Maximal Trades

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Abstract: We call a simple t-(v, k) trade with maximum volume a maximal trade. In this paper, except for $v = 6m + 5$, $m \geq 3$, maximal 2-$(v, 3)$ trades for all v's are determined. In the latter case a bound for the volume of these trades is given.

1. Introduction

Trades are combinatorial objects with interesting and useful properties which have captured the attention of some design theorists in recent years and have been utilized in various parts of design theory. Among the applications of trades one could include construction of signed designs [6], construction of block designs with repeated blocks and the study of the spectrum of support sizes of block designs [5], construction of nonisomorphic block designs [5], determination of some intersection numbers among triple systems with different indices [1], and determination of defining sets of some designs [4].

In general, working with trades is rather convenient and handy since they form a $\mathbb{Z}$-module [3]. We will call this module *trade space.* Although some families of generators for the trade space with very nice minimality properties and complete characterization have been discovered by some authors [2,8,9], nevertheless, their internal structures, to a great extent, remain unknown. Information about trade space could be helpful in various ways.

The aim of this paper is to study the simple trades (trades without repeated blocks) with block size three and with the maximum possible volume.

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2. Background and notations

Let v, k, and t be positive integers such that $v > k > t$. Let X be a v-set. For $0 \leq i \leq v$, the set of all i-subsets of X is denoted by $P_i(X)$. In what follows, the set $\{x_1, \cdots, x_i\} \in P_i(X)$ will be denoted by $x_1x_2\cdots x_i$. An incidence structure $D = (X, \mathcal{B})$ is called a t-(v, k, λ) design (or simply a t-design) if $\mathcal{B}$ is a collection of the elements of $P_k(X)$ (called blocks) such that every element of $P_t(X)$ appears in $\mathcal{B}$ exactly λ times. A t-design is called *simple* if $\mathcal{B}$ is a set. $(X, P_k(X))$ is called the *trivial design* and is denoted by D_{tr}. A t-design with $\lambda = 1$ is called a *Steiner system*. For given v, k, t, and λ, let $\lambda_i = \lambda\frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}$, for $i = 0, 1, \cdots, t-1$. A set of necessary conditions for the existence of a t-(v, k, λ) design is λ_i's to be integer for $i = 0, 1, \cdots, t-1$.

For given v, k, and t, let λ_* denote the minimum value of $\lambda's$ satisfying the above conditions. Let $m = \frac{\binom{v-t}{k-t}}{\lambda_*}$. If, for every $1 \leq i \leq m$, $D_i = (X, \mathcal{B}_i)$ is a simple t-(v, k, λ_*) design such that for $1 \leq i, j \leq m$, $\mathcal{B}_i \cap \mathcal{B}_j = \phi$ and $\bigcup_{i=1}^{m} D_i = D_{tr}$, then $\{D_i\}_{i=1}^{m}$ is called a *large set of designs.*

Let T^+ and T^- denote two disjoint collections of the elements of $P_k(X)$ such that the occurrences of every element of $P_t(X)$ in both T^+ and T^- are equal. Then $T = (T^+, T^-)$ is called a t-(v, k) *trade*. Clearly $|T^+| = |T^-|$, and $|T^+|$ is called the *volume* of T, denoted by vol(T). Both T^+ and T^- must cover the same subset of X which is called the *foundation* of T, denoted by found(T). The trade T is called *simple* if T^+ and T^- do not contain repeated blocks.

Hwang [7] has shown that for every arbitrary t-(v, k) trade T, we have

(i) vol $(T) \geq 2^t$,

(ii) |found $(T)| \geq k + t + 1$,

and a trade with volume 2^k and foundation size equal to $k+t+1$, is called a *minimal trade*. For given v, k, and t, a simple trade with maximum possible volume is called a *maximal* t-(v, k) *trade*.

The purpose of this paper is the determination of maximal 2-$(v, 3)$ trades for $v > 5$ which will be denoted by $T_M(v)$. By trades in this paper, we shall mean 2-$(v, 3)$ trades.

3. Some further notations

Let $D = (X, \mathcal{B})$ denote an incidence structure. For $x, y \in X$, the number of blocks in D which contain the pair $\{x, y\}$ is denoted by $\lambda_D(xy)$ and the number of blocks which contain $x \in X$ is denoted by $r_D(x)$. Let $D_{tr} = (X, P_3(X))$, then $\lambda_{D_{tr}}(xy)$ and $r_{D_{tr}}(x)$ will be denoted by λ^* and r^*, respectively. Clearly,

$$\lambda^* = v - 2, \quad r^* = \frac{(v-1)(v-2)}{2}.$$

We will also use the following notations:

$$\lambda_T(xy) = \lambda_{T^+ \cup T^-}(xy),$$
$$r_T(x) = r_{T^+ \cup T^-}(x).$$

