, Can. J. Math. 30 (1978), 997-1015.

10. A. Meir and J.W. Moon, *The asymptotic behaviour of coefficients of powers of certain generating functions*, Europ. J. Comb. 11 (1990), 581-587.

11. J.W. Moon, *On the bipartition numbers of random trees*, Ars Comb. 25C (1988), 3-10.

12. V.V. Petrov, *On the probabilities of large deviations for sums of independent random variables*, Th. Prob. Appl. 10 (1965), 287-298.

13. W. Richter, *Local limit theorems for large deviations*, Th. Prob. Appl. 2 (1957), 206-219.

14. J.-M. Steyaert and P. Flajolet, *Patterns and pattern-matching in trees: an analysis*, Inform. Control 58 (1983), 19-58.