Generalized Exponents of Primitive, Nearly Reducible Matrices*

Bolian Liu
Department of Mathematics
South China Normal University
Guangzhou
P.R. of China

ABSTRACT. In this paper, the k-exponent and the kth upper multiexponent of primitive nearly reducible matrices are obtained and bound on the kth lower multiexponent of this kind of matrices is given.

1 Introduction

A square nonnegative matrix A is *primitive* if $A^k > 0$ for some positive integer k. The smallest such k for a given matrix A is called the *exponent* of primitivity (sometimes called *index* of primitivity). A is *reducible* if there exists a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_1 & 0 \\ X & A_2 \end{bmatrix}$$

where A_1 the A_2 are square (nonvacuous) matrices. The matrix A is *irreducible* if it is not reducible. A is *nearly reducible* if A is irreducible but each matrix obtained from A by replacing a nonzero entry by zero is reducible. It is known that the combinatorial properties of A depend only upon the pattern of A. So the exponent of nonnegative matrices can be conveniently described by the study of the corresponding Boolean matrices.

With an $n \times n$ matrix $A = (a_{i,j})$, there is an associated digraph $D(A)$.The vertices of $D(A)$ are $1, 2, \ldots, n$ with an arc from i to j if and only if $a_{i,j} \neq 0$ $(i, j = 1, \ldots, n)$. The digraph $D(A)$ is termed *primitive*, if A is primitive. It is well known that $D(A)$ is primitive if and only if $D(A)$ is strongly

*This research was supported by NNSF of P.R. China.

connected and the greatest common divisor of the lengths of its (simple) cycles is 1. It is clear that a matrix A is nearly reducible if and only if $D(A)$ is a minimally strong digraph. The properties of minimally strong digraphs can be found in [1].

The problem of characterizing the set of exponents for primitive matrices has recently been completely settled ([2] [3] [4]). The study of exponents for primitive nearly reducible matrices has had great progress ([1] [6] [5]). From the background of a memoryless communication system, R.A. Brualdi and B.L. Liu recently generalized the concept of exponent for a primitive digraph (primitive matrix) and introduced some new parameters related to the exponent as follows ([7]).

Let Γ denote a digraph with n vertices $1, 2, \ldots, n$. Let

$$\exp_\Gamma(i,j) := \text{the smallest integer } p \text{ such that there is a walk of length } t \text{ from } i \text{ to } j \text{ for each integer } t \geq p \ (1 \leq i,j \leq n).$$

The digraph Γ is called primitive provided all of the numbers $\exp_\Gamma(i,j)$ are finite, and the number

$$\exp(\Gamma) := \mathrm{MAX}_{i,j}\{\exp_\Gamma(i,j)\}$$

is called the exponent of Γ. As in [7] let

$$\exp(n) := \mathrm{MAX}_\Gamma\{\exp(\Gamma)\},$$

where the maximum is taken over all the primitive digraphs Γ with n vertices. It is well known (see e.g. [1])

$$\exp(n) = n^2 - 2n + 2,$$

Let the exponent of vertex i be defined by

$$\exp_\Gamma(i) := \mathrm{MAX}\{\exp_\Gamma(i,j)\} \ (i = 1, \ldots, n).$$

Thus $\exp_\Gamma(i)$ is the smallest integer p such that there is a walk of length p from i to each vertex j of Γ. It follows also that

$$\exp(\Gamma) = \mathrm{MAX}_i\{\exp_\Gamma(i)\}.$$

We choose to order the vertices of Γ in such a way that

$$\exp_\Gamma(1) \leq \exp_\Gamma(2) \leq \cdots \leq \exp_\Gamma(n).$$

Hence $\exp(\Gamma) = \exp_\Gamma(n)$. We define

$$\exp(n,k) := \mathrm{MAX}_\Gamma\{\exp_\Gamma(k)\}, \ (k = 1, \ldots, n),$$

where the maximum is taken over all primitive digraphs with n vertices. The number $\exp(n, k)$ is called k-exponent of primitive digraphs (primitive matrices). It follows that $\exp(n, n) = \exp(n)$.