Clearly, $\lambda_T(xy)$ and $r_T(x)$ are even. Now let

$$T^c = P_3(X) \backslash (T^+ \cup T^-).$$

Then

$$\lambda_{T^c}(xy) = \lambda^* - \lambda_T(xy),$$
$$r_{T^c}(x) = r^* - r_T(x).$$

Proposition 1. Let T be an arbitrary trade and $x, y \in X$. Then

$$\lambda_{T^c}(xy) \equiv \lambda^* \pmod 2,$$
$$r_{T^c}(x) \equiv r^* \pmod 2.$$

4. Results for v even

Lemma 1. If $v = 4m$ and $m \geq 2$, then there exists a simple trade T such that

$$\text{vol}\,(T) = \frac{4m(4m+1)(m-1)}{3}.$$

Proof. For $m \geq 2$, assume that

$$X = \{1, 2, \cdots, 4m\}.$$

For $1 \leq i \leq m$, let

$$X_i = \{4i-3, 4i-2, 4i-1, 4i\}.$$

Table 1 shows that there exists a 2-$(8,3)$ trade with volume equal to 24 (i.e., the statement is true for $m = 2$.) It is easy to observe that in this trade blocks like xyz such that $\{x,y,z\} \subset X_1$ or $\{x,y,z\} \subset X_2$ do not occur. Now, let $m > 2$ and $I_m = \{1, 2, \cdots, m\}$. To construct a trade T with vol$(T) = 4m(4m+1)(m-1)/3$, we partition the elements of $P_3(X)$ into 3 types as follows:

Type 1. This type consists of all blocks xyz such that

$$\{x,y,z\} \subset X_i,$$

for some $i \in I_m$.

Type 2. This type consists of all blocks xyz such that

$$\{x,y,z\} \cap X_i \neq \phi,\ \{x,y,z\} \cap X_j \neq \phi,\ \{x,y,z\} \subset (X_i \cup X_j),$$

for some $i, j \in I_m$.

Type 3. This type consists of all blocks xyz such that

$$|\{x,y,z\} \cap X_{i_j}| = 1,$$

for some $i_j \in I_m, 1 \leq j \leq 3$.

Blocks of Type 1 do not occur in T. But, all the blocks of Type 2 appear in T. To assign the blocks to T^+ and T^-, for each i, j, we establish a one-to-one correspondence between X_i and X_1, and X_j and X_2. Now partitioning the blocks of T into T^+ and T^- can be done as in Table 1. Also all the blocks of Type 3, do belong to T. Let xyz be a block of Type 3. If $x+y+z$ is even, then $xyz \in T^+$, and otherwise $xyz \in T^-$.

The number of blocks of T is equal to $\binom{v}{3} - \binom{4}{3}m = \binom{v}{3} - v = \frac{8m(4m+1)(m-1)}{3}$. Therefore, it suffices to show that $T = (T^+, T^-)$ is a 2-$(v,3)$ trade.

From the construction of T^+ and T^-, it is clear that $T^+ \cap T^- = \phi$. Suppose that $\{x,y\} \in P_2(X)$. If for an $i \in I_m$, $x, y \in X_i$, then

$$\begin{aligned}\lambda_{T^+}(xy) &= |\{B \in T^+ | \{x,y\} \subset B\}| \\ &= |\{B \in T^+ | B = xyz, z \notin X_i\}|.\end{aligned}$$

Since every pair appears twice in the positive part of the 2-$(8,3)$ trade of Table 1, hence, $\lambda_{T^+}(xy) = 2(m-1)$. A similar argument implies that $\lambda_{T^-} = 2(m-1)$.

Now, suppose for each $i \in I_m$, $\{x, y\} \notin X_i$, then there exists $i, j \in I_m$ such that $x \in X_i$ and $y \in X_j$, and

$$\begin{aligned}
\lambda_{T^+}(xy) &= |\{B \in T^+ | \{x,y\} \subset B\}| = |\{xyz \in T^+ | z \in X\}| \\
&= \sum_{l=1}^{m} |\{xyz \in T^+ | z \in X_l\}| \\
&= |\{xyz \in T^+ | z \in X_i\}| + |\{xyz \in T^+ | z \in X_j\}| \\
&\quad + \sum_{l \in I_m \setminus \{i,j\}} |\{xyz \in T^+ | z \in X_l\}| \\
&= 3 + \sum_{l \in I_m \setminus \{i,j\}} |\{z \in X_l | x + y + z \stackrel{2}{\equiv} 0\}| \\
&= 3 + \sum_{l \in I_m \setminus \{i,j\}} |\{z \in X_l | z \stackrel{2}{\equiv} 0\}| \\
&= 3 + \sum_{l \in I_m \setminus \{i,j\}} 2 = 2m - 1.
\end{aligned}$$

Similarly, $\lambda_{T^-}(xy) = 2m - 1$. Therefore, for every $x, y \in X$, we have

$$\lambda_{T^+}(xy) = \lambda_{T^-}(xy).$$

Thus, T is a trade.

