Permutation graphs and Petersen graph

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ABSTRACT. It was proved by Ellingham (1984) that every permutation graph either contains a subdivision of the Petersen graph or is edge-3-colorable. This theorem is an important partial result of Tutte's Edge-3-Coloring Conjecture and is also very useful in the study of the Cycle Double Cover Conjecture. The main result in this paper is that every permutation graph contains either a subdivision of the Petersen graph or two 4-circuits and therefore provides an alternative proof of the theorem of Ellingham. A corollary of the main result in this paper is that every uniquely edge-3-colorable permutation graph of order at least eight must contain a subdivision of the Petersen graph.

1 Introduction

A *cubic graph* is a 3-regular simple graph. A *2-factor* of a graph G is a 2-regular spanning subgraph of G. The *underlying graph* of a graph G, denoted by $\overline{G}$, is the graph homeomorphic to G and containing no degree two vertex. A *chord* of a circuit C is an edge not in C with both endvertices in $V(C)$. A cubic graph G is called a *permutation graph* if G has a 2-factor F which is the union of two chordless circuits. All other graph-theoretic terms that are used in this paper can be found, for instance, in [6].

The following well-known conjecture due to Tutte is a generalization of the 4-color problem ([3, 4, 5, 11]).

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Conjecture 1 (The Edge-3-coloring Conjecture, Tutte [12]) *Every 2-edge-connected cubic graph containing no subdivision of the Petersen graph is edge-3-colorable.*

One of the more successful approaches to this conjecture is the following theorem due to Ellingham.

Theorem 2 (Ellingham [7]) *If G is a permutation graph containing no subdivision of the Petersen graph, then*

(i) *G contains a 4-circuit,*

(ii) *G contains a Hamilton circuit,*

(iii) *G is edge-3-colorable.*

Theorem 2 (iii) is useful in the study of cycle cover problems and is one of the key lemmas in the proof of the following result.

Theorem 3 (Alspach, Goddyn and Zhang [1, 2]) *A minimal counterexample to the Cycle Double Cover Conjecture must contain a subdivision of the Petersen graph.*

The main result of this paper (Theorem 4) is a strengthening of the conclusion of Theorem 2 (i) to get information about the structure of permutation graphs with no subdivision of the Petersen graph. This stronger theorem is then used to obtain a result about uniquely edge-3-colorable permutation graphs.

Definition 1 *Let G be permutation graph with at least four vertices and F be a 2-factor of G which is the union of two chordless circuits and $M = G \setminus E(F)$. A circuit of length four containing exactly two edges of M is called an M-C_4. A subdivision of the Petersen graph in G is called a P_{10}-*subgraph. *A P_{10}-subgraph which has a 2-factor consisting of all edges of F (so that it has a perfect matching consisting of five edges of M) is called an M-P_{10}*-subgraph.

Theorem 4 *Let G be permutation graph with at least four vertices and F be a 2-factor of G which is the union of two chordless circuits and $M = E(G) \setminus E(F)$. If G contains no M-P_{10} subgraph, then G contains at least two distinct M-C_4's.*

A family of cubic graphs will be constructed later which indicates that Theorem 4 is the best possible. We note that *if G in Theorem 4 has at least 8 vertices, then in fact G has two* disjoint *M-C_4's.* Otherwise, the underlying graph of the graph obtained by removing the common edge of the distinct M-C_4's would have only one M-C_4, a contradiction. We also note that if G has at least 10 vertices then each 4-circuit of G is an M-C_4.

2 Proof of the main theorem

It is easy check that Theorem 4 holds for $|V(G)| \leq 8$. Let G be a counterexample to Theorem 4 with the least number of vertices and we will derive a contradiction. Let $A = a_1 \cdots a_n a_1$, $B = b_1 \cdots b_n b_1$ be two chordless circuits comprising the 2-factor F. Let $M = E(G) \setminus [E(A) \cup E(B)]$.

Since G contains at most one M-C_4 and $|V(G)| \geq 10$, some edge of M is not contained in any M-C_4 of G. Without loss of generality, let $a_1b_1 \in M$ be an edge not contained in any M-C_4 of G. Since G is a smallest counterexample, the underlying cubic graph $\overline{G \setminus \{a_1b_1\}}$ of $G \setminus \{a_1b_1\}$ contains at least two M-C_4's. One of these M-C_4's of $\overline{G \setminus \{a_1b_1\}}$ is not an M-C_4 of G and must contain either the subdivided edge a_na_2 or the subdivided edge b_nb_2 but not both. Without loss of generality, let $a_na_2b_hb_{h+1}a_n$ be an M-C_4 of $\overline{G \setminus \{a_1b_1\}}$ containing the subdivided edge a_na_2. That is,