In [7] we introduced the exponent for a subset X of k vertices of Γ, where $1 \leq k \leq n$.

$$\begin{array}{c}\exp_\Gamma(X) = \text{the smallest integer } p \text{ such that for each} \\ \text{vertex } i \text{ of } \Gamma \text{ there exists a walk of length } p \\ \text{from at least one vertex in } X \text{ to } i.\end{array}$$

The number

$$f(\Gamma, k) := \mathrm{MIN}_X\{\exp(X)\},$$

where the minimum is taken over all subsets X of k of the vertices of Γ, is called the kth lower multiexponent of Γ. The number

$$F(\Gamma, k) := \mathrm{MAX}_X\{\exp_\Gamma(X)\}.$$

where the maximum is taken over all subset X of k of the vertices of Γ, is called the kth upper multiexponent of Γ. It follows that $F(\Gamma, 1) = \exp_\Gamma(n)$ and $f(\Gamma, 1) = \exp_\Gamma(1)$. We let

$$f(n, k) := \mathrm{MAX}_\Gamma f(\Gamma, k), \quad (k = 1, \ldots, n),$$
$$F(n, k) := \mathrm{MAX}_\Gamma F(\Gamma, k), \quad (k = 1, \ldots, n),$$

where the maximum is taken over all primitive digraphs with n vertices. We observe that $f(n, n) = 0$, $f(n, 1) = \exp(n, 1)$ and $F(n, 1) = \exp(n)$.

Applying the adjacency matrix of a digraph, we can understand the interpretation for the above parameters in terms of a matrix (see [7]). In [7], we have obtained bounds for the numbers introduced above and evaluate all of the corresponding number for primitive symmetric digraphs. In this paper, For primitive, nearly reducible matrices, we evaluate $\exp(n, k)$, $F(n, k)$ and estimate $f(n, k)$.

In the following, unless special statement, all digraphs Γ concerned in this paper are associated digraphs for primitive, nearly reducible matrices – primitive minimally strong digraphs and $\exp(n, k)$, $f(n, k)$, $F(n, k)$ represent corresponding parameters for primitive, minimally strong digraph.

We shall first investigate $\exp(n, k)$.

2 k-exponent of primitive, nearly reducible matrices

The following lemmas are readily established.

Lemma 1. ([3]). *Let Γ be a primitive digraph and $L = \{r_1, r_2\}$ be the set of the lengths of its (simple) cycles, where g.c.d. $(r_1, r_2) = 1$. $d_L(i, j)$*

denotes the length of the shortest walk from i to j which meets at least one circuit of each length r_k for $k = 1, 2$. Then

$$\exp_\Gamma(i,j) \le d_L(i,j) + (r_1 - 1)(r_2 - 1).$$

Lemma 2. ([1]). *For primitive, nearly reducible $n \times n$ matrices,*

$$\exp(n) = n^2 - 4n + 6.$$

Lemma 3. ([5]). *There exist gaps in the exponent set of primitive, minimally strong digraphs on n vertices (primitive, nearly reducible $n \times n$ matrices) $(n^2 - 6n + 12, n^2 - 5n + 9)$ and $(n^2 - 5n + 9, n^2 - 4n + 6)$.*

Lemma 4. ([1], [5]). *Let D_s be a primitive, minimally strong digraph on n vertices with the shortest cycle of length s. D_{n-2} is the unique digraph as Figure 1 and $\exp_{D_{n-2}}(n) = n^2 - 4n + 6$. There are exactly two D_{n-3} as $D^{(1)}_{n-3}$, $D^{(2)}_{n-3}$ in Figure 2 and $\exp_{D^{(1)}_{n-3}} = n^2 - 6n + 12$, $\exp_{D^{(2)}_{n-3}}(n) = n^2 - 5n + 9$.*