Theorem 1. If v is even and $v = 4m + l$, then

$$\text{vol}\,(T_M(v)) = \frac{m(4m+1)(4m+3l-4)}{3}.$$

Proof. Let $v = 4m + 2$. In [8] it is shown that for any m, a 2-$(4m + 2, 3, \lambda^*/2)$ design exists. Therefore, one can find two disjoint designs such that together they cover $P_3(X)$. Now, we consider one of these designs as T^+ and the other as T^-. Thus they naturally form a trade of volume $\binom{v}{3}/2$. Hence

$$\text{vol}\,(T_M(v)) = \frac{v(v-1)(v-2)}{12} = \frac{m(4m+1)(4m+2)}{3}.$$

Now, let $v = 4m$. In this case we have

$$\lambda^* = 4m - 2, \quad r^* = (2m-1)(4m-1).$$

Suppose that T is a trade. Based on Proposition 1, $\lambda_{T^c}(xy)$ is even and $r_{T^c}(x)$ is odd. Therefore, for every $x \in X$, we have $r_{T^c}(x) \geq 1$ and $\lambda_{T^c}(xy)$ is even. Hence

$$\lambda_{T^c}(xy) \geq 2 \Rightarrow r_{T^c}(x) \geq 2 \Rightarrow r_{T^c}(x) \geq 3$$
$$|T^c| = \sum_{x \in X} \frac{r_{T^c}(x)}{3} \geq \sum_{x \in X} \frac{3}{3} = |X| = v.$$

Therefore, $|T^+ \cup T^-| \leq \binom{v}{3} - v$, which implies that

$$\text{vol}\,(T) \leq \frac{\binom{v}{3} - v}{2}.$$

Consequently

$$\text{vol}\,(T_M(v)) \leq \frac{4m(4m+1)(m-1)}{3},$$

Now from Lemma 1,

$$\text{vol}\,(T_M(v)) \geq \frac{4m(4m+1)(m-1)}{3}.$$

Therefore, equality holds.

5. Results for v odd

Lemma 2. If v is odd, then

$$\text{vol}\,(T_M(v)) \leq \frac{v(v-1)(v-3)}{12}.$$

Proof. Let $v = 2m+1$. Hence $\lambda^* = 2m-1$. λ^* is odd and it follows from Proposition 1 that for every $x, y \in X$, $\lambda_{T^c}(xy)$ is odd. Hence $\lambda_{T^c}(xy) \geq 1$, and

$$|T^c| = \sum_{xy \in P_2(X)} \frac{\lambda_{T^c}(xy)}{3} \geq \sum_{xy \in P_2(X)} \frac{1}{3} = \frac{|P_2(X)|}{3} = \frac{\binom{v}{2}}{3}$$
$$\Rightarrow 2\text{vol}(T) \leq \binom{v}{3} - \frac{\binom{v}{2}}{3}$$
$$\Rightarrow \text{vol}(T) \leq \frac{v(v-1)(v-3)}{12}.$$

Since this is true for any arbitrary trade, so we have the results.

Theorem 2. If $v \equiv 1$ or $3 \pmod 6$ and $v > 7$, then

$$\text{vol}\,(T_M(v)) = \frac{v(v-1)(v-3)}{12}.$$

Proof. It is well known that for $v \equiv 1$ or $3 \pmod 6$ and $v > 7$ large sets exists [11]. This means that $P_3(X)$ can be partitioned into $v-2$, 2-$(v,3,1)$ disjoint designs. We define T^+ as the union of $\frac{v-3}{2}$ of these designs and T^- as the union of $\frac{v-3}{2}$ designs of the remaining $v-2-\frac{v-3}{2}=\frac{v-1}{2}$ ones. Now we have a trade with volume equal to $\frac{v-3}{2}b$, where b is the number of blocks of the related Steiner triple system which is equal to $\frac{v(v-1)}{6}$. Thus $\text{vol}(T) = \frac{v(v-1)(v-3)}{12}$. Hence, by

$$\text{vol}\,(T_M(v)) \geq \frac{v(v-1)(v-3)}{12},$$

together with Lemma 1, we obtain the result.

Lemma 3. If $v = 7$, Then there does not exist a trade T such that $\text{vol}(T) = 13$.

Proof. Suppose there exists a trade T with $\text{vol}(T) = 13$. Then $|T^c| = 9$. Suppose, for $1 \leq i \leq 21, \alpha_i$ is the number of ith pair in T^c. Then we have $\sum_{i=1}^{21} \alpha_i = |T^c|\binom{3}{2} = 27$ and by Proposition 1, $\alpha_i's$ are all odd, and $1 \leq \alpha_i \leq \lambda^* = 5$. Thus two cases could happen:
(i) $\alpha_i = 3, \alpha_j = 5, \alpha_l = 1, \forall l \neq i,j$
(ii) $\alpha_i = \alpha_j = \alpha_l = 3, \alpha_n = 1, \forall n \neq i,j,l$.