$$a_2b_h, a_nb_{h+1} \in E(G).$$

Since a_1b_1 is not contained in any M-C_4 of G, we have that

$$h > 2 \text{ and } h + 1 < n. \tag{1}$$

Let σ be a permutation on $\{1, \cdots, n\}$ so that $M = \{a_ib_{\sigma(i)} : i = 1, \cdots, n\}$. We claim that for each pair of integers i, j with $i \in \{3, \cdots, h\}$ and $j \in \{h+1, \cdots, n-1\}$, it is impossible that $\sigma(i) \in \{h+2, \cdots, n\}$ and $\sigma(j) \in \{2, \cdots, h-1\}$. For otherwise, the subgraph of G induced by

$$E(F) \cup \{a_1b_1, a_2b_h, a_nb_{h+1}, a_ib_{\sigma(i)}, a_jb_{\sigma(j)}\}$$

is an M-P_{10} subgraph (see figure 1).

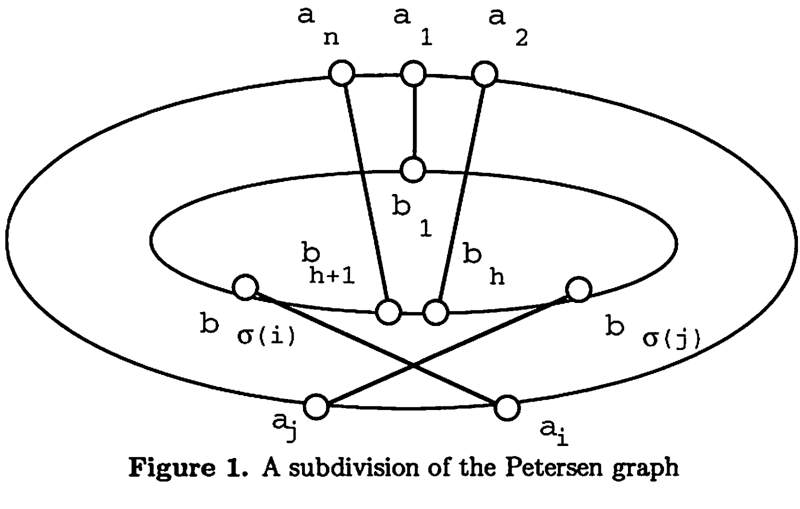

Figure 1. A subdivision of the Petersen graph

Thus,

$$\begin{cases} \sigma(\{3,\cdots,h\}) = \{2,\cdots,h-1\} \text{ and} \\ \sigma(\{h+1,\cdots,n-1\}) = \{h+2,\cdots,n\}. \end{cases}$$

Furthermore, we also see that the edge subsets

$$\{a_1a_2, b_1b_2, a_ha_{h+1}, b_hb_{h+1}\} \text{ and } \{a_1a_n, b_1b_n, a_ha_{h+1}, b_hb_{h+1}\}$$

are edge-cuts of G. Denote

$$H_1 = G \setminus \{a_ib_{\sigma(i)} : i = h+1,\cdots,n\},$$

$$H_1' = H_1 \setminus \{a_1b_1\},$$

$$H_2 = G \setminus \{a_ib_{\sigma(i)} : i = 2,\cdots,h\},$$

$$H_2' = H_2 \setminus \{a_1b_1\}.$$

By (1), the underlying graph of each of $\{H_1, H_1', H_2, H_2'\}$ has at least four vertices. We claim that each of $\{\overline{H_1}, \overline{H_2}\}$ contains an M-C_4 which is also an M-C_4 of G. Since G is a smallest counterexample, the underlying cubic graph $\overline{H_1}$ of H_1 contains at least two M-C_4's. If each of these M-C_4's of $\overline{H_1}$ is not an M-C_4 of G, then each must contain at least one of the subdivided edges in $\{a_1a_h, b_1b_h\}$. Since $a_1b_1, a_2b_h \in E(G)$, these M-C_4's of $\overline{H_1}$ must be $a_1b_1b_ha_2a_1$ and $a_1b_1b_2a_ha_1$. Hence we have that

$$a_hb_2 \in E(G).$$

Furthermore, in $\overline{H_1'}$, the 4-circuit $a_2b_hb_2a_ha_2$ is an M-C_4 and contains both subdivided edges $\{a_2a_h, b_2b_h\}$. Thus, all M-C_4's of $\overline{H_1'}$ other than $a_2b_hb_2a_ha_2$ do not contain either of the subdivided edges $\{a_2b_h, a_hb_2\}$. Note that $\overline{H_1'}$ has at least two M-C_4's. Therefore H_1' must contain an M-C_4 of G and H_1. Similarly, H_2 also contains an M-C_4 which is also an M-C_4 of G. Thus, G contains two distinct M-C_4's, a contradiction. □

3 Graphs with two 4-circuits

Theorem 4 is the best possible. Here we construct a family $\mathcal{C}$ of cubic graphs satisfying the conditions described in Theorem 4, each member of $\mathcal{C}$ having exactly two 4-circuits.

