Some Results on Packing Graphs in their Complements

T. Gangopadhyay
XLRI Jamshedpur
Post Box 222
Jamshedpur 831 001
India

ABSTRACT. A supergraph H of a graph G is called tree-covered if $H - E(G)$ consists of exactly $|V(G)|$ vertex-disjoint trees, with each tree having exactly one point in common with G. In this paper, we show that if a graph G can be packed in its complement and if H is a tree-covered supergraph of G then H itself is self-packing unless H happens to be a member of a specified class of graphs. This is a generalisation of earlier results that almost all trees and unicyclic graphs can be packed in their complements.

1 Introduction and definitions

The notion of packing a pair of graphs is now fairly old. According to Schuster [10], this notion first appears in 1977 and 1978 (see [1], [2], [9]). We quote the definition from Schuster [10]:

Definition. *Let G and G be two graphs of the same order p. A* packing of G_1 with G_2 *is an isomorphic embedding of G_1 in $\overline{G}_2$, the complement of G_2.*

We say that G is *self-packing* if there is a packing of G with G itself. For a self-packing graph G, let σ denote an injection from G into $\overline{G}$ such that if $uv \in E(G)$ then $\sigma(u)\sigma(v) \in E(\overline{G})$. Such a σ is called an isomorphic embedding (or an isomorphism) from G into $\overline{G}$. The image of σ in $\overline{G}$ is denoted by G^* and is called a copy of G in $\overline{G}$ under σ. In what follows, letters G, H will always denote simple undirected graphs.

Let G be a self-packing graph on n points. We say that a supergraph H of G is *tree-covered* if $H - E(G)$ consists of exactly n vertex-disjoint trees (maybe trivial), with each tree having exactly one point in common with G. (i.e. H is obtained by attaching one tree to each point of G).

Packing of trees has received considerable attention (see [8], [10], [11]). In particular, Burns and Schuster [3] has proved that every non-trivial tree other than a star is self-packing. Also Faudree et al [4] have characterised all unicyclic graphs that are self-packing.

It is easy to see that every spanning subgraph of a self-complementary graph is self-packing. So are spanning subgraphs of r-partite self-complementary graphs (see [6]) for every $r \geq 2$, and those of t-self-complementary graphs (see [5]), for every $t \geq 3$. Thus the class of self-packing graphs is very rich and interesting. In particular it is a generalisation of self-complementary, r-partite self-complementary and t-self-complementary graphs.

For all terms not defined in this paper we refer to Harary [7].

We define the class of labelled trees $T_{m,r}$ for $m \geq 1$, $r \geq 1$, as follows :

$$V(T_{m.r}) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_r\},$$

and

$$E(T_{m,r}) = (\cup_{i=1}^{m-1}\{u_i u_{i+1}\}) \cup (\cup_{i=1}^{r}\{u_m v_i\}).$$

Figure 1.1 illustrates $T_{3,5}$ and $T_{1,3}$.

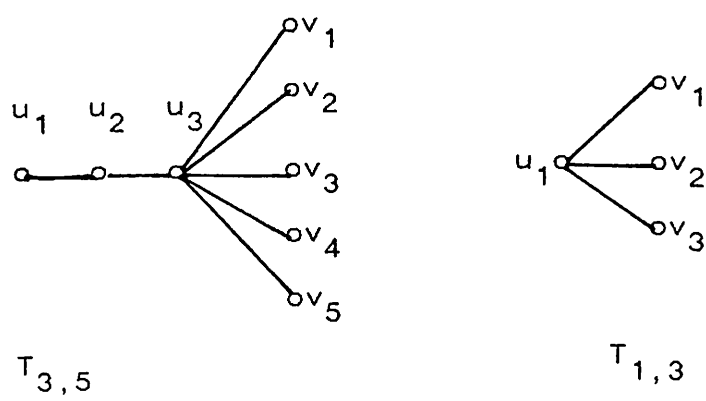


Figure 1.1

For a given graph G, we denote by $G \circ T_{m,r}$ any graph obtained by attaching the tree $T_{m,r}$ to some point of G, with u_1 of $T_{m,r}$ being used as the point of contact. The points of $T_{m,r}$ retain their original labels in $G \circ T_{m,r}$. In Figure 1.2, for a given graph G, we draw all possible versions of $G \circ T_{2,2}$.

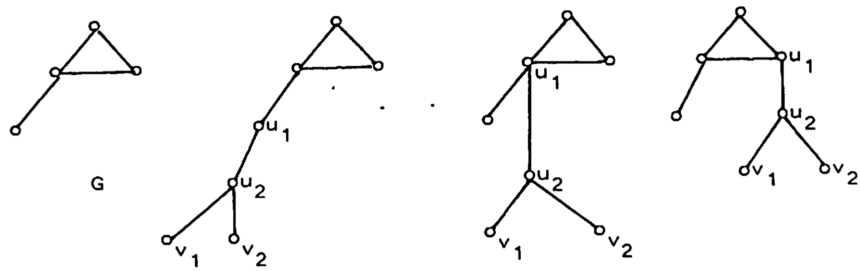


Figure 1.2. $G \circ T_{2,2}$

In the present paper we show that if H is a tree-covered supergraph of a self-packing graph G, then H itself is self-packing unless $H = G \circ T_{1,r}$ for some $r \geq 1$, in which case more stringent conditions on G are required. This is a generalisation of earlier results that almost all trees and unicyclic graphs are self-packing, (see [12], [4]). It thus provides for a unifying approach to proving such results.

We say that a pair of points (u, v) in a graph G is an *end-pair*, if u, v are both end-points, $uv \notin E(G)$ and u and v are not adjacent to a common point in G. Two end-pairs (u, v) and (u_1, v_1) are called *distinct* if $\{u_1, v_1\} \cap \{u_2, v_2\} = \phi$.

Finally if u is a point of G then $d_G(u)$ denotes the degree of u in G and $N_G(u)$ denotes the set of points to which u is adjacent in G. Also if $X \subseteq v(G)$ then $G[x]$ denotes the subgraph of G induced by the set X.

2 The main result

We first prove some preliminary lemmas.

