On Complementary Consecutive Labelings of Octahedron

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ABSTRACT. There are only two kinds of non-isomorphic consecutive vertex labelings of octahedron, and each of them can be deduced from the other. There is an algorithm to construct consecutive edge labelings. It is shown that there exist many non-isomorphic complementary consecutive edge labelings of octahedron.

1 Introduction

The notion of magic labeling can be traced back to a treatise published in 1275 AD by the Chinese mathematician Yang Hui [1,2]. The topic, so called complementary consecutive edge labeling [1], emerged from the study of extended magic squares and has developed rapidly in the past 100 years. A number of such magic and consecutive labelings have been defined by Ko-Wei Lih using modern notions of graph theory [1]. Martin Baca has also published several papers on the subject [3-7].

Let $\mathbf{G} = (V, E, F)$ be the plane graph. The symbols $\mathbf{V}(G)$, $\mathbf{E}(G)$, $\mathbf{F}(G)$ will denote the vertex set, the edge set and the face set of $\mathbf{G}$. Let p, q and t be the number of vertices, edges and faces of $\mathbf{G} = (V, E, F)$, respectively. A *vertex labeling* of $\mathbf{G}$ is a one-to-one mapping of the set $\{1, 2, \dots, p\}$ onto the vertices of plane graph $\mathbf{G}$. An *edge labeling* of $\mathbf{G}$ is a bijection from the set $\{1, 2, \dots, q\}$ onto the edges of $\mathbf{G}$. The *weight* of a face under vertex labeling or edge labeling is the sum of the labels of vertices or edges surrounding that face, respectively.

This paper describes the magic labelings for graph of octahedron from the family of platonic polyhedrals. A labeling of octahedron is said to be *magic* if all 3-sided faces have the same weight. We say that a labeling of octahedron is *consecutive* if the weight of all 3-sided faces constitute a set of consecutive integers.

Two labelings f and f′ of octahedron are said to be *complementary* if the sum of the f-weight and f′-weight of each 3-sided face is a constant. Complementary consecutive labelings played interesting roles in obtaining magic labelings.

The first consecutive vertex labeling of octahedron was discovered by a virtually unknown Chinese amateur mathematician named Bao Qishou (e.g. Pao Chhi-Shou, c. 1880, see [8]).

In the present paper, it is shown that there exist exactly two non-isomorphic consecutive vertex labelings of octahedron. We also show that there exist many non-isomorphic consecutive edge labelings of octahedron which are complementary to the consecutive vertex labeling.

2 Theorem

Theorem. *There exist exactly two non-isomorphic consecutive vertex labelings of octahedron.*

Proof: Let $a_1, a_2, \ldots, a_6$ denote the labels of vertices of octahedron. Let b_i $(i = 1, 2, \ldots, 8)$ denote the weight of 3-sided faces. It is not difficult to check that the vertex labeling of octahedron will be consecutive if the weights of all 3-sided faces constitute a set of consecutive integers $\{b_i : 1 \le i \le 8\} = \{7, 8, \ldots, 14\}$.

We have the following expressions:

$$\begin{aligned} b_1 &= a_1 + a_2 + a_3 & \qquad b_5 &= a_2 + a_3 + a_6 \\ b_2 &= a_1 + a_2 + a_5 & b_6 &= a_2 + a_5 + a_6 \\ b_3 &= a_1 + a_3 + a_4 & b_7 &= a_3 + a_4 + a_6 \\ b_4 &= a_1 + a_4 + a_5 & b_8 &= a_4 + a_5 + a_6 \end{aligned}$$

There are also some relationships between a_i and b_i:

$$\begin{aligned} 21 = b_1 + b_8 = b_2 + b_7 &= b_3 + b_6 = b_4 + b_5 \\ a_1 - a_6 &= b_1 - b_5 = b_2 - b_6 \\ a_2 - a_4 &= b_1 - b_3 = b_2 - b_4 \\ a_3 - a_5 &= b_1 - b_2 = b_3 - b_4 \end{aligned}$$

We can use the principles of symmetry and rotation about a fixed axis (180°) to construct numerous isomorphic plane graphs. If label of vertex a_1

see (figure 1) hold the center position, then we can draw 4 different vertex labelings which are isomorphic each other. It is easy to see that if we write a_i $(i = 2, 3, \ldots, 6)$ instead a_1, we obtain 24 different vertex labelings which are isomorphic to each other. If we rotate (180°) a vertex labeling about a fixed axis (a_1a_6, or a_2a_4, or a_3a_5), we obtain another vertex labeling and they are isomorphic. Thus for each vertex labeling of octahedron we can obtain 48 different vertex labelings which are isomorphic each other.

