BALANCED TERNARY DESIGNS FROM ANY (v, b, r, k) DESIGN

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ABSTRACT

Balanced ternary and generalized balanced ternary designs are constructed from any (v, b, r, k) designs. These results generalise the earlier results of Diane Donovan (1985).

1. Introduction

Tocher (1952) defined a balanced n-ary design to be a collection of B multisets called blocks, of size K, taken from V elements, where elements may occur 0,1, . . . or n-1 times in any block. Moreover, if n_{im} denotes the number of times the i-th element occurs in the m-th block then

$$\sum_{n=1}^{B} n_{im}\, n_{jm} = \Lambda$$

is the index and is constant for all i and j such that $1 \leq i \leq j \leq v$. Let ρ_g denote the number of blocks in which an element occurs g times, for $1 \leq g \leq n-1$ and if ρ_g is constant then it can easily be seen that

$$R = \sum_{g=1}^{n-1} g\rho_g \qquad \text{...(1.1)}$$

$$VR = BK \qquad \text{...(1.2)}$$

$$\Lambda\,(V-1) = \sum_{g=1}^{n-1} g\rho_g \quad (K-g) \qquad \text{...(1.3)}$$

A balanced n-ary design is a balanced ternary design if n = 3. A generalized balanced ternary design is one in which an element may occur 0, 1 or m times in any one block, that is $\rho_g = 0$ for $1 < g \leq n-1$, $g \neq m$. In the case of a balanced ternary design identity (1.3) can be restated,

$$\Lambda\,(V-1) = R\,(K-1) - 2\rho_2$$

and for a generalized balanced ternary design :

$$\Lambda\,(V-1) = R\,(K-1) - m\,(m-1)\,\rho_m\,.$$

Since ρ_g is constant in both these designs we can write the parameters (V, B, R, ρ_1, ρ_2 , K, Λ) and (V, B, ρ_1 , ρ_m , R, K, Λ) respectively. A balanced quarternary design is a balanced n-ary design with n = 4. A generalized balanced quarternary design has elements occurring 0, 1, α or β times in any block, that is, $\rho_g = 0$ for $1 < g \leq n - 1, g \neq \alpha, \beta$.

Theorem 1. A balanced ternary design with parameters V = v, B = v (b–r) (v–1), R = v (v–1) (b–r), K = v and Λ = (v + 1). (v–2). (v–r) can be constructed from any (v, b, r, k) design.

Proof : Let us start with a design D: (v, b, r, k). Take a block b* of D and let b* be the set complimentary to b′ consisting of v–k treatments. It is well known that out of t treatments, if we write t blocks each of size t–1, by omitting successively one treatment for each block, then we get a SBIBD (v, v–1, v–2). Now construct a SBIBD (v–k, v–k–1, v–k–2) on treatments of b* and to each such block of size v–k–1, we embed the original block b*. Thus getting (v–k) blocks each of size v–1 and if we duplicate the treatments one by one for each such block of size (v–1), we get (v–1) (v–k) blocks corresponding to each block of (v, b, r, k) and in total b (v–1) (v–k) blocks for the corresponding ternary design. Repeat the process for each and every block of the (v, b, r, k) design but b (v–1) (v–k) = v (b–r) (v–1) blocks. The replication numbers will count to 2 [r (v–k) + (v–k–1) (b–r)] + (v–2) [r (v–k) + (v–k–1) (b–r)] = v (v–1) (b–r). In order to prove the constancy of Λ we assume say elements x and y occur together in i blocks of the (v, b, r, k) design where $0 \leq i \leq r$. Then Λ = i (v–k)(v + 1) + (b–2 r + i) (v–k–2) (v + 1) + 2(r–i) (v–k–1)(v + 1) or Λ = (v + 1)(v–2) (b–r). This relation is independent of i. Hence the design is balanced.

Corollary 1.1 : By deleting the repeated blocks from the above theorem, we get a balanced ternary design having the parameter V = v, B = v (v–1), R = v(v–1), K = v and Λ = (v + 1)(v – 2).

Corollary 1.2 : By choosing a BIB (v, b, r, k, λ) in place of (v, b, r, k) and by choosing a BIB (v′ = v–k, b′, r′, k′, λ') in place of SBIBD used in the proof of Theorem 1 with v′ = v–k, we get a balanced ternary design with parameters

$$V = v,\ B = bb'\ (k + k'),$$
$$R = (k + k' + 1)\ \{rb' + r'\ (b-r)]$$
$$K = (k + k' + 1),\ \Lambda = \lambda\ b' + \lambda'\ (b-r) + (2r' - \lambda)(r-\lambda).$$

Corollary 1.3 : If we consider an m-associate PBIB design [v′ = (v–k), b′, r′, k′ = (v–k–1), $\lambda_1, \lambda_2, \ldots, \lambda_m$] in place of a SBIB (v–k, v–k–1, v–k–2) design then we can construct an m-associate PB ternary design having the same association scheme.