Lemma 2.1. *Let G be self-packing. Let $H = G \circ T_{m,r}$ for some m, r with $m \geq 2$, $r \geq 1$. Then H is self-packing.*

Proof: Let σ_1 be an isomorphism from G into $\overline{G}$ and let G^* be a copy of G in $\overline{G}$ under σ_1. Let $x = \sigma_1(u_1)$ and $N_1 = N_{G^*}(x)$. We consider four cases depending on the value of m. In each case, we construct an isomorphism σ from H in $\overline{H}$ to show that the graph H is self-packing. We also specify H^*, a copy of H in $\overline{H}$ under σ. In each case $V(H^*) = V(H)$. So we only specify $E(H^*)$. For each case, the construction of H^* is illustrated in a diagram for $H = C_5 \circ T_{m,r}$.

Case 1. $m = 2$.

Let $N_2 = N_{G^*}(u)$ and $y = \sigma_1^{-1}(u_1)$.

Then

$$E(H^*) = E(G^*) \cup \{xv_1, v_1u_1\} \cup (\cup_{i=2}^{r}\{v_1v_i\})$$
$$\cup (\cup_{w\in N_2}\{u_2w\}) - \cup_{w\in N_2}\{u_1w\}.$$

and the isomorphism σ is defined as follows :

$\sigma(y) = u_2, \sigma(u_2) = v_1, \sigma(v_1) = u_1,$
$\sigma(v_i) = v_i$ for all $i, 2 \le i \le r,$
$\sigma(w) = \sigma_1(w)$ for all $w \in V(G) - \{y\}$ (See Figure 2.1)

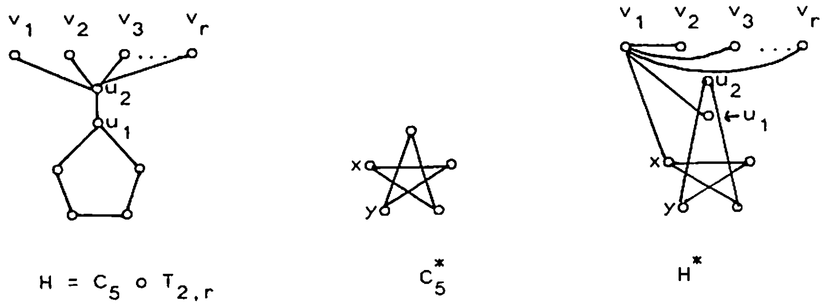


Figure 2.1

Case 2. $m = 3$.
Then

$$E(H^*) = E(G^*) \cup \{u_3x, xv_1, v_1u_2\} \cup (\cup_{i=2}^{r}\{v_1v_i\})$$
$$\cup (\cup_{w\in N_1}\{u_3w\}) - \cup_{w\in N_1}\{xw\}.$$

and the isomorphism σ is defined as follows:

$\sigma(u_1) = u_3, \sigma(u_2) = x, \sigma(u_3) = v_1, \sigma(v_1) = u_2,$
$\sigma(v_i) = v_i$ for all $i, 2 \le i \le r,$
$\sigma(w) = \sigma_1(w)$ for all $w \in V(G) - \{u\}$. (See Figure 2.2.)

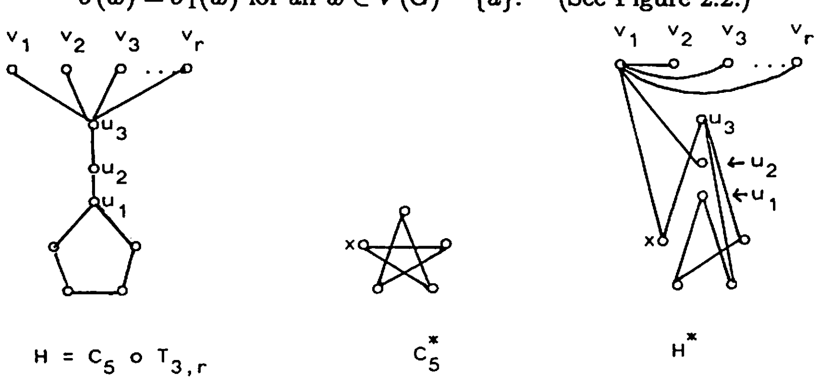


Figure 2.2

Case 3. $m = 4$.

Then

$$E(H^*) = E(G^*) \cup \{xu_4, u_4u_2, u_2v_1, v_1u_3\} \cup (\cup_{i=2}^{r}\{v_1v_i\}).$$

and

$$\sigma = \sigma_1(u_2u_4v_1u_3) \quad \text{(See Figure 2.3)}$$

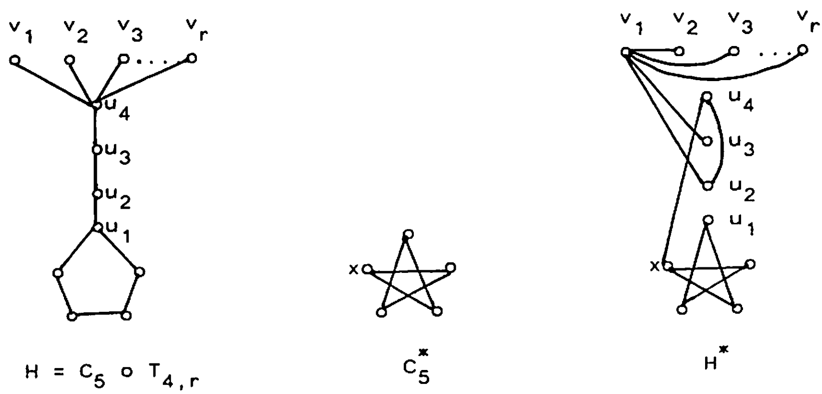


Figure 2.3

Case 4. $m \geq 5$.