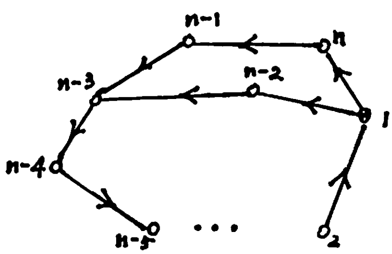
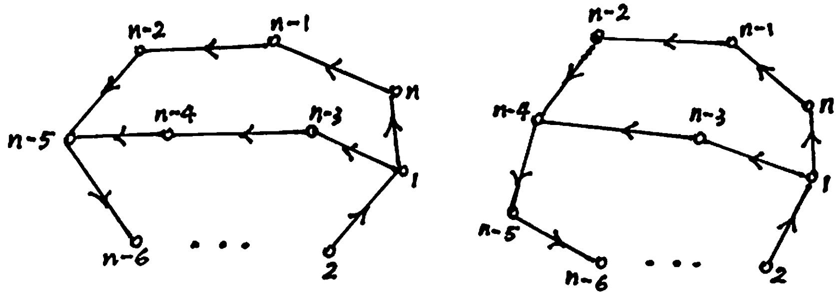


Figure 1. D_{n-2}



$D^{(1)}_{n-3}$ $\quad$ $D^{(2)}_{n-3}$ $n \equiv 0 \pmod 2$

Figure 2.

Lemma 5. ([7]). *Let* Γ *be a primitive digraph on* n *vertices. Then* $\exp_\Gamma(k) \le \exp_\Gamma(k-1)+1$ $(2 \le k \le n)$. *Hence* $\exp(n,k) \le \exp(n,k-1)+1$.

Lemma 6. ([7]). *Let* Γ *be a primitive digraph on* n *vertices with the shorter cycle of length* s. *Then*

$$\exp_\Gamma(k) \le \begin{cases} s(n-1) & \text{if } k \le s, \\ s(n-1+k-s) & \text{if } k > s. \end{cases}$$

Lemma 7. *Let* Γ *be a primitive, minimally strong digraph on* n *vertices. Then*

$$\exp_\Gamma(1) \le n^2 - 5n + 8.$$

Proof: Let the length of the shortest cycle of Γ be s. Then $2 \le s \le n-2$.

(1) If $2 \le s \le n-4$, according to lemma 6

$$\exp_\Gamma(1) \le (n-4)(n-1) = n^2 - 5n + 4 < n^2 - 5n + 8.$$

(2) If $s = n-3$, by lemma 4, there are exactly two digraphs $D_{n-3}^{(1)}$ and $D_{n-3}^{(2)}$ as in Figure 2.

Case 1. $\Gamma = D_{n-3}^{(1)}$, $L = \{n-3, n-2\}$, $n > 1$. By lemma 1

$$\begin{aligned} \exp_\Gamma(1) &\le (n-4)(n-3) + \max_j \ d_L(1,j) \\ &= (n-4)(n-3) + d_L(1,2) \\ &= (n-4)(n-3) + n - 4 \\ &= n^2 - 6n + 8 < n^2 - 5n + 8. \end{aligned}$$

For other vertices $i \ne 1$, it is easy to see that

$$\exp_\Gamma(i) \ge \exp_\Gamma(1).$$

Thus when $\Gamma = D_{n-3}^{(1)}$

$$\exp_\Gamma(1) \le n^2 - 6n + 9 < n^2 - 5n + 8.$$

Case 2. $\Gamma = D_{n-3}^{(2)} n \equiv 0 \pmod 2$, $L = \{n-3, n-1\}$. By lemma 1

$$\begin{aligned} \exp_\Gamma(1) &\le (n-4)(n-2) + \max_j \ d_L(1,j) \\ &= (n-4)(n-2) + d_L(1,2) \\ &= (n-4)(n-2) + n - 4 \\ &= n^2 - 5n + 4 < n^2 - 5n + 8. \end{aligned}$$