Define a function (a bijection) $f : Z \mapsto Z$ as follows:

$$f(i) = \begin{cases} i & \text{if } i \text{ is even} \\ -i & \text{if } i \text{ is odd.} \end{cases}$$

Let n be a positive integer and construct a cubic graph H_n as follows. Let $A = a_{-n}\cdots a_0 \cdots a_na_{-n}$ and $B = b_{-n}\cdots b_0\cdots b_nb_{-n}$ be two disjoint circuits, let $M = E(H_n) \setminus [E(A) \cup E(B)] = \{a_i b_{f(i)} : -n \leq i \leq n\}$, let $V(H_n) = V(A) \cup V(B)$ and let $E(H_n) = M \cup E(A) \cup E(B)$ (See figure 2).

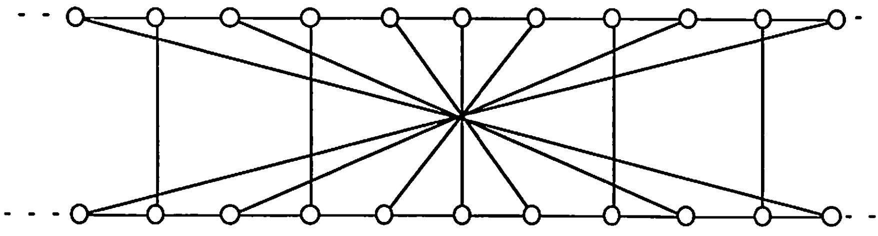

Figure 2. H_n

For each $m \geq 4$, we now construct a graph $G \in \mathcal{C}$ of order $2m$. If $m = 2k$ is even, then $G = H_k \setminus \{a_0 b_0\}$; if $m = 2k-1$ is odd, then $G = H_k \setminus \{a_0 b_0, a_2 b_2\}$. Obviously, $a_1 b_{-1} b_1 a_{-1} a_1$ and $a_n b_n b_{-n} a_{-n} a_n$ (when n is even) or $a_n b_{-n} b_n a_{-n} a_n$ (when n is odd) are the only 4-circuits of G (not just M-C_4's).

We also can prove that the graph $G \in \mathcal{C}$ constructed above contains no M-P_{10} subgraph. Denote the underlying graph of $H_i \setminus \{a_0 b_0\}$ by L_i for each positive integer i. Note that *the underlying graph of the graph obtained from L_k by deleting the edges $\{a_{-1} b_1, a_1 b_{-1}\}$ is isomorphic to L_{k-1}.*

It is sufficient to show that L_k contains no M-P_{10} subgraph. We assume inductively that L_{k-1} contains no M-P_{10} subgraph. If $L_k \setminus \{a_1 b_{-1}\}$ contains an M-P_{10} subgraph P, then the subgraph P must contain the edge $a_{-1} b_1$. But P cannot contain the edge $a_2 b_2$ since $a_{-1} b_1, a_2 b_2$ are contained in a 4-circuit in the underlying cubic graph of $L_k \setminus \{a_1 b_{-1}\}$ whereas P contains no 4-circuit. Thus we obtain an M-P_{10} subgraph $[P \cup \{a_2 b_2\}] \setminus \{a_{-1} b_1\}$ in L_{k-1}. This is a contradiction. Similarly, we can prove that L_k contains no M-P_{10} subgraph since $L_k \setminus \{a_1 b_{-1}\}$ contains no M-P_{10} subgraph and $\{a_1 b_{-1}, a_{-1} b_1\}$ are contained in a 4-circuit of L_k.

In [9], we show two further properties about the L_k's:

(1) *Each L_k contains no subdivision of the Petersen graph (not just M-P_{10} subgraph).*

(2) *Every permutation graph containing no M-P_{10} and containing precisely two 4-circuits must be homeomorphic to a subgraph of some member of $\mathcal{C}$.*

4 Unique edge-3-coloring

The following is a well-known open problem regarding uniquely edge-3-colorable cubic graphs.

Conjecture 5 (Fiorini and Wilson 1978 [8]) *Let G be a planar cubic graph with at least 4 vertices. If G is a uniquely edge-3-colorable cubic graph, then G has a triangle.*

The planarity condition is required in Conjecture 5 since Tutte found that the generalized Petersen graph $P(9,2)$ is uniquely edge-3-colorable and triangle-free ([13], see Figure 3). The following is a refinement of Conjecture 5.