Then

$$E(H^*) = E(G^*) \cup (\cup_{i=2}^{m-2}\{u_iu_{i+2}\}) \cup (\cup_{i=1}^{r}\{u_3v_i\}) \\ \cup \{xu_{m-\delta}, u_2u_{m+\delta-1}\}$$

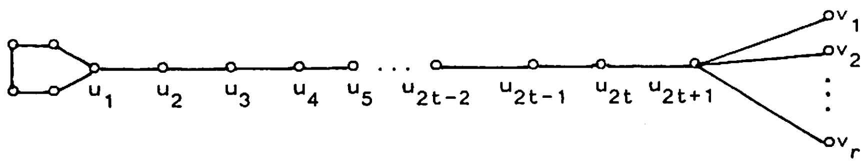
and the isomorphism σ is defined as follows:

$$\sigma(w) = \sigma_1(w) \text{ if } w \in V(G),$$
$$\sigma(v_i) = v_i \text{ for all } i, 1 \leq i \leq r,$$
$$\sigma(u_i) = u_{m-\delta-2i+4} \text{ if } 2 \leq i \leq \frac{m-\delta}{2} + 1,$$
$$\sigma(u_i) = u_{2(m-i)+3} \text{ if } \frac{m-\delta}{2} + 2 \leq i \leq m,$$

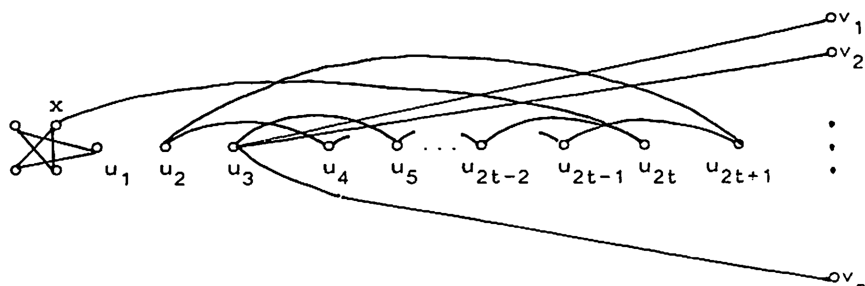
where

$$\delta = \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{cases}$$

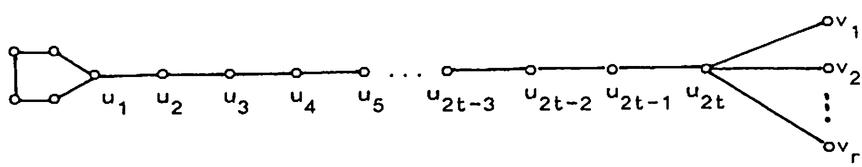
(See Figure 2.4)



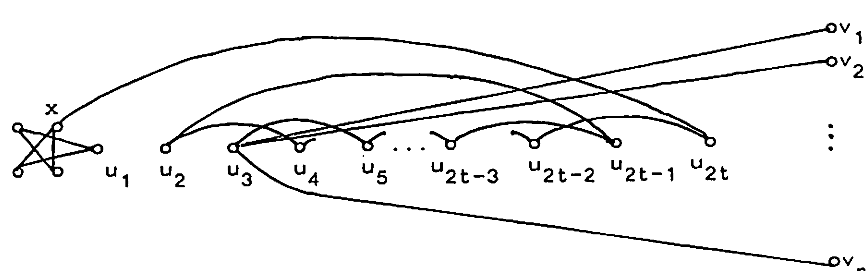
$H = C_5 \circ T_{2t+1,r} \quad (t \geq 2)$



H^*



$H = C_5 \circ T_{2t,r} \quad (t \geq 3)$



H^*

Figure 2.4

Lemma 2.2. *Let H be a tree-covered supergraph of a self-packing graph G. Let (u, v) be an end-pair of H satisfying*

(i) *$\{u, v\}$ does not contain any point of G,*

(ii) $H - \{u, v\} = G \circ T_{1,r}$ *for some* $r \geq 1$.

Then H is self-packing.

Proof: As per convention, we label the points of $T_{1,r}$ in $H - \{u, v\}$ as $u_1, v_1, v_2, \ldots, v_r$. Now, let σ_1, be an isomorphism from G into $\overline{G}$ and let G^* be a copy of G in $\overline{G}$ under σ. Let $z = \sigma_1(u_1)$ and $N_1 = N_{G^*}(z)$. Let $ux, vy \in E(H)$. We now consider three cases. In each case, we construct an isomorphism σ from H into $\overline{H}$ to show that the graph H is self-packing. We also specify H^*, a copy of H in $\overline{H}$ under σ. In each case $V(H^*) = V(H)$. So we only specify $E(H^*)$. Wherever necessary, the construction of H^* is illustrated in a diagram for some specific graph G.

Case 1. $x, y \in \{v_1, v_2, \ldots, v_r\}$.

Then w.l.g. let $x = v_1$ and $y = v_2$. Now

$$E(H^*) = E(G^*) \cup \{vv_1, v_1v_2, vu, uz\} \cup (\cup_{i=3}^{r}\{vv_i\}) \\ \cup (\cup_{w \in N_1}\{vw\}) - \cup_{z \in N_1}\{zw\}.$$

and the isomorphism σ is defined as follows :

$$\sigma(w) = \sigma_1(w) \text{ if } w \in V(G) - u_1,$$
$$\sigma(v_i) = v_i \text{ for all } i, 3 \leq i \leq r,$$
$$\sigma(u_1) = v, \sigma(v_1) = u, \sigma(u) = z, \sigma(v_2) = v,_1 \sigma(v) = v_2.$$

(See Figure 2.5.)

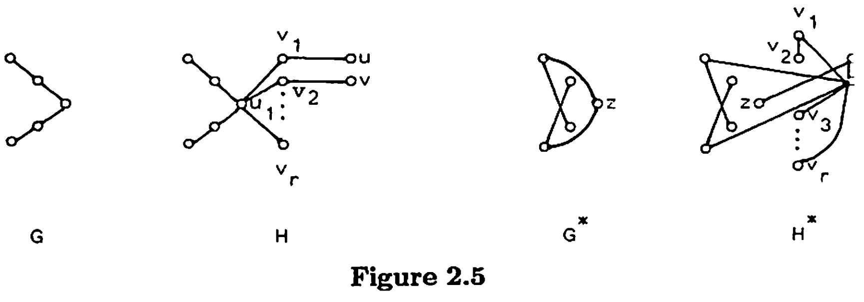


Figure 2.5

Case 2. $|\{x, y\} \cap \{v_1, v_2, \ldots, v_r\}| = 1$. Then w.l.g. let $x = v_1$. Now,