For other vertices $i \neq 1$,

$$\exp_\Gamma(i) \geq \exp_\Gamma(1).$$

Thus when $\Gamma = D_{n-3}^{(2)}$,

$$\exp_\Gamma(1) \leq n^2 - 5n + 5 < n^2 - 5n + 8.$$

(3) If $s = n-2$, by lemma 4, there is the only digraph D_{n-2} (see Figure 1), $L = \{n-2, n-1\}$. Similarly, we have

$$\exp_\Gamma(1) \leq (n-2)(n-3) + 2 = n^2 - 5n + 8.$$

According to (1) (2) (3), the lemma holds. □

Lemma 8. *If $\Gamma = D_{n-2}$, then*

$$\exp_\Gamma(k) = \begin{cases} n^2 - 5n + 7 + k & \text{if } 1 \leq k \leq n-2, \\ n^2 - 4n + 5 & \text{if } k = n-1, \\ n^2 - 4n + 6 & \text{if } k = n. \end{cases}$$

Proof: Observing digraph $\Gamma = D_{n-2}$ (Figure 1), we can verify

$$\exp_\Gamma(1) \leq \cdots \leq \exp_\Gamma(2) \leq \cdots \leq \exp_\Gamma(n-2) \leq \exp_\Gamma(n-1) \leq \exp_\Gamma(n).$$

By lemma 1

$$\exp_\Gamma(k) = (n-2)(n-3) + 2 + (k-1) = n^2 - 5n + 7 + k, 1 \leq k \leq n-2,$$
$$\exp_\Gamma(n-1) = (n-2)(n-3) + 2 + (n-3) = n^2 - 4n + 5,$$
$$\exp_\Gamma(n) = n^2 - 4n + 6.$$

□

Lemma 9. *If $\Gamma = D_{n-3}^{(2)}$, $n \equiv 0 \pmod 2$, then*

$$\exp_\Gamma(n-1) \leq n^2 - 5n + 8.$$

Proof: The following relation is straightforward to verify

$$\exp_\Gamma(1) \leq \cdots \leq \exp_\Gamma(2) \leq \cdots \leq \exp_\Gamma(n-2) \leq \exp_\Gamma(n-1) \leq \exp_\Gamma(n).$$

By lemma 1, $\exp_\Gamma(n-1) \leq (n-4)(n-2) + n = n^2 - 5n + 8$. □

Theorem 1. *For primitive, nearly reducible $n \times n$ matrices*

$$\exp(n,k) = \begin{cases} n^2 - 5n + 7 & \text{if } 1 \leq k \leq n-2, \\ n^2 - 4n + 5 & \text{if } k = n-1, \\ n^2 - 4n + 6 & \text{if } k = n. \end{cases}$$

Proof: By lemma 5 and lemma 7, for all the primitive minimally strong digraphs Γ with n vertices

$$\exp_\Gamma(k) \le \exp_\Gamma(1) + k - 1 \le n^2 - 5n + 8 + k - 1 = n^2 - 5n + 7 + k.$$

By lemma 8, there exists a primitive minimally strong digraph D_{n-2} such that

$$\exp_{D_{n-2}}(k) = n^2 - 5n + 7 + k, 1 \le k \le n-2.$$

So $\exp(n,k) = \text{MAX}_\Gamma\{\exp_\Gamma(k)\} = n^2 - 5n + 7 + k$, $1 \le k \le n-2$, where the maximum is taken over all the primitive minimally strong digraph Γ with n vertices. In addition by lemma 2, $\exp(n,n) = n^2 - 4n + 6$.