Now, we examine both cases.

(i) Suppose for $t_1, t_2 \in X$, we have $\lambda_{T^c}(t_1t_2) = 5$. Now, since for any $x \in X$, $\lambda_{T^c}(t_1x) \geq 1$, therefore,

$$\{t_1t_2t_3, t_1t_2t_4, t_1t_2t_5, t_1t_2t_6, t_1t_2t_7\} \subseteq T^c,$$

where $X = \{t_1, t_2, \cdots, t_7\}$. Since for any $x \in X$, $\lambda_{T^c}(t_3x) \geq 1$, with no loss of generality, we let $t_3t_6t_7$ and $t_3t_4t_5$ to be blocks of T^c. In this case, blocks $t_4t_6t_7$ and $t_5t_6t_7$ must belong to T^c. This says that $r_{T^c}(t_6) = 4$. This is contrary to the assumption of oddness of $r_{T^c}(x)$ for any x.(See Proposition 1.)

(ii) Assume that π_1, π_2, and π_3 are elements of $P_2(X)$ such that $\lambda_{T^c}(\pi_i) = 3$, and hence for any $\pi \in P_2(X)$ and $\pi \neq \pi_i$, we have $\lambda_{T^c}(\pi) = 1$. Based on the intersection among π_i's, we realize five cases.

Case 1. $\pi_1 \cap \pi_2 = \pi_2 \cap \pi_3 = \pi_3 \cap \pi_1 = \phi$.

With no loss of generality, let $\pi_1 = \{1,2\}$, $\pi_2 = \{3,4\}$, and $\pi_3 = \{5,6\}$. The blocks of T^c should have the following form:

$B_1 := 12\ \boxed{7}$, $B_2 := 12\ \Box$, $B_3 := 12\ \Box$, $B_4 := 34\ \boxed{7}$, $B_5 := 34\ \Box$, $B_6 := 34\ \Box$, $B_7 := 56\ \boxed{7}$, $B_8 := 56\ \Box$, $B_9 := 56\ \Box$.

Since $\forall x \in X$, $\lambda_{T^c}(7x) = 1$, therefore one can assume that B_1, B_4, and B_7 contain 7. With no loss of generality, assume that $B_2 = 123$. Since 3 has appeared with the elements of the set $\{1,2,4,7\}$, we are forced to assume that

$$\{B_4, B_5\} = \{346, 345\}.$$

If $\Box$ of B_3 is 5, then 5 appears with all the elements of X. Thus, there remains no choice for B_8 and B_9. Hence we assume that $B_3 \neq 125$ and with a similar argument $B_3 \neq 126$. So, $B_3 = 124$. Consequently $\{B_8, B_9\} = \{561, 562\}$ and contrary to Proposition 1, $r_{T^c}(1) = 4$.

Case 2. $\pi_1 \cap \pi_2 \cap \pi_3 \neq \phi$.

Suppose that $\pi_1 = \{1,2\}$, $\pi_2 = \{1,3\}$, and $\pi_3 = \{1,4\}$ and let B_1, B_2, and B_3 be blocks of T^c which contain π_1. If for $1 \leq i \leq 3$, $3 \notin B_i$, then the three blocks B_4, B_5, and B_6 must contain 13. Therefore, there are six positions in $B_1, \cdots, B_6$ which have to be filled only with $4,5,6,7$. With a similar argument, if $4 \notin B_i$, a contradiction is attained. With no loss of generality assume that $B_1 = 123$, $B_2 = 124$, and $B_3 = 125$. If $134 \notin T^c$, then there should be 4 different blocks in T^c, which are different from B_1, B_2, and B_3, such that two of them contain 13 and the other two contain 14. Since 1 has appeared with the elements of $\{2, \cdots, 5\}$ in T^c, 4 unoccupied spots of these 4 blocks have to be filled with 6 and 7, which is impossibility. Thus $134 \in T^c$. But then both of 13 and 14 have to appear in another block too. Hence $r_{T^c}(1) = 6$. Again a contradiction to Proposition 1 is obtained.

Case 3. $\pi_1 \cap \pi_2 \cap \pi_3 = \phi$, and $\pi_i \cap \pi_j \neq \phi$, $1 \leq i, j \leq 3$.

Let $\pi_1 = \{1,2\}, \pi_2 = \{1,3\}$, and $\pi_3 = \{2,3\}$. If $123 \notin T^c$, then there are exactly three blocks which contain 12, three blocks containing 13, and three blocks containing 23. In this case $r_{T^c}(1) = 6$ and this is in contradiction with Proposition 1. Therefore, $B_1 = 123 \in T^c$. In this case two blocks (say B_2 and B_3) contain 12, two blocks (say B_4 and B_5) contain 13, and two blocks (say B_6, B_7) contain 23. The unfilled spots of B_2, B_3, B_4, and B_5 (which all contain 1) should be filled with the elements of $\{4,5,6,7\}$. Therefore, all the elements have appeared with one of the two elements 2 or 3. Thus for the unfilled spots of B_6 and B_7 there remain no choices.

