A Note on the Cycle Structures of Automorphisms of 2-$(v, k, 1)$ Designs

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Let $\mathcal{D}$ be a 2-$(v, k, 1)$ design, and consider an automorphism g of $\mathcal{D}$. For any natural number n, we define p_n and b_n to be the number of cycles of length n of the permutations induced on the points and blocks, respectively. We call the points permuted in n-cycles by g *n-points*, and the blocks in n-cycles *n-blocks*.

If g is an automorphism of a projective plane, then the permutations induced on the points and blocks are similar [1]. Therefore $b_n = p_n$, for all n. We can see that $p_n \leq b_n$, whenever $n > 1$, for any automorphism g of an affine plane. In [4] it was shown that $p_n \leq b_n$ whenever $n > k$, for any automorphism g of a 2-$(v, k, 1)$ design. In this note we show that this inequality is also true for many values of n with $n \leq k$. Our main result is the following.

Theorem 1 *Let g be an automorphism of a 2-$(v, k, 1)$ design $\mathcal{D}$. Let n be a natural number. Suppose that every block containing two n-points is either an n-block or is fixed, and that there are at least two such blocks. Then $p_n \leq b_n$.*

In general, a block containing two n-points is permuted by g in a cycle of length dividing n. However, if n is a prime, any block containing two n-points is fixed or permuted in an n-cycle, proving the following corollary.

Corollary 1 *Let g be an automorphism of a 2-$(v, k, 1)$ design and let n be a prime. Suppose that not all n-points are contained in one block. Then $p_n \leq b_n$.*

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The condition that not all the n-points are collinear is necessary as there exist examples of Steiner triple systems [4] with automorphisms satisfying $b_n = 0$ and $p_n = 1$ for $n = 1, 2$ or 3, or $b_1 \in \{1, 2\}$ and $p_1 = 3$. In these examples, the n-points are collinear. One example with $b_3 = 0$ and $p_3 = 1$ is the 2-$(15, 3, 1)$ design with point-set $\{x, y, z\} \cup Z_{12}$ and automorphism of the points, $\pi = (x, y, z)(0, 1, \ldots, 11)$. This automorphism acts on the blocks with one fixed block, one 4-cycle, one 6-cycle and two 12-cycles given by the following starter blocks: $\{x, y, z\}$; $\{0, 4, 8\}$; $\{x, 0, 6\}$; $\{x, 1, 8\}$ and $\{0, 1, 3\}$. We do not know any example of a design satisfying $p_n < b_n$ for some integer n and such that the n-points are not collinear. In the following corollary we summerize the results of this paper and [4].

Corollary 2 *Let g be an automorphism of a 2-$(v, k, 1)$ design $\mathcal{D}$. If n is a positive integer, then $p_n \leq b_n$ whenever one of the following conditions is satisfied.*

(a) *Every n-cycle of points contains three non-collinear points.*

(b) *Each n-cycle consists of collinear points, and furthermore $np_n > k$.*

(c) *n is a prime.*

Proof of Theorem 1. We define $\mathcal{L}_0$ to be the substructure of $\mathcal{D}$ whose points are the n-points of $\mathcal{D}$ with respect to g and whose blocks are the blocks of $\mathcal{D}$ incident with two or more of the n-points. By the hypotheses of the theorem, $\mathcal{L}_0$ has more than one block and so is a linear space. Let $v_0 = n \cdot p_n$ be the number of points of $\mathcal{L}_0$ and let b_0 be the number of blocks of $\mathcal{L}_0$. We call the blocks of $\mathcal{L}_0$ *lines*.

If $n = 1$, then all points of $\mathcal{L}_0$ are fixed and so all lines of $\mathcal{L}_0$ are fixed. Since, by a result of de Bruijn and Erdős [2], every linear space has at least as many points as lines, we have $b_0 \geq v_0$ and thus $b_n = b_0 \geq v_0 = p_n$.

Now consider the case $n \geq 2$. Then the fixed lines of $\mathcal{L}_0$ are disjoint. We add a point q to $\mathcal{L}_0$ to form a new linear space $\mathcal{L}$. We do this in such a way that q lies on each of the fixed lines of $\mathcal{L}_0$ and for each point x that is not on a fixed line, we add the new line $\{q, x\}$ of length two, joining q with x. Then g extends uniquely to an automorphism of $\mathcal{L}$, which we also denote by g. It fixes q and maps each new line $\{q, x\}$ to the line $\{q, x^g\}$. We define v and b to be the number of points and lines of $\mathcal{L}$. Then $v = v_0 + 1$.

Assume that $p_n > b_n$. We shall derive a contradiction. Let r denote the number of lines of $\mathcal{L}$ on q. Then $b = b_n \cdot n + r$. Thus $b \leq (p_n - 1)n + r = v_0 - n + r = v + r - n - 1$. Since $n \geq 2$, it follows that $b < v + r - 2$. We can use the following result, which is a special case of Theorem 1 of [3]. In order to state the result, we need some terminology.

A set D consisting of some points of $\mathcal{L}$ is a *subspace*, if it contains the line joining any pair of its points. We call D *non-trivial*, if D contains three non-collinear points but does not contain all the points of $\mathcal{L}$. Given a non-trivial subspace D of a linear space $\mathcal{L}$, one can construct a new linear space $\mathcal{L}'$ by *smoothing* D, that is by removing all lines in D and by adjoining the set D itself as a new line.