We now consider $\exp(n, n-1)$. By lemma 3 there are no such primitive, nearly minimally strong digraphs Γ that

$$n^2 - 6n + 12 < \exp_\Gamma(n) < n^2 - 5n + 9,$$
$$n^2 - 5n + 9 < \exp_\Gamma(n) < n^2 - 4n + 6.$$

Since $\exp_\Gamma(n-1) \le \exp_\Gamma(n)$, by lemma 4, there is the unique primitive minimally strong digraph D_{n-2} such that $\exp_{D_{n-2}}(n) = n^2 - 4n + 6$ and the unique $D^{(2)}_{n-3}$ such that $\exp_{D^{(2)}_{n-3}}(n) = n^2 - 5n + 9$, where $n \equiv 0 \pmod 2$. Hence for all primitive minimally strong digraphs Γ with n vertices except D_{n-2} and $D^{(2)}_{n-3}$ we have

$$\exp_\Gamma(n-1) \le \exp_\Gamma(n) \le n^2 - 6n + 12.$$

By lemma 8, $\exp_{D_{n-2}}(n-1) = n^2 - 4n + 5$.

By lemma 9, $\exp_{D^{(2)}_{n-3}}(n-1) \le n^2 - 5n + 8$.

So $\exp_{D^{(2)}_{n-3}}(n-1) < \exp_{D_{n-2}}(n-1) = n^2 - 4n + 5$.

Hence $\exp(n, n-1) = \text{MAX}_\Gamma\{\exp_\Gamma(n-1)\} = \exp_{D_{n-2}}(n-1) = n^2 - 4n + 5$. □

3 On the lower and upper multiexponent of primitive, nearly reducible matrices

The problem to evaluate the kth lower and upper multiexponent for primitive matrices seems to be rather difficult. In [7], we have given its bounds as follows.

Lemma 10. *Let Γ be a primitive digraph with n vertices having a cycle of length s. Then*

$$f(\Gamma, k) \le \begin{cases} n-k & \text{if } s \le k \le n \\ 1 + s(n-k-1) & \text{if } k < s \end{cases}.$$

Thus we can estimate the $f(\Gamma, k)$ for all primitive minimally strong digraphs Γ as follow.

Theorem 2. *For all primitive nearly reducible matrices,*

$$f(n,k) \leq n^2 - (3+k)n + 2k + 3, \;\; 1 \leq k \leq n-1.$$

Proof: By lemma 10, when $1 \leq k \leq n-1$

$$f(n,k) \leq \max(n-k, 1+s(n-k-1)) = 1 + s(n-k-1).$$

Since Γ is primitive minimally strong digraph, Γ must contain a cycle of length s where $2 \leq s \leq n-2$. Thus

$$f(n,k) \leq 1 + (n-2)(n-k-1) = n - (3+k)n + 2k + 3.$$

□

Corollary 2.1. *For all nearly reducible matrices*

$$f(n, n-1) = 1.$$

Proof: By theorem 2, $f(n, n-1) \leq 1$. It is easy to see that when $\Gamma = D_{n-2}$, $f(\Gamma, n-1) = 1$. □

For the k-upper multiexponent, from [7] we have

Lemma 11. *Let Γ be a primitive minimally strong digraph with n vertices having a cycle of length s. Then*

$$F(\Gamma, k) \leq \begin{cases} s(n-1) & \text{if } k > n-s \\ s(2n-s-k) & \text{if } k \leq n-s \end{cases}$$

where $2 \leq s \leq n-2$.

The bounds for $f(n,k)$ and $F(n,k)$ given in theorem 2 and lemma 11 are not perfect. It is well known (see [1]) that the digraph D_{n-2} is the unique primitive minimally strong digraph with n vertices for which $\exp(D_{n-2}) = \exp(n) = n^2 - 4n + 6$. According to above results in this paper, for all primitive minimally strong digraph, $\exp_{D_{n-2}}(k) = \exp(n,k)$. We believe that D_{n-2} is also the unique digraph with n vertices for which $f(D_{n-2}, k) = f(n,k)$ and $F(D_{n-2}, k) = F(n,k)$ for all primitive minimally strong digraphs and $1 \leq k \leq n$.

We note that the proof of theorem 2.4 in [7] can be easily modified to give the following result.