Remark : Theorem 2.2 of Diane Donovan (1985) is a particular case of above theorem by substituting $v = k + 2$.

Theorem 2 : A generalized balanced ternary design with parameters $V = v$, $B = v(b-r)(v-1)$, $K = v + m-2$, $R = (v-1)(b-r)(v + m-2)$ and $\Lambda = (v + 2m-3)(v-2)(b-r)$; $m \geq 2$, can be constructed from any (v, b, r, k) design.

Proof : This theorem can be proved by repeating each element of (v–k) block of size (v–1), m times serially in place of doubling the treatment in each block as in Theorem 1.

Corollary 2.1 : By deleting the repeated blocks from the above design we get a generalized balanced ternary design having the parameters $V = v$, $B = v(v-1)$, $K = (v + m-2)$, $R = (v-1)(v + m-2)$ and $\Lambda = (v-3 + 2m)(v-2)$.

Corollary 2.2 : If we consider an m associate PBIB design in place of BIB design then we can always construct a generalized m-associate PBIB design.

Remark : If we substitute $v = k + 2$ in the Theorem 2, we get Corollary No.(2.3) of Diane Donovan (1985).

Theorem 3 : Existence of a BIB design whith parameters ($v' = v-k$, b', $k' = k-2$, r', λ') and any (v, b, r, k) design implies the existence of a balanced ternary design with parameters $V=v$, $B=b[k(b'-\lambda') + 4r']$, $R=r[r'(v-k + 4) + (2b'-k\lambda')]$, $K=k$ and $\Lambda = r[\lambda'(v-2k) + 4r']$.

Proof : Let b* be a block of the given (v, b, r, k) design D. We construct blocks of ternary balanced design from b* as in Theorem 1, by using blocks of a BIB design D′ with parameters [$v = v-k$, b', $k'=k-2$, r', λ']. As before, if we duplicate the treatments and add the b′ blocks each of size k–2 of D′, to these duplicate treatments, then we get b′ blocks of size k and there are k treatments in the blocks. So total number of blocks are b′k′. Repeat this process for each block of the (v, b, r, k) design. We get bb′k blocks in all. Now adjoin to these blocks ($4r'-k\lambda'$) copies of the original (v, b, r, k) design. Thus constructing the design BTD. The values of V, B, R, K can easily be obtained. To obtain Λ, we assume x and y occur together in i blocks of (v, b, r, k) design $0 \leq i \leq r$. Then $\Lambda = 4r'(r-i) + (b-2r + i)k\lambda' + (4r'-k\lambda')i = r[\lambda'(v-2k) + 4r']$. Thus Λ is independent of elements chosen.

Example : Consider a $v = 8$, $b = 4$, $r = 2$, $k = 4$ design constructed as follows :

1	2	3	4
3	4	5	6
5	6	7	8
7	8	1	2

Then the above method by using the BIB design $v' = 4$, $k' = 2$, $r' = 3$, $b' = 6$ and $\lambda' = 1$ and 8 copies of the above design, yields the following balanced ternary design, with parameters (8, 128, 64, 4, 24)

1156	5512	7734	1157	5514	7735
1158	5518	7736	1167	5527	7745
1168	5528	7746	1178	5578	7756
2256	6612	8834	2257	6614	8835
2258	6618	8836	2267	3356	3357
2268	2278	3358	3367	3368	3378
4456	4457	4458	4467	3312	3314
4468	4478	3318	3327	3328	3378
4412	4414	4418	7812	4478	6627
4427	4428	6628	6678	5512	5513
5514	5523	5524	5535	6614	6623
6612	6613	6624	6634	7712	7713
7714	7723	7724	7734	8814	8823
8812	8813	8824	8834	8845	8846
8856	1134	1135	1136	1156	2234
1145	1146	2235	2236	2245	2246
2256	1234	3456	5678	3456	5678
7812	1234	7812	1234	3456	5678
7812	1234	3456	5678	3456	5678
7812	1234	7812	1234	3456	5678
7812	1234	3456	5678	3456	5678
7812	1234				

On a similar line of Theorem 2, this result can also be extended to obtain a result for generalized balanced ternary design.

Theorem 4 : A generalized balanced ternary design with parameters $V = v$, $B = b'\,bk + (2mr' - \lambda'k)b = b\,[(K(b' - \lambda') + 2mr']$, $K = k$, $R = (b'm + 2mr' - \lambda'k)\,r + (b-r)r'$ and $\Lambda = 2mrr' + (b-2r)\,\lambda'\,k$, where $m > 2$ can be constructed from any (v, b, r, k) design and a BIB design having the parameter $[v' = (v-k), b', r', k' = (k-m), \lambda']$.