Case 4. $\pi_1 \cap \pi_2 \cap \pi_3 = \phi$, $\pi_1 \cap \pi_2 \neq \phi$, $\pi_1 \cap \pi_3 = \phi$, and $\pi_2 \cap \pi_3 = \phi$.

Let $\pi_1 = \{1,2\}$, $\pi_2 = \{1,3\}$, and $\pi_3 = \{4,5\}$. If $123 \notin T^c$, then three blocks will contain 12 and three other blocks will contain 13. Therefore, there are altogether six blocks which contain 1, and we are left with 6 spots and 4 elements to choose from. Thus $123 \in T^c$. Besides the block 123, there are four other blocks which contain 1 and at the same time they have to be completed with $4,5,6,7$. With no loss of generality, assume that $124 \in T^c$, and also assume that B_1, B_2, and B_3 are 3 blocks of T^c which contain $\{4,5\}$. Now, if $135 \in T^c$, then for 3 unoccupied positions of B_1, B_2, and B_3, there are two choices, 6 and 7 which create the same problem. Thus, $135 \notin T^c$ and $125 \in T^c$. Therefore, $123, 124, 125, 136, 137, 345, 456, 457$ are the 8 blocks of T^c and consequently the 9th block should be 267, which implies that $r_{T^c}(2) = 4$. This is in contradiction with Proposition 1.

Case 5. $\pi_1 \cap \pi_2 \cap \pi_3 = \phi$, $\pi_1 \cap \pi_2 \neq \phi$, $\pi_1 \cap \pi_3 \neq \phi$, and $\pi_2 \cap \pi_3 = \phi$.

Let $\pi_1 = \{1,2\}$, $\pi_2 = \{1,3\}$, and $\pi_3 = \{2,4\}$. If $123 \notin T^c$, then there will be 6 blocks which contain 1, which is impossible. Likewise, $124 \notin T^c$ brings up a contradiction. Thus 123, $124 \in T^c$. There should be another block from T^c which contains 12, (with no loss of generality), suppose $125 \in T^c$. Then the unoccupied spots of the two blocks containing 13 (and the unoccupied spots which contain 14) are necessarily 6 and 7. Since the pairs $34, 35, 45, 56, 57, 67$ should occur in T^c, therefore, two more blocks of T^c must be of the form 345 and 567. Then $r_{T^c}(3) = 4$, which is again a contradiction to Proposition 1. Therefore, if $v = 7$, then there exists no simple trade of volume 13.

Theorem 3. $\mathrm{vol}(T_M(7)) = 12$.

Proof. Form Lemma 2, $\mathrm{vol}(T_M(7)) \leq 14$. If $\mathrm{vol}(T) = 14$, then $|T^c| = 7$. Since $\lambda^* = 5$, then with Proposition 1, for every $xy \in P_2(X)$, $\lambda_{T^c}(xy)$ is odd, therefore, $\lambda_{T^c}(xy)$ is at least 1. Since $|T^c| = 7$, therefore,

$$\sum_{xy \subset B \in T^c} \lambda_{T^c}(xy) = 7 \times 3 = 21.$$

From other hand $|P_2(X)| = 21$. Hence, for all $xy \in X$, $\lambda_{T^c}(xy) = 1$. Hence T^c is a 2-$(7,3,1)$ design.

$$\lambda_T(xy) = \lambda^* - \lambda_{T^c}(xy) = 4 \Rightarrow \lambda_{T^+}(xy) = 2.$$

Thus each of T^+ and T^- are a 2-$(7,3,2)$ design, and since every simple 2-$(7,3,2)$ is the union of two disjoint 2-$(7,3,1)$ designs [8], therefore, there is a large set for $v = 7$ which is impossible [11]. A trade with volume 12 is given in Table 2.

Now, by Lemma 3, the proof is complete.

Theorem 4. If $v = 6m + 5$, then

$$\text{vol}\,(T_M(v)) \leq 18m^3 + 33m^2 + 19m + 2.$$

Proof. With Lemma 2 we have

$$|T^c| \geq \frac{\binom{v}{2}}{3} = \frac{(6m+5)(3m+2)}{3},$$

and

$$(6m+5)(3m+2) \overset{3}{\equiv} 5.2 \overset{3}{\equiv} 1.$$

Therefore,

$$|T^c| \geq \frac{(6m+5)(3m+2)+l}{3},$$

where $l \equiv 2 \pmod 3$ and $l \geq 2$. Thus

$$\begin{aligned}\text{vol}\,(T) &= \frac{\binom{v}{3} - |T^c|}{2} \leq \frac{3(6m+5)(3m+2)(2m+1) - (6m+5)(3m+2)+2}{6}\\ &= 18m^3 + 33m^2 + 19m + 3.\end{aligned}$$