Lemma 12.

$$f(D_{n-2},k)=\begin{cases}1 & n-1\le k\le n\\ 2+(2n-4-k)\lfloor (n-2)/k\rfloor-\lfloor (n-2)/k\rfloor^2 k, & 1\le k\le n-2\end{cases},$$

$$F(D_{n-2},k)=\begin{cases}n^2-4n+6 & k=1\\ (n-1)^2-k(n-2) & 2\le k\le n\end{cases}.$$

Hence in view of lemma 12 we make the following conjecture.

Conjecture. For all primitive nearly reducible $n\times n$ matrices

$$F(n,k)=F(D_{n-2},k),\quad \text{(Conjecture 1)}$$
$$f(n,k)=f(D_{n-2},k),\quad \text{(Conjecture 2)}$$

We will show that conjecture 1 is true.

We first establish the following lemma.

Lemma 13. *$F(n,k)=F(D_{n-2},k)$ for $k=n$, $n-1$.*

Proof: It is obvious by definition that $F(n,n)=1$. By lemma 12 $F(D_{n-2},n)$ $=(n-1)^2-n(n-2)=1$. Hence $F(n,n)=F(D_{n-2},n)$.

In [7], we had the following conclusion (Lemma 5.2, [7]): Let Γ be a primitive digraph with n vertices and let s and t be, respectively, the lengths of the shortest and longest cycles of Γ, Then $F(\Gamma,n-1)\le \max\{n-s,t\}$.

Now Γ is a primitive minimally strong digraph, $2\le s\le n-2$, $t\le n-1$. So $F(\Gamma,n-1)\le \max\{n-2,n-1\}=n-1$. And by lemma 12 $F(D_{n-2},n-1)=(n-1)^2-(n-1)(n-2)=n-1$, then $F(n,n-1)=F(D_{n-2},n-1)=n-1$. □

In [8] we have shown the following lemma.

Lemma 14. ([8]). *For any primitive digraph Γ with the shortest cycle of length s, $F(\Gamma,k)\le n-s+s(n-k)$, $1\le k\le n-1$.*

We now show conjecture 1 as follows.

Theorem 3. *For all primitive nearly reducible $n\times n$ matrices*

$$F(n,k)=F(D_{n-2},k)=\begin{cases}n^2-4n+6 & k=1\\ (n-1)^2-k(n-2) & 2\le k\le n\end{cases}.$$

Proof: $F(n,1)=\exp(n)=n^2-4n+6$. By lemma 12 $F(D_{n-2},1)=n^2-4n+6$. Hence $F(n,1)=F(D_{n-2},1)=n^2-4n+6$.

By lemma 13 $F(n,k)=F(D_{n-2},k)$ for $k=n$, $n-1$. Thus we consider only the case that $2\le k\le n-2$.

Let Γ be a primitive minimally strong digraph with the shortest cycle of length s. Clearly, $s \le n-2$.

If $s \le n-3$, by lemma 14

$$\begin{aligned} F(\Gamma, k) &\le n - s + s(n-k) \\ &= n + s(n-k-1) \\ &\le n + (n-3)(n-k-1) \\ &= 3 + (n-3)(n-k) \\ &\le 1 + (n-2)(n-k) \\ &= (n-1)^2 - k(n-2) \end{aligned}$$

for $2 \le k \le n-2$.

If $s = n-2$, there is the only minimally strong digraph D_{n-2} (Figure 1). Thus

$$F(\Gamma, k) = F(D_{n-2}, k) \text{ for } 2 \le k \le n-2.$$

To sum up, we complete the proof of theorem 3. □

Next we will show that conjecture 2 is true for $k = 1,\ n,\ n-1,\ n-2$.

(1) $k = 1$.

It has been known (theorem 1) that $f(n,1) = \exp(n,1) = n^2 - 5n + 8$. By lemma 12, $f(D_{n-2}, 1) = 2 + (2n-5)(n-2) - (n-2)^2 = n^2 - 5n + 8$. Hence $f(n,1) = f(D_{n-2}, 1)$.