Proof : As in Theorem 3 instead of repeating twice, repeat each element of the block m times and adjoin $(2mr' - \lambda'k)$ copies of the original (v, b, r, k) design, then we get the required generalized balanced ternary design.

Corollary 4.1 : If we consider an m-associate PBIB design having the parameter $v' = (v-k)$, b', r', $k' = (k-m)$, $\lambda_1, \ldots, \lambda_m$ instead of BIB design, then we can obtain a generalized m-associate partially balanced ternary design.

Remark : Corollary (3.2) of Diane Donovan (1985) can easily be obtained by substituting $k = m + 1$ in Theorem 4.

This method can be used to construct a generalized balanced quaternary design. Recall that an element may occur 0, 1, α or β times in any one block.

Theorem 5: A generalized balanced quarternary design with parameters $V = v$, $B = b\,[k(v' - \lambda') + 2r'\alpha\beta]$, $R = r\,[(b\alpha + 2r'\alpha\beta - \lambda'k + r'\beta k\,(b-r)]$, $K = k$ and $\Lambda = 2r'\alpha\beta\,(b-2r)\,\lambda'k$ can be constructed from any $(v, b, r, k = \alpha + t\beta)$ design and a BIB design $v' = (v-k)$, b', r', $k' = t$, λ', where α, β, t are positive integers.

Proof : Take any block b_0 of the (v, b, r, k) design D and any element x of b_0. Construct a new block by repeating the element x, α times, and then adjoining the b blocks of BIB design by repeating each element β times. Repeat this procedure for each element of b_0 and then for each block of the (v, b, r, k) design such as b_0. To these newly created blocks adjoin $(2r'\alpha\beta - \lambda'k)$ copies of the (v, b, r, k) design. Assume x and y occur together in i blocks of the (v, b, r, k) design $0 \leq i \leq r$, then

$$\Lambda = 2rr'\alpha\beta - 2ir'r + (b-2r)\lambda'k + i\lambda'k + (2r'\alpha\beta - \lambda'k)i$$
$$= rr'\alpha\beta + (b-2r)\lambda'k.$$ Hence the design is balanced.

For $\alpha = 2$ and $\beta = 1$ (or $\alpha = m$ and $\beta = 1$), we obtain balanced ternary design (or generalised balanced ternary design) with parameters as given in Theorem 3 and Theorem 4.

The following theorem uses a slight variation of the above technique to produce a balance quarternary design.

Theorem 6 : A balanced quarternary design with parameters

$$V = v,\quad B = bb'k(k-1) + b\,[(k-1)(10r'-k\lambda')] - 12b'b$$
$$= b\,[(k-1)(b'k + 10r' - k\lambda') - 12b']$$
$$R = r[b'(5k-17) + (k-1)\{r'(v-k+10)-k\lambda'\}]$$

$K = k$ and $\Lambda = r(k-1)\ [10r' + (v-2k)\lambda']$ can be constructed from any (v, b, r, k) design.

Proof : Take $(k-1)(10r' - k\lambda') - 12b'$ copies of any (v, b, r, k) design. To this, adjoin blocks constructed in the following way :

Assume that $(x_1, x_2, \ldots, x_k)$ is a block of any (v, b, r, k) design, select any element of that block, say x_1 and repeat x_1 three times and then repeat two times each of remaining $x_2, x_3, \ldots, x_k$ serieally, we get (k–1) such blocks. Adjoin to each of these blocks to the blocks of the BIB design $(v' = v-k, b', r', k' = k-5, \lambda')$. Thus, we get b (k–1) blocks. Now choose x_2 three times and other x_i's two times and so on, we get b'k (k–1) blocks. Repeat this procedure for each block of (v, b, r, k) design, the parameters V, B, R, K could be verified. If elements x and y occur together in i blocks of the (v, b, r, k) design for $0 \leq i \leq r$, then $\Lambda = i\ (k-1)\ [10r' - k\lambda'] - 12b'i + 12b'i + 2(r-i)5r'(k-1) + (b-2r + i)\ k(k-1)\lambda' = r(k-1)\ [10r' + (v-2k)\lambda']$, which is independent of occurence of x and y in i-blocks. Hence the design is balanced.

Corollary 6.1 : In place of a BIB $(v', b', r', k', \lambda')$ design, if we consider an m-associate PBIB design, we get an m-associate PB quarternary design.

Remark : Corollary No. (3.4) of Diane Donovan (1985) is a particular case of Theorem 6. Her corollary is not applicable for $k > 6$, though our theorem is valid for any k.

Acknowledgement : We wish to thank Prof. H.L. Agrawal, Senior Professor, Department of Statistics for his help during the preparation of this paper.

References :

Diane Donovan : Balanced ternary designs from 1-designs, ARS Combinatoria, Vol.19 (1985), pp.95-103.

K.D. Tocher : The design and analysis of block experiment. J.Roy.Statist. Soc.Ser. 14 (1952), 45-100.

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