Now, if $l = 2$, then

$$\exists t_1, t_2 \in X, \lambda_{T^c}(t_1 t_2) = 3,$$

and for every $\pi \in P_2(X)$, $\pi \neq \{t_1, t_2\}$, we have

$$\lambda_{T^c}(\pi) = 1.$$

Let $t_3, t_4, t_5 \in X$ be such that

$$\{t_1t_2t_3, t_1t_2t_4, t_1t_2t_5) \subseteq T^c.$$

t_1 should appear with the rest of the elements of X in T^c. The number of these elements is $v - 5$, i.e., $6m$. Therefore, $3m$ other blocks of T^c must contain t_1, and hence

$$r_{T^c}(t_1) = 3m + 3.$$

We know that $r^* = (3m+2)(6m+3)$. Now if m is odd, then r^* is odd and $r_{T^c}(t_1)$ is even (contradiction to Proposition 1), and if m is even, then r^* is even, whereas $r_{T^c}(t_1)$ is odd (again a contradiction to Proposition 1).

Therefore, for $l > 2$, we have

$$\begin{aligned}&\text{vol}\,(T) < 18m^3 + 33m^2 + 19m + 3\\ &\Rightarrow \text{vol}\,(T_M(v)) \leq 18m^3 + 33m^2 + 19m + 2.\end{aligned}$$

Note. For $m = 1$ and 2 (i.e, $v = 11$ and $v = 17$), Tables 3 and 4, show that the inequality of Theorem 4 is in fact equality. It remains to show that the equality holds in general.

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maximal 2-(8,3) trade							
T^+				T^-			
1 2 5	1 5 7	2 4 7	3 5 6	1 2 6	1 5 6	2 4 8	3 5 7
1 2 8	1 6 7	2 5 6	3 5 8	1 2 7	1 5 8	2 5 7	3 6 7
1 3 5	1 6 8	2 6 7	3 7 8	1 3 6	1 7 8	2 5 8	3 6 8
1 3 7	2 3 6	2 7 8	4 5 7	1 3 8	2 3 5	2 6 8	4 5 6
1 4 6	2 3 8	3 4 6	4 5 8	1 4 5	2 3 7	3 4 5	4 6 7
1 4 8	2 4 5	3 4 8	4 6 8	1 4 7	2 4 6	3 4 7	4 7 8

Table 1.

maximal 2-(7,3) trade							
T^+				T^-			
1 2 6	1 3 7	2 3 4	3 4 6	1 2 4	1 3 6	2 3 7	3 4 7
1 2 7	1 4 5	2 4 7	3 6 7	1 2 5	1 5 7	2 4 6	3 5 6
1 3 5	1 4 6	2 5 6	5 6 7	1 3 4	1 6 7	2 6 7	4 5 6

Table 2.

maximal 2-(11,3) trade							
T^+				T^-			
1 2 4	1 9 11	3 4 11	4 7 10	1 2 6	1 8 11	3 4 6	4 9 10
1 2 5	1 10 11	3 5 6	4 8 9	1 2 7	1 9 10	3 4 8	5 6 10
1 2 8	2 3 4	3 5 7	4 8 10	1 2 9	2 3 5	3 4 10	5 6 11
1 2 10	2 3 6	3 5 8	4 9 11	1 2 11	2 3 8	3 5 9	5 7 8
1 3 4	2 3 7	3 5 10	5 6 8	1 3 5	2 3 9	3 6 7	5 7 10
1 3 6	2 3 11	3 6 9	5 6 9	1 3 7	2 3 10	3 6 8	5 8 9
1 3 8	2 4 7	3 7 8	5 7 9	1 3 10	2 4 8	3 6 11	5 8 10
1 3 9	2 5 7	3 7 10	5 7 11	1 3 11	2 4 10	3 7 9	5 9 11
1 4 5	2 5 9	3 8 11	5 8 11	1 4 6	2 4 11	3 7 11	6 7 8
1 4 10	2 5 10	3 9 10	5 0 11	1 4 7	2 5 6	3 8 10	6 7 10
1 5 9	2 6 8	3 10 11	6 7 9	1 4 8	2 5 8	3 9 11	6 8 9
1 5 11	2 6 10	4 5 6	6 7 11	1 4 9	2 5 11	4 5 7	6 9 10
1 6 7	2 6 11	4 5 8	6 8 10	1 5 6	2 6 7	4 5 9	6 9 11
1 6 8	2 7 11	4 5 10	6 10 11	1 5 7	2 6 9	4 5 11	7 8 9
1 6 10	2 8 9	4 6 7	7 8 10	1 5 10	2 7 9	4 6 8	7 8 11
1 7 8	2 9 10	4 6 9	7 9 10	1 6 11	2 7 10	4 6 10	7 10 11
1 7 9	2 9 11	4 6 11	8 9 10	1 8 9	2 10 11	4 7 9	8 10 11
1 7 11	3 4 9	4 7 8	8 9 11	1 8 10	3 4 5	4 7 11	9 10 11