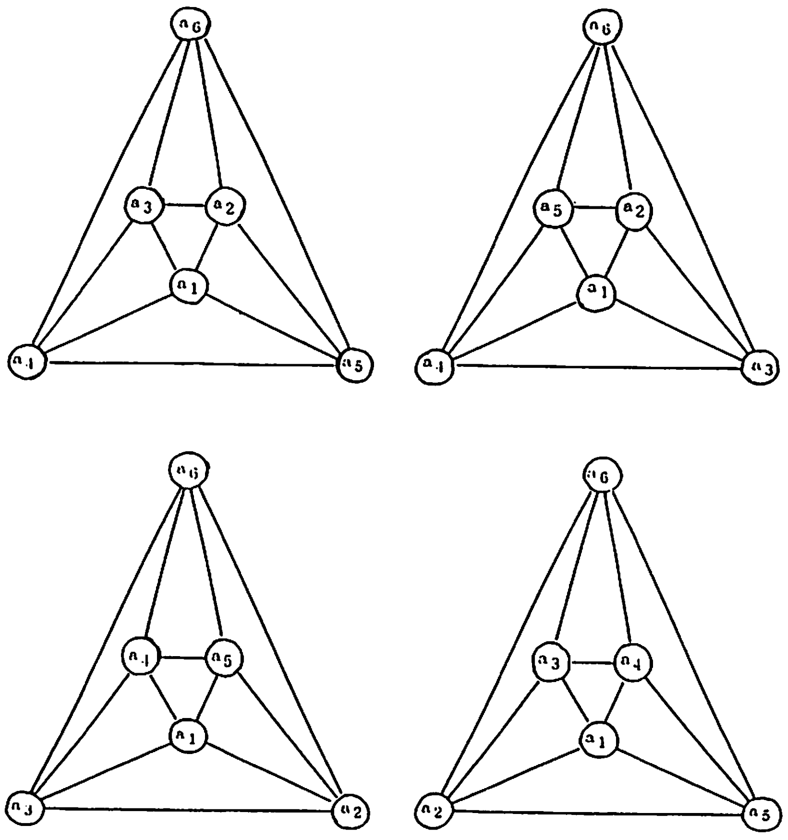

Figure 1. Four kinds of isomorphic vertex labelings of octahedron

Because each a_i is one of 1, 2, 3, 4, 5, and 6, and each is different from the others, $|a_i - a_j|$ must be one of 1, 2, 3, 4, and 5 provided that i does not equal to j.

Now let us prove that $|a_1 - a_6|$ cannot be 3 nor 5.

If $a_1 - a_6 = 3$, then $b_i = b_{4+i} + 3$, $1 \le i \le 4$. It is simple to verify that for numbers b_i, $1 \le i \le 8$, there exists no set of consecutive integers.

Analogously, if we suppose now that $a_1 - a_6 = 5$, then $b_i = b_{4+i} + 5$, $1 \le i \le 4$. It is easy to see that for numbers b_i, $1 \le i \le 8$, there exists no set of consecutive integers too.

According to the principle of symmetry, we also know that $a_1 - a_6$ cannot be -3 nor -5. It is easy to deduce that $|a_2 - a_4|$ and $|a_3 - a_5|$ can be neither 3 nor 5. So $|a_1 - a_6|$, $|a_2 - a_4|$ and $|a_3 - a_5|$ are the permutations of 1, 2, and 4.

There are only two kinds of non-isomorphic permutations of 1, 2, and 4. A representation of them is as follows:

$$(a_1, a_6) = (1,5); \quad (a_2, a_4) = (2,3); \quad (a_3, a_5) = (4,6) \qquad \text{(i)}$$
$$(a_1, a_6) = (2,6); \quad (a_2, a_4) = (4,5); \quad (a_3, a_5) = (1,3) \qquad \text{(ii)}$$

It is simple to verify that the vertex labelings of octahedron (described above by permutations) are consecutive labelings and they are non-isomorphic.

This completes the proof of the theorem.

In fact, the former from these two kinds of non-isomorphic vertex labelings of octahedron ((i) (ii)) can be deduced by the latter (or on the contrary).