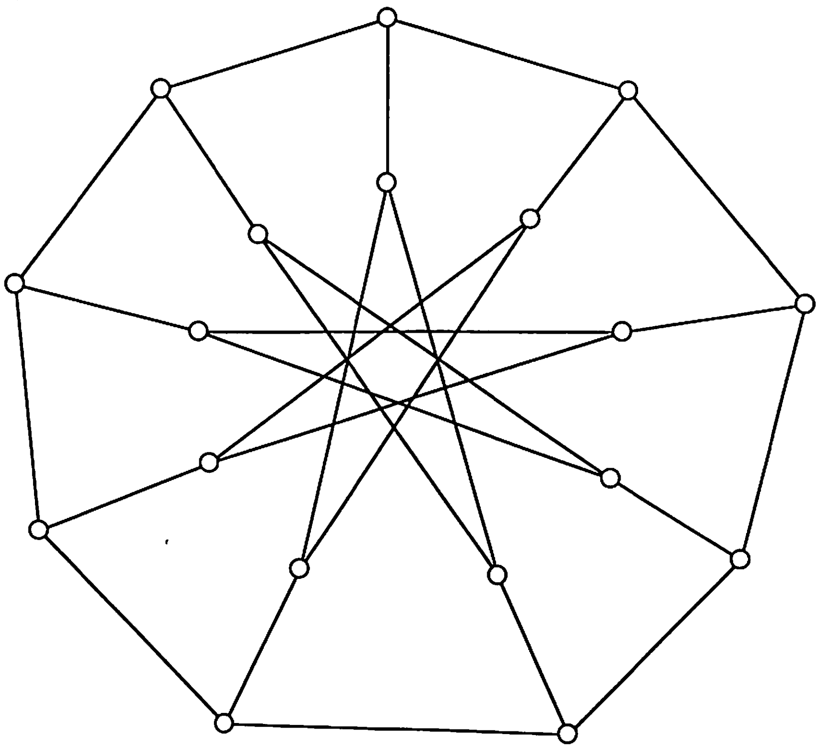

Figure 3. $P(9,2)$

Conjecture 6 ([14]) *Every uniquely edge-3-colorable, triangle-free, cubic graph of order at least four must contain a subdivision of the Petersen graph.*

Note that $P(9,2)$ is a permutation graph and contains a subdivision of the Petersen graph. Using Theorem 4, we obtain a partial result for Conjecture 6.

Theorem 7 *Every uniquely edge-3-colorable permutation graph of order at least eight must contain a subdivision of the Petersen graph.*

Proof. Let G be a permutation graph containing no subdivision of the Petersen graph. By Theorem 4, the graph G contains two distinct M-C_4's and hence, G contains two distinct Hamilton circuits. It is easy to see that G has two distinct 1-factorizations and therefore G is not uniquely edge-3-colorable. □

In the proofs of Theorem 2 (ii) and Theorem 7, the 4-circuits play a central role. The following result was proved by Hind ([10]).

Theorem 8 (Hind [10]) *If there exists a uniquely edge-3-colorable, triangle-free, (planar) cubic graph, then there exists a uniquely edge-3-colorable, triangle-free and 4-circuit-free (planar) cubic graph.*

The authors believe the following.

Conjecture 9 *The girth of every uniquely edge-3-colorable, triangle-free, cubic graph of order at least four is at least five.*

5 Remarks

By considering a minimal counterexample G to Conjecture 1, one may easily see that G has a 2-factor which consists of only two odd-circuits and all others are even-circuits. A permutation graph with $2k$ vertices (k odd) is a special case of this kind of cubic graph. The following refinement of Theorem 2 has been considered by a few mathematicians,

Conjecture 10 *Let G be a bridgeless cubic graph such that G has a 2-factor F consisting of two odd circuits. Then either G contains a subdivision of the Petersen graph or G is edge-3-colorable.*

If this conjecture were proved, it is possible that the lengthy proof of a theorem by Alspach, Goddyn and Zhang ([2]) could be simplified (with a similar argument as that of [1]). The family $\mathcal{C}$ of permutation graphs constructed in Section 3 indicates that the approach of finding M-C_4's might be very difficult, since an additional chord can eliminate all (only two) 4-circuits in the graphs. It is surprising that even with the additional condition of planarity, there is as yet no proof of Conjecture 10 without using the 4-color theorem. Here we propose two problems (they are certainly true because of the 4-color theorem) that were posted on GraphNet in 1994 by one of the authors.

Problem 11 *Let G be a bridgeless cubic planar graph such that G has a 2-factor F consisting of two odd circuits. Prove that G is edge-3-colorable without applying the 4-color theorem.*

If one component of the 2-factor is a 3-circuit, then this circuit can be contracted to a vertex. Thus, we have the following extremal case of Problem 11.

Problem 12 *Let G be a bridgeless cubic planar graph of order n such that G has a circuit of length $n-1$. Prove that G is edge-3-colorable without applying the 4-color theorem.*

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