Table 3.

maximal 2-(17,3) trade							
T^+				T^-			
1 2 5	2 6 15	4 7 13	6 14 17	1 2 3	2 7 9	4 7 8	6 13 17
1 2 7	2 7 10	4 7 14	6 15 17	1 2 4	2 7 12	4 7 9	6 14 15
1 2 8	2 7 13	4 7 16	6 16 17	1 2 6	2 7 14	4 7 12	6 15 16
1 2 10	2 7 15	4 8 9	7 8 9	1 2 9	2 7 16	4 7 15	7 8 10
1 2 11	2 7 17	4 8 10	7 8 12	1 2 13	2 8 10	4 7 17	7 8 11
1 2 14	2 8 9	4 8 11	7 8 14	1 2 15	2 8 11	4 8 13	7 8 13
1 2 16	2 8 12	4 8 12	7 8 15	1 2 17	2 8 13	4 8 14	7 8 16
1 3 4	2 8 14	4 9 10	7 8 17	1 3 6	2 8 15	4 8 16	7 9 13
1 3 5	2 8 16	4 9 12	7 9 11	1 3 8	2 8 17	4 8 17	7 9 15
1 3 7	2 9 11	4 10 13	7 9 12	1 3 9	2 9 10	4 9 11	7 9 17
1 3 10	2 9 14	4 10 14	7 9 14	1 3 12	2 9 12	4 9 13	7 10 11
1 3 11	2 9 15	4 10 15	7 9 16	1 3 15	2 9 13	4 9 14	7 10 13
1 3 13	2 10 12	4 11 12	7 10 12	1 3 17	2 9 16	4 9 15	7 10 14
1 3 14	2 10 13	4 11 16	7 10 15	1 4 5	2 9 17	4 9 16	7 10 16
1 4 6	2 10 15	4 11 17	7 10 17	1 4 7	2 10 14	4 9 17	7 11 12
1 4 8	2 10 16	4 12 13	7 11 13	1 4 11	2 11 12	4 10 11	7 11 15
1 4 10	2 10 17	4 12 17	7 11 14	1 4 14	2 11 14	4 10 12	7 11 17
1 4 12	2 11 15	4 13 14	7 11 16	1 4 15	2 11 16	4 10 16	7 12 17
1 4 13	2 12 13	4 13 17	7 12 13	1 4 17	2 11 17	4 11 13	7 13 15
1 4 16	2 12 15	4 14 15	7 12 14	1 5 8	2 12 14	4 11 14	7 14 16
1 5 6	2 12 16	4 14 16	7 12 16	1 5 9	2 12 17	4 11 15	7 14 17
1 5 7	2 13 16	4 15 16	7 13 14	1 5 11	2 13 14	4 12 15	7 15 16
1 5 10	2 13 17	4 15 17	7 13 16	1 5 12	2 13 15	4 12 16	7 15 17
1 5 15	2 14 17	4 16 17	7 14 15	1 5 13	2 14 15	4 13 15	8 9 10
1 5 17	2 15 17	5 6 7	7 16 17	1 5 16	2 16 17	4 13 16	8 9 12
1 6 7	3 4 5	5 6 11	8 9 14	1 6 10	3 4 7	4 14 17	8 9 13
1 6 8	3 4 6	5 6 12	8 10 12	1 6 12	3 4 8	5 6 8	8 9 15
1 6 9	3 4 9	5 6 13	8 10 15	1 6 14	3 4 10	5 6 9	8 9 16
1 6 13	3 4 12	5 6 16	8 10 17	1 6 15	3 4 13	5 6 14	8 9 17
1 6 17	3 4 15	5 7 8	8 11 13	1 6 16	3 4 14	5 6 15	8 10 11
1 7 11	3 4 17	5 7 11	8 11 14	1 7 9	3 4 16	5 6 17	8 10 13
1 7 12	3 5 9	5 7 15	8 11 16	1 7 10	3 5 6	5 7 9	8 10 14
1 7 17	3 5 12	5 7 17	8 11 17	1 7 13	3 5 7	5 7 10	8 11 12
1 8 9	3 5 14	5 8 9	8 12 13	1 7 14	3 5 8	5 7 12	8 11 15
1 8 11	3 5 16	5 8 10	8 13 15	1 7 15	3 5 10	5 7 13	8 12 14
1 8 12	3 6 10	5 8 12	8 13 16	1 7 16	3 5 11	5 7 14	8 12 15
1 8 17	3 6 11	5 8 13	8 14 15	1 8 10	3 5 15	5 7 16	8 12 16
1 9 12	3 6 12	5 8 15	8 14 17	1 8 13	3 5 17	5 8 11	8 12 17
1 9 13	3 6 13	5 8 16	8 15 16	1 8 14	3 6 7	5 8 14	8 13 17
1 9 15	3 6 14	5 9 11	8 16 17	1 8 15	3 6 8	5 8 17	8 14 16