Result 1 *Let $\mathcal{L}$ be a finite linear space with v points and b lines. Suppose that $\mathcal{L}$ has a point q such that $b < v + r - 2$, where r is the number of lines on q. Then one of the following cases occurs.*

(1) *$\mathcal{L}$ is a near-pencil.*

(2) *$\mathcal{L}$ can be obtained from a projective plane of order m by removing at most $m - 2$ points.*

(3) *$\mathcal{L}$ has $s \geq 1$ non-trivial subspaces $D_1, \ldots, D_s$ that pairwise meet in q. If one smoothes each subspace D_i, then the resulting linear space can be obtained from a projective plane of some order m by removing at most $m - 2$ points.*

Notice that Theorem 1 of [3] also covers the situation when $b = v + r - 2$. This is the reason that Theorem 1 of [3] has two more cases, and that it states that $m - 1$ instead of $m - 2$ points can be removed from a projective plane in the above cases (2) and (3).

It is obvious that $\mathcal{L}$ cannot be a near-pencil. Consider now the case that $\mathcal{L}$ can be obtained from a projective plane $\mathcal{P}$ of order m by removing a set T of at most $m - 2$ points. Then $r = m + 1$, $b = m^2 + m + 1$, and $v > b + 2 - r = m^2 + 2$. Since $|T| \leq m - 2$, all lines have at least three points in $\mathcal{L}$. This implies that all lines of $\mathcal{L}$ on q are fixed by g.

The automorphism g of $\mathcal{L}$ can be extended to an automorphism of $\mathcal{P}$. This can be seen as follows. Consider a point $p \in T$. Since $|T| \leq m - 2$, there exists a line L that meets T only in p. Then L^g meets T in a unique point p'. If H is a line other than L, then $p \in H$ iff H and L are disjoint in $\mathcal{L}$ iff H^g and L^g are disjoint in $\mathcal{L}$ iff $p' \in H^g$. Putting $p^g := p'$ for all points $p \in T$ gives the desired extension. The extended automorphism fixes the $m + 1$ lines through q. Since q is the only point of $\mathcal{L}$ fixed by g, the extension has at most $1 + |T| \leq m - 1$ fixed points. This provides a contradiction as every automorphism of a projective plane fixes an equal number of points and lines.

Finally we assume that $\mathcal{L}$ satisfies (3) of the above result. Let $D_1, \ldots, D_s$ be the non-trivial subspaces according to (3). Smoothing each D_i, we obtain a linear space $\mathcal{L}'$ that can be obtained from a projective plane $\mathcal{P}$ by removing a set T of at most $m - 2$ points, but no line. This implies that $\mathcal{L}'$ has $m^2 + m + 1$ lines. Therefore $\mathcal{L}'$ has exactly m^2 lines that do not contain q. Each subspace D_i of $\mathcal{L}$ contains at least one line that does not contain q. Thus $\mathcal{L}$ has at least $m^2 + s$ lines that do not contain q. Since q lies on r lines of $\mathcal{L}$, it follows that $m^2 + s + r \leq b$.

Put $|D_i| = m+1-d_i$, and $d := \sum_{i=1}^{s} d_i$. The point q lies on $m+1-s$ lines $L_1, \ldots, L_{m+1-s}$ that do not belong to any of the subspaces D_i. Let $m+1-l_i$ be the number of points of $\mathcal{L}$ on L_i, and put $l := \sum_i l_i$. Then $\mathcal{L}$ has $v = m^2+m+1-d-l$ points. Since $b \leq v+r-3$, we obtain $m^2+s+r \leq b \leq m^2+m-2-d-l+r$ and thus $d+l+s \leq m-2$.

Let D be the set consisting of the lines of $\mathcal{L}$ that are in one of the subspaces D_i. Each line L_i of $\mathcal{L}$ on q has $m+1-l_i \geq m+1-l \geq 3$ points and is therefore fixed by the action of g. Consider a line L of $\mathcal{L}$ that does not contain q and that is not in D. In $\mathcal{P}$, the line L meets each line $L_1, \ldots, L_{m+1-s}$. Since $l \leq m-2-s$ (so that at most $m-2-s$ lines L_i have less than $m+1$ points), it follows that L meets at least three of the lines L_i in a point of $\mathcal{L}$. Since the lines L_i are fixed, it follows that the image L^g contains at least three points that are not in any of the subspaces D_i. Hence $L^g \notin D$. Hence g maps any line that is not in D to a line that is also not in D. Therefore g fixes D.

Now consider a subspace D_i and a point $p \in D_i$ with $p \neq q$. Since the lines L_i are fixed by g, the point p^g lies in some subspace D_j. Consider a second point $x \in D_i$ with $x \neq p, q$. Then the line $X := px$ is a line of D, and hence the image $X^g = p^g x^g$ is a line of D, that is, X^g lies in some of the subspaces $D_1, \ldots, D_s$. As $p \in D_j$, this implies that X^g is a line of D_j and hence that x^g is a point of D_j. Hence g maps every point of D_i to a point of D_j. Since this holds for every subspace D_i, it follows that g permutes the subspaces D_i.

Since $\mathcal{L}'$ is the linear space obtained from $\mathcal{L}$ by smoothing the subspaces D_i, this implies that g induces an automorphism g' of $\mathcal{L}'$. This automorphism fixes the lines L_i and thus g' fixes at least $m+1-s$ lines of $\mathcal{L}'$, so that the number of non-fixed lines of $\mathcal{L}'$ under the action of g' is at most m^2+s. The number of points of $\mathcal{L}'$ permuted by g' in cycles of length n is $v-1 = m^2+m-d-l \geq m^2+m-(m-1-s) = m^2+s+1$. Thus g' has more n-cycles on points than on lines. Since $\mathcal{L}'$ satisfies the conclusion in (2) of the above result, we obtain a contradiction in the same way as before (that is, we extend g' to an automorphism of the projective plane $\mathcal{P}$, where it then has more fixed points than fixed lines).

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