$$E(H^*) = E(G^*) \cup \{uv, vv_1, uz\} \cup (\cup_{i=2}^{r}\{uv_i\}) \cup (\cup_{w \in N_1}\{uw\}) \\ - \cup_{w \in N_1}\{zw\} \text{ if } y = u_1, \quad \text{(See Figure 2.6(a))}$$
$$= E(G^*) \cup \{uu_1, u_1v, v_1\sigma_1(y)\} \cup (\cup_{i=2}^{r}\{uv_i\}) \cup (\cup_{w \in N_1}\{uw\}) \\ - \cup_{w \in N_1}\{zw\} \text{ if } y \neq u_1 \text{ and } z = u_1 \\ (\geq \sigma_1(y) \neq \sigma_1(u_1) = u_1), \quad \text{(See Figure 2.6(b))}$$
$$= E(G^*) \cup \{u\sigma_1(y), v_1v\} \cup (\cup_{i=1}^{r}\{zv_i\}) \\ \text{if } y \neq u, \text{ and } z \neq u_1. \quad \text{(See Figure 2.6(c))}$$

and the isomorphism σ is defined as follows :

$$\sigma(v_i) = v_i \text{ if } 2 \le i \le r,$$
$$\sigma(w) = \sigma_1(w) \text{ if } w \in V(G) - u_1$$

with

$$\sigma(u_1) = u, \sigma(v_1) = v, \sigma(u) = v_1, \sigma(v) = z \text{ if } y = u_1,$$
$$\sigma(u_1) = u, \sigma(v_1) = u_1, \sigma(u) = v, \sigma(v) = v_1 \text{ if } y \ne u_1, \text{ and } z = u_1,$$
$$\sigma(u) = v, \sigma(v) = u, \sigma(u_1) = u, \sigma(v_1) = v_1 \text{ if } y \ne u_1, \text{ and } z \ne u_1.$$

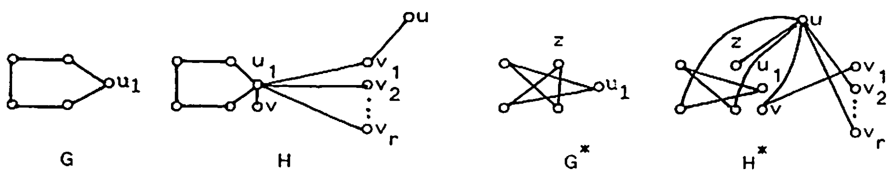


Figure 2.6 (a)

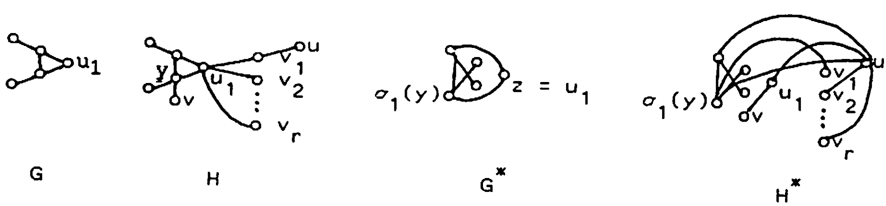


Figure 2.6 (b)

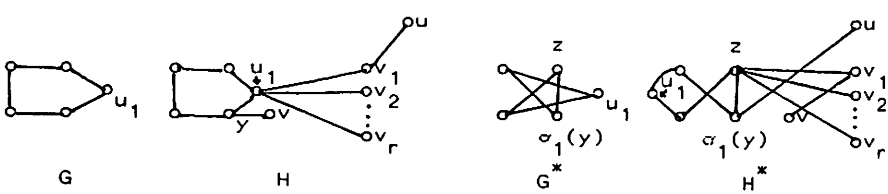


Figure 2.6 (c)

Case 3. $\{x, y\} \cap \{v_1, v_2, \ldots, v_r\} = \phi$.

We consider two subcases.

Case 3.1. $z \ne u_1$

Then let

$$E(H^*) = \begin{cases} E(G^*) \cup \{v\sigma_1(x), u\sigma_1(y)\} \cup (\cup_{i=1}^{r}\{zv_i\}) & \text{if } \sigma_1(x) = x \\ & \text{or } \sigma_1(y) = y, \\ E(G^*) \cup \{u\sigma_1(x), v\sigma_1(y)\} \cup (\cup_{i=1}^{r}\{zv_i\}) & \text{otherwise.} \end{cases}$$

and

$$\sigma = \sigma_1(\prod_{i=1}^{r}(v_i))\sigma_2$$

where

$$\sigma_2 = \begin{cases} (uv) & \text{if } \sigma_1(x) = x \text{ or } \sigma_1(y) = y, \\ (u)(v) & \text{otherwise.} \end{cases}$$

Case 3.2. $z = u$. We then consider two subcases:

Case 3.2.1. $u \notin \{x, y\}$. Then let

$$\begin{aligned} E(H^*) &= E(G^*) \cup \{uu_1, v\sigma_1(x), v_1\sigma_1(y)\} \cup (\cup_{i=2}^{r}\{uv_i\}) \\ &\cup (\cup_{w \in N_1}\{uw\}) - \cup_{w \in N_1}\{zw\} \text{ if } \sigma_1(x) = x \text{ or } \sigma_1(y) = y, \\ &= E(G^*) \cup \{uu_1, v\sigma_1(y), v_1\sigma_1(x)\} \cup (\cup_{i=2}^{r}\{uv_i\}) \\ &\cup (\cup_{w \in N_1}\{uw\}) - \cup_{w \in N_1}\{zw\} \text{ otherwise.} \end{aligned}$$

and

$$\sigma = \sigma_2(\prod_{i=1}^{r}(v_i))\sigma_3$$

where σ_2 is the restriction of σ_1 on $V(G) - \{u_1\}$ and

$$\begin{aligned} \sigma_3 &= (u_1uvv_1) \text{ if } \sigma_1(x) = x \text{ or } \sigma_1(y) = y \\ &= (u_1uv_1)(v) \text{ otherwise.} \end{aligned}$$

Case 3.2.2. $u_1 \in \{x, y\}$. Since (u, v) is an end-pair it follows that $x \neq y$. W.l.g. let $u_1 = y$.