1 9 16	3 6 16	5 9 12	9 10 11	1 8 16	3 6 9	5 9 10	8 15 17
1 9 17	3 7 8	5 9 13	9 10 12	1 9 10	3 6 17	5 9 14	9 10 14
1 10 11	3 7 9	5 10 11	9 10 13	1 9 11	3 7 11	5 9 15	9 10 16
1 10 14	3 7 10	5 10 12	9 10 15	1 9 14	3 7 12	5 9 17	9 11 14
1 10 15	3 7 15	5 10 13	9 10 17	1 10 12	3 7 13	5 10 15	9 11 17
1 11 15	3 7 16	5 10 14	9 11 13	1 10 16	3 8 12	5 10 17	9 12 13
1 11 16	3 7 17	5 10 16	9 11 15	1 10 17	3 8 14	5 11 13	9 12 15
1 12 14	3 8 9	5 11 12	9 12 14	1 11 12	3 8 16	5 11 14	9 12 17
1 12 15	3 8 10	5 11 17	9 12 16	1 11 13	3 9 10	5 11 16	9 13 16
1 12 16	3 8 11	5 12 15	9 13 14	1 11 14	3 9 11	5 12 13	9 14 15
1 13 15	3 8 13	5 13 14	9 13 17	1 11 17	3 9 12	5 12 14	10 11 12
1 13 16	3 8 15	5 13 16	9 14 16	1 12 13	3 9 14	5 12 16	10 11 14
1 13 17	3 9 13	5 14 15	9 14 17	1 12 17	3 9 16	5 13 15	10 11 15
1 14 15	3 9 15	5 14 17	9 15 16	1 13 14	3 10 11	5 13 17	10 12 13
1 14 16	3 9 17	5 15 17	9 15 17	1 15 16	3 10 13	5 14 16	10 12 15
1 14 17	3 10 14	5 16 17	10 11 13	1 16 17	3 10 15	5 15 16	10 12 16
2 3 5	3 10 16	6 7 9	10 11 16	2 3 4	3 11 12	6 7 8	10 13 15
2 3 8	3 10 17	6 7 10	10 11 17	2 3 6	3 11 15	6 7 11	10 13 17
2 3 11	3 11 13	6 7 13	10 12 14	2 3 7	3 11 16	6 7 12	10 14 16
2 3 12	3 11 14	6 7 15	10 12 17	2 3 10	3 12 13	6 7 14	10 14 17
2 3 13	3 11 17	6 8 10	10 13 14	2 3 15	3 12 14	6 7 17	10 15 17
2 3 14	3 12 15	6 8 13	10 13 16	2 3 17	3 13 14	6 8 9	10 16 17
2 3 16	3 12 16	6 8 14	10 15 16	2 4 5	3 13 16	6 8 11	11 12 16
2 4 6	3 12 17	6 8 16	11 12 13	2 4 10	3 13 17	6 8 15	11 13 15
2 4 7	3 13 15	6 8 17	11 12 14	2 4 12	3 14 15	6 9 11	11 13 16
2 4 8	3 14 17	6 9 10	11 12 15	2 4 13	3 14 16	6 9 12	11 13 17
2 4 9	3 15 16	6 9 16	11 12 17	2 4 15	3 15 17	6 9 13	11 14 15
2 4 11	4 5 9	6 9 17	11 13 14	2 5 10	3 16 17	6 9 15	11 16 17
2 4 14	4 5 11	6 10 11	11 14 16	2 5 11	4 5 6	6 10 12	12 13 14
2 4 17	4 5 13	6 10 14	11 15 16	2 5 12	4 5 8	6 10 13	12 14 15
2 5 6	4 5 14	6 10 16	11 15 17	2 5 13	4 5 10	6 10 15	12 14 17
2 5 7	4 5 15	6 11 12	12 13 15	2 5 15	4 5 12	6 10 17	12 15 16
2 5 9	4 5 16	6 11 14	12 13 17	2 5 16	4 5 17	6 11 13	12 15 17
2 5 14	4 6 7	6 11 15	12 14 16	2 6 7	4 6 10	6 11 16	13 14 16
2 5 17	4 6 8	6 12 14	12 16 17	2 6 8	4 6 11	6 11 17	13 14 17
2 6 9	4 6 9	6 12 15	13 14 15	2 6 10	4 6 12	6 12 13	13 15 16
2 6 11	4 6 15	6 12 17	13 15 17	2 6 14	4 6 14	6 12 16	14 15 17
2 6 12	4 7 10	6 13 15	13 16 17	2 6 16	4 6 16	6 13 14	14 16 17
2 6 13	4 7 11	6 14 16	14 15 16	2 7 8	4 6 17	6 13 16	15 16 17

Table 4.