(2) $k = n$.

By definition $f(n,n) = 1$. By lemma 12 $f(D_{n-2}, n) = 1$. Hence $f(n,n) = f(D_{n-2}, n)$.

(3) $k = n-1$.

By corollary 2.1 $f(n, n-1) = 1$. By lemma 12 $f(D_{n-2}, n-1) = 1$. Thus $f(n, n-1) = f(D_{n-2}, n-1)$.

(4) $k = n-2$.

By lemma 12 we know that $f(D_{n-2}, n-2) = 2$. We now show that $f(n, n-2) = 2$.

Proof: Let Γ be a primitive minimally strong digraph. Then Γ contains at least one subgraph containing two cycles whose lengths are different as Figure 3.

The two paths from C to D are denoted by P_1 and P_2 respectively. Their lengths are denoted by $L(P_1)$ and $L(P_2)$ respectively. Since $L(P_1) \ne L(P_2)$, say $L(P_1) > L(P_2) \ge 0$. We denote by $R_t(X)$ the set of vertices of Γ which can be reached by a walk of length t which begins at every vertex of X.

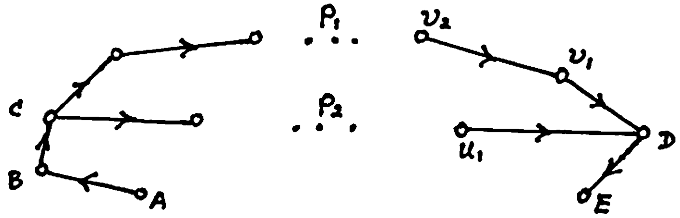


Figure 3

(1) If $L(P_2) = 0$, it means that C and D coincide. In this case, we knew ([1]) that $n \geq 4$; Without loss of generality, let the length of a cycle containing P_2 be not less than 3. And let $X = V(\Gamma) \setminus \{A, B\}$ where $|X| = n - 2$. Clearly $R_2(X) = V(\Gamma)$.

(2) If $L(P_2) > 0$, according to the properties of the minimally strong digraph (see [1]), $L(P_2) \geq 2$, $L(P_1) \geq 3$. Let v_1, v_2 be two vertices nearest to vertex D on path P_1. And let $X = V(\Gamma) \setminus \{v_1, v_2\}$, then $|X| = n-2$ and $R_2(X) = V(\Gamma)$. Hence $f(n, n-2) = 2$ and $f(n, n-2) = f(D_{n-2}, n-2)$.

According to (1), (2), (3), (4), it follows that conjecture 2 is true for $k = 1, n, n-1, n-2$.

References

[1] R.A. Brualdi and J.A. Ross, On the exponent of a primitive nearly reducible matrix, *Math. Oper. Res.* **5** (1980), 229–241.

[2] M. Lewin and Y. Vitek, A system of gaps in the exponent set of primitive matrices, *Illinois J. Math.*, **25** (1981), 87–98.

[3] J.Y. Shao, On a conjecture about the exponent set of primitive matrices, *Linear Alg. Appl.* **65** (1985), 91–123.

[4] K.M. Zhang, On Lewin and Vitek's conjecture about the exponent set of primitive matrices, *Linear Alg. Appl.* **96** (1987), 101–108.

[5] J.A. Ross, On the exponent of a primitive, nearly reducible matrix II, *SIAM J. Alg. Disc. Math.* **3** (1982), 398–410.

[6] Li Qiao and Shao Jiayu, The index set problem for Boolean (or nonnegative) matrices, *Disc. Math.* **123** (1993)1–3, 75–92.

[7] R.A. Brualdi and Bolian Liu, Generalized exponents of primitive directed graphs, *J. Graph Theory* **4** (1990), 483–499.

[8] Bolian Liu and Li Qiaoliang, On a conjecture about the generalized exponent of primitive matrices, *J. Graph Theory* **18** (1994), 177–179.