Now, if $\sigma_1(x) = x$, then $x \in N_1 \Rightarrow xu_1 \in E(G^*) \Rightarrow \sigma_1^{-1}(x)\sigma_1^{-1}(u_1) \in E(G) \Rightarrow xu_1 \in E(G) \Rightarrow xu_1 \notin E(\overline{G}) \Rightarrow xu_1 \notin E(G^*) \Rightarrow x \notin N_1$, a contradiction.

So $x \notin N_1$. Then

$$\begin{aligned} E(H^*) = E(G^*) &\cup \{uu_1, vx\} \cup (\cup_{i=1}^{r}\{uv_i\}) \\ &\cup (\cup_{w \in N_1}\{uw\}) - \cup_{w \in N_1}\{zw\}. \end{aligned}$$

and

$$\sigma = \sigma_2(\prod_{i=1}^{r}(v_i)(u_1uv))$$

where σ_2 is the restriction of σ_1 on $V(G) - \{u\}$.

However, if $\sigma_1(x) \neq x$, then let $N_2 = N_1 - \{x, \sigma_1(x)\}$, $N_3 = N_{G^*}(\sigma_1(x)) - \{u_1, x\}$ and $N_4 = N_{G^*}(x) - \{u, \sigma_1(x)\}$. Then

$$\begin{aligned} E(H^*) = E(G^*) &\cup \{uu_1, v_1x, v_1\sigma_1(x)\} \cup (\cup_{i=2}^{r}\{v_1v_i\}) \\ &\cup (\cup_{w \in N_2}\{v_1w\}) \cup (\cup_{w \in N_3}\{uw\}) \\ &\cup (\cup_{w \in N_4}\{vw\}) \cup S - \cup_{w \in N_1}\{u_1w\} \\ &- \cup_{w \in N_3 \cup \{u_1, x\}}\{w\sigma_1(x)\} - \cup_{w \in N_4 \cup \{u_1, \sigma_1(x)\}}\{xw\} \end{aligned}$$

where $S \subseteq \{uv, vv_1, uv_1\}$, with

$$\begin{cases} uv \in S & \text{iff } x\sigma_1(x) \in E(G^*), \\ vv_1 \in S & \text{iff } xu_1 \in E(G^*), \\ uv_1 \in S & \text{iff } u_1\sigma_1(x) \in E(G^*). \end{cases}$$

and

$$\sigma = \sigma_2(uu_1v_1xv\sigma_1(x))(\prod_{i=2}^{r}(v_i))$$

where σ_2 is the restriction of σ_1 on $V(G) - \{u_1, x, \sigma_1(x)\}$. We illustrate the construction of H^* in Figure 2.7.

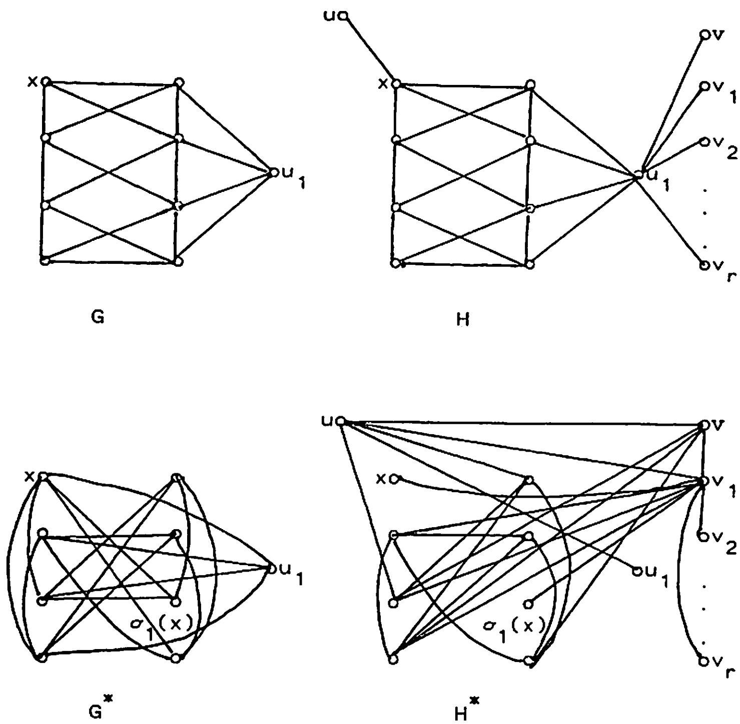


Figure 2.7

This completes all cases and proves the lemma.

We now state and prove the main result of this section.

Theorem 2.3. *Let H be a tree-covered supergraph of a self-packing graph G. Then H is self-packing if either*

(i) $H \neq G \circ T_{1,r}$ *for any* $r \geq 1$, *or*

(ii) $H = G \circ T_{1,r}$ *for some* $r \geq 1$, *and there is an isomorphism* σ_1 *from* G *into* $\overline{G}$ *such that* $\sigma_1(u_1) \neq u_1$, *where* u_1 *is the point in* G *to which* $T_{1,r}$ *is attached.*

Proof: We say that an end-pair of H contains a point of G if at least one of the end-points in the end-pair is also a point of G.

Suppose first H satisfies (ii). Then let H^* be the spanning subgraph of $\overline{H}$ having

$$E(H^*) = E(G^*) \cup (\cup_{i=1}^{r} \{v_i \sigma_1(u_1)\}).$$

Then clearly H^* is a copy of H in $\overline{H}$ under the isomorphism $\sigma = \sigma_1 \prod_{i=1}^{r}(v_i)$. So H is self-packing in this case.

Suppose next H satisfies (i). In this case we prove that H is self-packing by induction on $t(H)$, where $t(H)$ is the maximum number of distinct end-pairs (u, v) in H such that

1° $\{u, v\}$ does not contain any point of G,

2° $H - \{u, v\} \neq G \circ T_{1,r}$ for any $r \geq 1$.

Let $t(H) = 0$. We consider two exhaustive cases and show that H is self-packing in each case.

Case 1. There is an end-pair (u, v) of H that does not contain any point of G.