Let us now consider the first representation (i) of vertex labeling. Define the vertex labeling of octahedron as follows:

$$a_i' = 7 - a_i \text{ for } i = 1, 2, \ldots, 6$$

So we get the representation

$$(a_1', a_6') = (6,2); \quad (a_2', a_4') = (5,4); \quad (a_3', a_5') = (3,1) \qquad \text{(iii)}$$

Now to obtain the representation (ii) we exchange arrangement in the pairs of representation (iii).

3 Algorithm

In this section, we describe an algorithm for obtaining the consecutive edge labelings of octahedron which are complementary to the consecutive vertex labeling.

Let c_i, $i = 1, 2, \ldots, 12$, denote the labels of edges of octahedron (see figure 2). The weights of 3-sided faces under the consecutive vertex labeling are given. They are $\{b_i \colon 1 \le i \le 8\} = \{7, 9, 8, 10, 11, 13, 12, 14\}$. If there exists a consecutive edge labeling of octahedron which is complementary to the consecutive vertex labeling, then the weights of 3-sided faces have to constitute the set $\{16, 17, 18, 19, 20, 21, 22, 23\}$ and the edge labeling must accord with following relations:

$$23 = c_{10} + c_{11} + c_{12} \qquad 16 = c_1 + c_2 + c_3$$
$$21 = c_5 + c_9 + c_{12} \qquad 18 = c_1 + c_4 + c_8$$
$$22 = c_6 + c_8 + c_{11} \qquad 17 = c_2 + c_5 + c_7$$
$$19 = c_4 + c_7 + c_{10} \qquad 20 = c_3 + c_6 + c_9$$

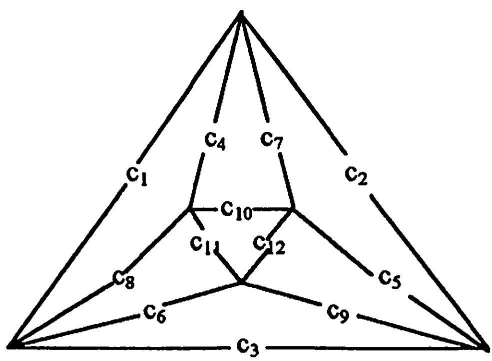

Figure 2. Label of edges of octahedron

It is easy to see that there exist 13 triples of integers which are solutions of the equation $23 = c_{10} + c_{11} + c_{12}$. They are:

12,10,1; 12,9,2; 12,8,3; 12,7,4; 12,6,5;
11,10,2; 11,9,3; 11,8,4; 11,7,5; 10,9,4;
10,8,5; 10,7,6; 9,8,6.

Let us further suppose that $c_{10} + c_{11} + c_{12} = 12 + 10 + 1$. We can get the following 10 kinds of combinations of c_i. Each complementary consecutive edge labeling in the case that $c_{10} + c_{11} + c_{12} = 12 + 10 + 1$ must be one of them.

c_{10}	c_{11}	c_{12}	$c_4 + c_7 = 19 - c_{10}$	$c_6 + c_8 = 22 - c_{11}$	$c_5 + c_9 = 21 - c_{12}$	$c_1 + c_2 + c_3$	consecutive labeling
1	10	12	11+7	3+9	4+5	2+6+8	yes
1	10	12	11+7	4+8	3+6	2+5+9	no
1	12	10	11+7	2+8	5+6	3+4+9	no
1	12	10	11+7	4+6	3+8	2+5+9	no
1	12	10	11+7	4+6	2+9	3+5+8	yes
10	12	1	2+7	4+6	11+9	3+5+8	no
10	12	1	4+5	3+7	11+9	2+6+8	yes
10	12	1	3+6	2+8	11+9	4+5+7	yes
12	10	1	3+4	5+7	11+9	2+6+8	yes
12	10	1	2+5	4+8	11+9	3+6+7	no

The steps for obtaining a complemelltaly consecutive edge labeling of octahedron are as follows:

1. Assign the values 12, 10, 1 to c_{10}, c_{11}, and c_{12};
2. Assign some values to c_4 and c_7 so that they are the solutions of the equation $c_4 + c_7 = 19 - c_{10}$;
3. Because $c_1 + c_8 = 18 - c_4$ and $c_6 + c_8 = 22 - c_{11}$, the values for c_1, c_8 and c_6 could be determined. If it is impossible then we reverse assignment the values for c_4 and c_7; If we still can not determine the values for c_1, c_8 and c_6, we turn to the next combination of c_{10}, c_{11}, and c_{12};
4. According to $c_2+c_5 = 17-c_7$, and $c_5+c_9 = 21-c_{12}$, we can determine c_2, c_5, and c_9;

Finally, $c_3 = 16 - c_1 - c_2$.