Now, if $H - \{u, v\} \neq G \circ T_{1,r}$ for any $r \geq 1$, then clearly $t(H) = 1$, a contradiction. So let $H - \{u, v\} = G \circ T_{1,r}$ for some $r \geq 1$. Then by Lemma 2.2, H is self-packing.

Case 2. (not case 1). For every end-pair (u, v) of H either $u \in V(G)$ or $v \in V(G)$.

Then clearly $H = G$ or $H = G \circ T_{m,r}$ for some $m \geq 1$, $r \geq 1$ (This also holds if H has no end-pair). If $H = G$ then H is self-packing. If $H = G \circ T_{m,r}$ for some $m \geq 1$, $r \geq 1$ then since H satisfies (i) we have $m \geq 2$. It then follows by Lemma 2.1, that H is self-packing.

Let $t(H) \geq 1$. Let (u, v) be an end-pair in H. Let x, y be two points in H such that $ux, vy \in E(H)$. Then $x \neq y$. Let $H = H - \{u, v\}$. Then $t(H_1) = t(H) - 1$. So since H_1 is also a tree-covered supergraph of G satisfying (i) (i.e., $H_1 \neq G \circ T_{1,r}$ for any $r \geq 1$), it follows by the induction hypothesis that H_1 is self-packing. Let σ_1 be an isomorphism from H_1 into $\overline{H}_1$. Then let H^* be the spanning subgraph of $\overline{H}$ with

$$E(H^*) = \begin{cases} E(H_1^*) \cup \{u\sigma_1(y), v\sigma_1(x)\} & \text{if either } \sigma_1(x) = x \text{ or } \sigma_1(y) = y, \\ E(H_1^*) \cup \{u\sigma_1(x), v\sigma_1(y)\} & \text{otherwise.} \end{cases}$$

Then it can be easily seen that H^* is a copy of H in $\overline{H}$ under σ where

$$\sigma = \begin{cases} \sigma_1(uv) & \text{if either } \sigma_1(x) = x \text{ or } \sigma_1(y) = y, \\ \sigma_1(u).(v) & \text{otherwise.} \end{cases}$$

Thus H is self-packing. This completes the induction.

Thus H is self-packing in either case. This proves the theorem.

We now deduce a string of corollaries to show that except for some forbidden classes of graphs, all unicyclic graphs are self-packing. This result had been earlier proved independently by Faudree et al [4].

Lemma 2.4. *The cycle C_n is self-packing for all n, $n \geq 5$. Also, for any point u_1 in C_n, there is an isomorphism σ_1 from C_n into $\overline{C}_n$ such that $\sigma_1(u_1) \neq u_1$.*

Proof: Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Let H be the spanning subgraph of $\overline{C}_n$ with

$$E(H) = \begin{cases} \cup_{i=1}^{n}\{u_i u_{i+2}\} & \text{if } n \text{ is odd.} \\ \cup_{i=1}^{n}\{u_i u_{i+2}\} \cup \{u_n u_{n-3}, u_{n-2}u_{n-5}\} & \\ \quad -\{u_{n-2}u_n, u_{n-5}u_{n-3}\} & \text{if } n \text{ is even,} \end{cases}$$

where addition is taken modulo n.

Let σ_1 be defined by

$$\sigma_1(u_i) = \begin{cases} u_{2i \pmod n}, & \text{if } 1 \leq i \leq n-1, \\ u_n, & \text{if } i = n. \end{cases}$$

if n is odd, and

$$\sigma_1(u_i) = \begin{cases} u_{2i}, & \text{if } 1 \leq i < \frac{n}{2}, \\ u_{n-5-2i \pmod n}, & \text{if } \frac{n}{2} \leq i < n, \\ u_n, & \text{if } i = n. \end{cases}$$

if n is even.

Then it is easy to see that H is a copy of C_n in $\overline{C}_n$ under the isomorphism σ_1. Also $\sigma_1(u_1) = u_2$. This proves the lemma.

Corollary 2.5. *Let H be a unicyclic graph with a C_n, for some $n \geq 5$. Then H is self-packing.*

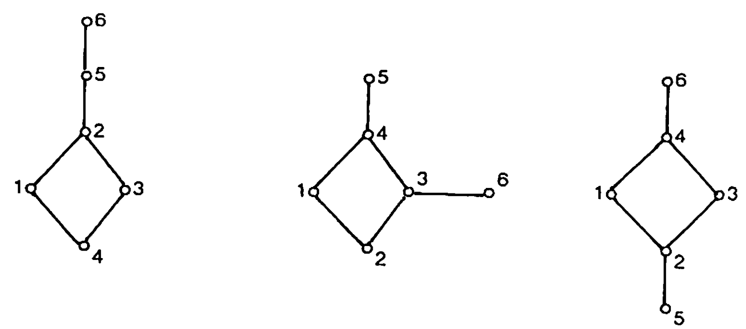
Proof: If $H \neq C_n \circ T_{1,r}$ for any $r \geq 1$, then H is self-packing by Lemma 2.4 and Theorem 2.3 (ii). Suppose $H = C_n \circ T_{1,r}$ for some $r \geq 1$. By Lemma 2.4, C is self-packing and there is an isomorphism from C_n into $\overline{C}_n$ such that $\sigma_1(u_1) \neq u_1$, where u_1 is the point in C_n to which $T_{1,r}$ is attached. Then by Theorem 2.3 (i), H is self-packing. This proves the corollary.

The following corollary follows directly from Theorem 2.3.

Corollary 2.6. *Let G be self-packing and H be a tree-covered supergraph of G. If there is an isomorphism σ_1 from G into $\overline{G}$ having no fixed point, then H is self-packing.*

Corollary 2.7. *Let H be a unicyclic graph with a C_4. Then H is self-packing if $H \neq C_4$ and $H \neq C_4 \circ T_{1,r}$ for some $r \geq 1$.*

Proof: It is easy to see that H is a tree-covered supergraph of one of the three graphs G_1, G_2, G_3 (see Figure 2.8). But for each $i = 1,2,3$, G_i is self-packing with an isormorphism σ_{1i} that has no fixed point, as is shown in Figure 2.8. The result then follows by Corollary 2.6.