In the case, when $c_{10} + c_{11} + c_{12} = 12 + 10 + 1$, there exist 10 different possibilities to determine the values c_i, $1 \leq i \leq 12$, as listed above. Only five of the describe the complementary consecutive edge labelings and they are non-isomorphic each other. We can write them as follows:

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
8	2	6	7	4	9	11	3	5	1	10	12
3	8	5	11	2	6	7	4	9	1	12	10
6	2	8	5	11	3	4	7	9	10	12	1
4	5	7	6	9	2	3	8	11	10	12	1
8	2	6	3	11	5	4	7	9	12	10	1

4 Discussion

1. If we are going to find a consecutive edge labeling of octahedral which is complementary to the consecutive vertex labeling, then the weights of 3-sided faces are fixed. Therefore the number of complementary consecutive edge labelings is less than the number of non-complementary consecutive edge labelings. Figure 3 shows a non-complementary consecutive edge labeling.

2. Under the vertex labeling of octahedron the 3-sided faces can have the common weight (e.g. $b_i = b_j$ for $i \neq j$; $i, j = 1, 2, \ldots, 8$) if and only if $a_1 = a_6$, $a_2 = a_4$, and $a_3 = a_5$. This contradicts the fact that the vertex labeling of octahedron is a bijection from the set $\{1, 2, \ldots, 6\}$ onto the vertices of octahedron. Therefore we can formulate the next proposition: *An octahedron has not a magic vertex labeling.*

3. We define a new labeling: em An odd vertex labeling of plane graph G with p vertices is a one-to-one mapping of the set $\{1, 2, \ldots, 2p-1\}$ onto

the vertices of G. It is not difficult to verify that there exist exactly two non-isomorphic consecutive odd vertex labelings of octahedron. They are illustrated in Figure 4.

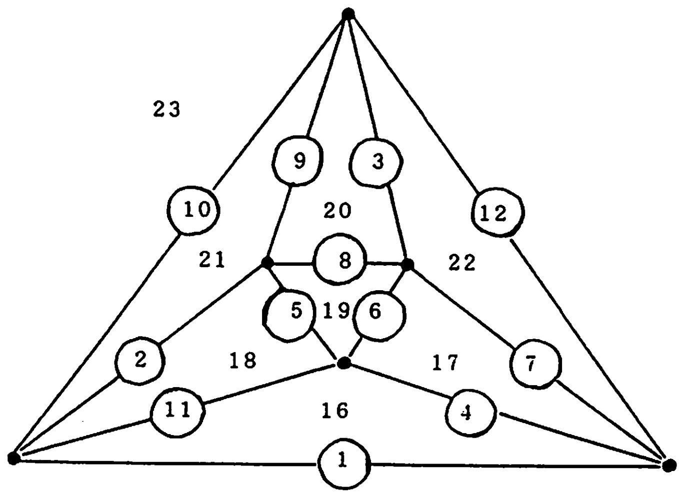

Figure 3. An non-complementary consecutive edge labeling

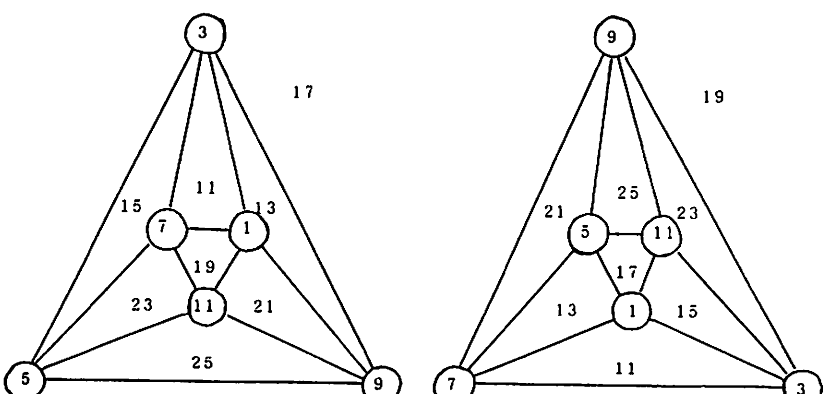

Figure 4. Two kinds of non-isomorphic consecutive odd vertex labelings

Acknowledgments. The Li Foundation has provided me with a fellowship which has permitted this paper to be written at the Needham Research Institute in Cambridge, UK. Mr. Lowell Skar and Professor Lih Ko-Wei read through an earlier draft of this paper and made some suggestions in the style of the paper.

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