G_1 $\sigma_{11} = (1\,3)(2\,5\,4\,6)$ G_2 $\sigma_{12} = (1\,2\,4\,5\,3\,6)$ G_3 $\sigma_{13} = (1\,3)(2\,6\,4\,5)$

Figure 2.8

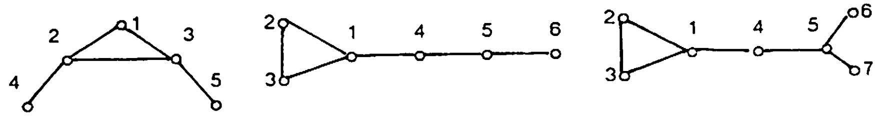
Corollary 2.8. *Let H be a unicyclic graph with a C_3. Then H is self-packing if*

(i) $H \neq C_3$,

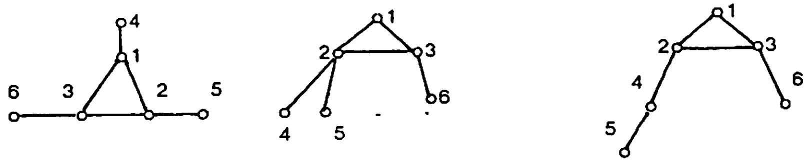
(ii) $H \neq C_3 \circ T_{m,r}$ *for any* $m \leq 2$,

(iii) $H \neq C_3 \circ T_{3,r}$ *for any* $r \geq 3$.

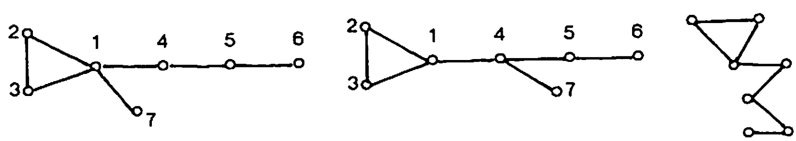
Proof: An easy enumeration shows that H is either one of the graphs G_1, G_2, G_3 in Figure 2.9 or a tree-covered supergraph of some G_i, $4 \leq i \leq 9$ (see Figure 2.9). But as shown in Figure 2.9, G_1, G_2, G_3 are self-packing; moreover, for each i, $4 \leq i \leq 9$, the graph G_i is self-packing with an isomorphism σ_{1i} that has no fixed point. The result then follows by Corollalry 2.6.



G_1 $\sigma_{11} = (1)(2435)$ G_2 $\sigma_{12} = (2)(5)(1436)$ G_3 $\sigma_{13} = (143726)(5)$



G_4 $\sigma_{14} = (143625)$ G_5 $\sigma_{15} = (153624)$ G_6 $\sigma_{16} = (125436)$



G_7 $\sigma_{17} = (1653247)$ G_8 $\sigma_{18} = (1653247)$ G_9 $\sigma_{19} = (1563)(274)$

Figure 2.9

Combining the results of Corollary 2.4, 2.5 and 2.6 we obtain

Theorem 2.9. (Faudree [4]) *Let G be a unicyclic graph. Then G is self-packing if*

(i) $G \neq C_n$ *for any* $n \leq 4$,

(ii) $G \neq C_3 \circ T_{m,r}$ *for any* $m \leq 2$,

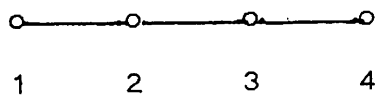
(iii) $G \neq C_3 \circ T_{3,r}$ *for any* $r \geq 3$,

(iv) $G \neq C_4 \circ T_{1,r}$ *for any* $r \geq 1$.

The result that every non-trivial tree other than a star is self-packing also follows as a Corollary to Theorem 2.3, as is shown below.

Corollary 2.10. (Straight [12]) *If H is a non-trivial tree and H is not a star, then H is self-packing.*

Proof: It is easy to see that H is some tree-covered supergraph of the graph G shown in Figure 2.10. However, G is self-packing with an isomorphic σ_1 that has no fixed point (see Figure 2.10). Hence by Corollary 2.6, H is self-packing.



$\sigma_1 = (1\,3\,4\,2)$
Figure 2.10

Thus we see that Theorem 2.3 is a very powerful tool for showing that classes of graphs are self-packing. In particular, it contains a unifying approach for proving that except for certain forbidden classes of graphs, all unicyclic graphs and all trees are self-packing.

We may also generate new classes of self-packing graphs using Theorem 2.3. This is explored in the following group of Corollaries.

Corollary 2.11. *Let G be a self-complementary graph having an even number of points. If H is a tree-covered supergraph of G, then H is self-packing.*

Proof: Follows from Corollary 2.6, since no complementing permutation of G has a fixed point.

Corollary 2.12. *Let G be either self-complementary, r-partite self-complementary for some $r \geq 2$, or t-self-complementary for some $t \geq 3$. Let H be a tree-covered supergraph of G. If $H \neq G \circ T_{1,r}$ for any $r \geq 1$, then H is self-packing.*

Proof: Follows from Theorem 2.3, since G is self-packing.

3 An infinite class of graphs that are not self-packing

In this section, we construct an infinite class of graphs that are not self-packing to illustrate that the conditions in Theorem 2.3 cannot be relaxed. This shows that the results of Theorem 2.3 are the best possible.

Construction 3.1. Let $m \geq 3$ and G_m be the following graph on $4m+1$ points:

$$V(G_m) = \{u_1\} \cup X_1 \cup X_2 \cup Y_1 \cup Y_2$$

where

$$X_1 = \{x_1, x_2, \ldots, x_m\}, \quad X_2 = \{x_{m+1}, x_{m+2}, \ldots, x_{2m}\},$$
$$Y_1 = \{y_1, y_2, \ldots, y_m\}, \quad Y_2 = \{y_{m+1}, y_{m+2}, \ldots, y_{2m}\}$$

and

$$\begin{aligned} E(G_m) = {} & (\cup_{i=1}^{2m}\{u_1x_i\} \cup (\cup_{i<j=1}^{2m}\{x_ix_j\}) \cup (\cup_{i=1}^{m} \cup_{j=1}^{m} \{x_iy_j\} \\ & \cup (\cup_{i=m+1}^{2m} \cup_{j=m+1}^{2m} \{x_iy_j\}) \cup (\cup_{i=1}^{2m-1}\{y_iy_{i+1}\} \cup \{y_{2m}y_1\} \\ & - \{x_{2m}x_1\} \cup (\cup_{i=1}^{2m-1}\{x_ix_{i+1}\}). \end{aligned}$$

For $m = 3$, we illustrate the construction in Figure 3.1.

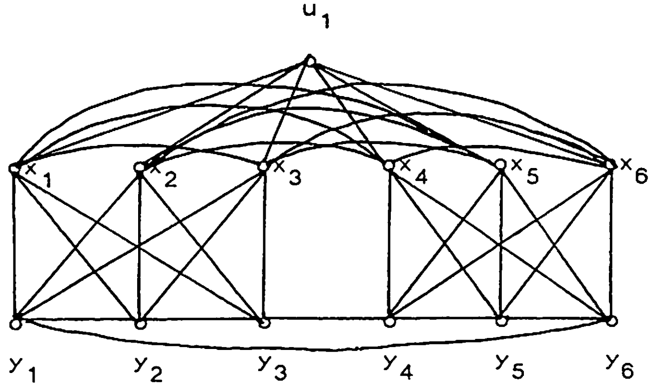


Figure 3.1

Theorem 3.1. *Let $m \geq 3$. Let G_m be as in construction 3.1 and let H_m be the tree-covered supergraph of G_m with*

$$V(H_m) = V(G_m) \cup \{v_1\},$$
$$E(H_m) = E(G_m) \cup \{u_1 v_1\}.$$

Then H is not self-packing, although G_m is.

Proof: We first prove that G_m is self-packing. This follows since G_m is self-complementary with a complementing permutation σ_1 given by

$$\sigma_1(u_1) = u,$$
$$\sigma_1(x_1) = y_i, \ 1 \leq i \leq 2m,$$
$$\sigma_1(y_1) = x_{2m+1-i}, \ 1 \leq i \leq 2m.$$

We next show that H_m is not self-packing. Suppose it is. Let σ be an isomorphism from H_m into $\overline{H}_m$ and let H^*_m be a copy of H_m in $\overline{H}_m$ under σ. We now consider two cases:

Case 1. $\sigma(u_1) \neq v_1$.
Since $d_{H_m}(u_1) = 2m + 1$ and $m \geq 3$, it follows that $\sigma(u_1) \in Y_1 \cup Y_2$. W.l.g let $\sigma(u_1) = y_1$. Now, the number of points of degree $3m - 1$ that are adjacent to u_1 in H_m is $2m$ (namely the points in $X_1 \cup X_2$). Thus in H^*_m, the point y_1 has to be adjacent to exactly $2m$ points of degree $3m - 1$. But $N^*_{H_m}(y_1) \subseteq X_2 \cup Y_1 \cup Y_2 \cup \{v_1\} - \{y_1, y_2, y_{2m}\}$, and for all $x \in X_2$, $d^*_{H_m}(x) \leq m + 3 \leq 3m - 1$. So the number of points of degree $3m - 1$ to which y_1 is adjacent in $H^*_{H_m}$ is at most $|Y_1 \cup Y_2| - 2 = 2m - 2$. This gives us a contradiction in this case.

Case 2. $\sigma(u_1) = v_1$.
Now for all $x \in X_1 \cup X_2$, $d_{H_m}(x) = 3m - 1$. Since $d_{\overline{H}_m}(u_1) = 2m < 3m - 1$ and for all $x \in x_1 \cup x_2$, $d_{\overline{H}_m}(x) = m + 3 < 3m - 1$, it follows that $\sigma(X_1 \cup X_2) = Y_1 \cup Y_2$. Now, $\sigma(v_1) \neq u_1$ since $u_1 v_1 \notin E(\overline{H}_m)$. So $\sigma(v_1) \in X_1 \cup X_2$. W.l.g. let $\sigma(v_1) = x_1$. Then

$$\sigma(Y_1 \cup Y_2) = X_1 \cup X_2 \cup \{u_1\} - \{x_1\}.$$

But $H_m[Y_1 \cup Y_2] = C_{2m}$ whereas $H_{\overline{m}}[X_1 \cup X_2 \cup \{u_1\} - \{x_1\}]$ contains no C_{2m}, a contradiction!

Thus, we obtain a contradiction in each case. This proves the theorem.

References

[1] B. Bollobas and S.E. Eldridge, Packing of graphs and applications to computational complexity. *Proc. Fifth British Combinatorics Conference* (Aberdeen, 1978), Utilitas Mathematica Publishing, Inc., Winnipeg.

[2] D. Burns and S. Schuster, Every $(p, p-2)$ graph is contained in its complement, *J. Graph Theory* **1** (1977), 277–279.

[3] D. Burns and S. Schuster, Embedding $(p, p-1)$ graphs in their complements, *Israel J. of Math.* **30** (1978), 313–320.

[4] R.J. Faudree, C.C. Rousseau, R.H. Schelp and S. Schuster, Embedding graphs in their complements, *Czechoslovak Math J.* **31**(106), (1981), 53–62.

[5] T. Gangopadhyay, The Class of $t - sc$ Graphs and their Stable Complementing Permutations, *Ars Combinatoria* **43** (1996), 49–63.

[6] T. Gangopadhyay and S.P. Rao Hebbare, Multipartite self-complementary Graphs, *Ars Combinatoria* **13** (1982), 87–114.

[7] F. Harary, *Graph Theory*, Addison Wesley, Reading, Massachusetts, 1969.

[8] S.M. Hedetniemi, S.T. Hedetniemi and P.J. Slater, A note on packing two trees in K_n, *Ars Combinatoria* **11** (1981), 149–153.

[9] N. Sauer and J. Spencer, Edge-disjoint placement of graph, *J. of Combinatorial Theory*, Ser. B. **25** (1978), 295–302.

[10] S. Schuster, Packing a tree of order p with a (p, p) Graph, *Graph Theory with Applications to Algorithms and Computer Science*, John Wiley & Sons, New York, 1985.

[11] P.J. Slater, S.K. Teo, and H.P. Yap, Packing a tree with a graph of the same size, *J. Graph Theory*, accepted for publication.

[12] H.J. Straight, stated as private communication (1976) in [9] above.