A Characterization of Uniquely 2-List Colorable Graphs

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ABSTRACT. Let G be a graph with ν vertices. If there exists a list of colors $S_1, S_2, \ldots, S_\nu$ on its vertices, each of size k. Such that there exists a unique proper coloring for G from this list of colors, then G is called a *uniquely k-list colorable graph*. We prove that a connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph. For each k, a uniquely k-list colorable graph is introduced.

1 Introduction and Preliminaries

We consider simple graphs which are finite, undirected, with no loops or multiple edges. For the necessary definitions and notations we refer the reader to standard texts, such as [2]. Following that book we usually use ν for $|V(G)|$. In this section we mention some of the definitions and results which are referred to throughout the paper.

Let G be a graph with the vertex set $\{1, 2, \ldots, \nu\}$, and $S_1, S_2, \ldots, S_\nu$ a list of colors on its vertices. If there exists a proper coloring c for G such that $c(v) \in S_v$ for all $v \in V(G)$; then G is called to have a *list coloring*. By a proper coloring we mean for adjacent vertices u and v we have $c(u) \neq c(v)$. The idea of list coloring of graphs is due independently to V.G. Vizing [7] and to P. Erdös, A.L. Rubin, and H. Taylor [3]. For a recent survey on list coloring we refer the interested reader to N. Alon [1]. It is interesting to note that if the assumed graph is the complete graph K_ν, then a list coloring of G is nothing but a system of distinct representatives -SDR- for the sets $S_1, S_2, \ldots, S_\nu$. The following theorem of M. Hall, which is a corollary of the celebrated Marriage Theorem of P. Hall, is of a great interest to us.

ARS COMBINATORIA 51(1999), pp. 295-305

Theorem. [4] *If ν sets $S_1, S_2, \ldots, S_\nu$ have an SDR and the smallest of these sets contains t objects, then if $t \geq \nu$, at least there are $t(t-1)\ldots(t-\nu+1)$ different SDRs, and if $t < \nu$, there are at least $t!$ different SDRs.*

Corollary. *If ν sets $S_1, S_2, \ldots, S_\nu$ have an SDR and the smallest of these sets is at least of size 2, then there are at least two different SDRs. Or, equivalently, if there exists a list coloring for the complete graph K_ν with the list of colors $S_1, S_2, \ldots, S_\nu$, each of size at least 2, then there are at least two different colorings of K_ν with this list of colors.*

The following natural question is the motivation of this note.

Question 1. *For which graphs does the result of the above corollary hold?*

We say that G, a graph with ν vertices, *has the property* $M(2)$ (M for Marshal Hall), if for any list of colors $S_1, S_2, \ldots, S_\nu$, with $|S_i| \geq 2$; having a coloring for G implies that there exists also a different coloring for G. Note that in this definition, without loss of generality, we can assume $|S_i| = 2$; for $i = 1, 2, \ldots, \nu$. Conversely, a graph G with ν vertices, is called a *uniquely 2-list colorable graph*, if there exists $S_1, S_2, \ldots, S_\nu$, a list of colors on its vertices, each of size 2, such that there is a unique coloring for G from this list of colors. So G is uniquely 2-list colorable if and only if it does not have the property $M(2)$.

From the above corollary we see the following

Example 1. *Complete graphs K_ν have the property $M(2)$.*

Example 2. *The graph $K_4 - e$ is a uniquely 2-list colorable graph. Thus it does not have the property $M(2)$.*

To see this we give a list of colors, each of size two, as in Figure 1.

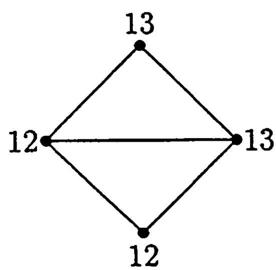

Figure 1. $K_4 - e$

Our main result is the following.

Theorem. (Main) *A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.*

In Section 2 we present the basic results which end up in the proof of the above theorem. In Section 3 we discuss further questions and problems which arise in this regards.

2 Basic Results

In this section we provide results concerning graphs which are uniquely 2-list colorable. These include some lemmas which will deduce the proof of the main theorem. The following proposition is clear from the definition of the property $M(2)$.

Proposition 1. *A graph has the property $M(2)$ if and only if at least one of its components has the property $M(2)$.*

In the following proposition we introduce some families of graphs which have the property $M(2)$.

Proposition 2. *Cycles C_ν, complete graphs K_ν, and complete bipartite graphs $K_{m,n}$ have the property $M(2)$.*

Proof: (i) Let $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_\nu, b_\nu\}$ be a list of assigned colors to the vertices of a cycle $C_\nu = (v_1, v_2, \ldots, v_\nu)$. And assume that there is a unique coloring c for C_ν with, say $c(v_i) = a_i$. We show that there is another coloring also. If $|\bigcup\{a_i, b_i\}| = 2$ then ν is even and by changing the color of each vertex from the used color to the unused one, we obtain another coloring. So, assume that $|\bigcup\{a_i, b_i\}| > 2$. If $|\bigcup\{a_i\}| = 2$; then there exists a color b_j which is not used in any vertex. Thus by recoloring v_j with this unused color we obtain a new coloring. So we further assume that $|\bigcup\{a_i\}| > 2$. Thus, there exist three consecutive vertices, say without loss of generality v_1, v_2, and v_3, which have distinct colors a_1, a_2, and a_3. Now by starting from vertex v_1 we can obtain a new coloring c' for the path $C_\nu - \{v_2\}$, with $c'(v_1) = b_1$. If $\{a_2, b_2\} \neq \{b_1, c'(v_3)\}$, then we can choose $c'(v_2)$ in such a way that c' be a new coloring for C_ν. Thus, assume that $\{a_2, b_2\} = \{b_1, c'(v_3)\}$.

If $c'(v_3) = a_3$, then $a_2 = b_1$ and $b_2 = a_3$. By defining $c''(v_3) = b_3$ and $c''(v_2) = b_2 = a_3$, we can recolor vertices of C_ν to obtain a new coloring c''.

If $c'(v_3) = b_3$, then from $\{a_2, b_2\} = \{b_1, c'(v_3)\}$, without loss of generality, we may assume that $a_2 = b_1$ and $b_2 = b_3$. In this case if $a_1 \neq b_2$ $(= b_3)$, again by defining $c''(v_3) = a_3$ and $c''(v_2) = b_2$, we can recolor vertices of C_ν to obtain a new coloring also.

So the worst case appears to be when $c'(v_3) = b_3$, $a_1 = b_2 = b_3$, and $a_2 = b_1$. We introduce a new coloring c'' for the latter case also. For simplicity we let $a_1 = b_2 = b_3 = 1$, and $a_2 = b_1 = 2$, and $a_3 = 3$. So $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c'(v_1) = 2$ and $c'(v_3) = 1$. Let k be the largest number between 3 and ν such that $3 \in \{a_i, b_i\}$, for $3 \leq i \leq k$; and

$3 \notin \{a_{k+1}, b_{k+1}\}$ (we assume $\nu + 1 \equiv 1$). There exists such a k, because $3 \in \{a_3, b_3\}$ and $3 \notin \{a_1, b_1\}$. Our new coloring c'', depends on the parity of k; i.e. the parity of the length of path $v_3 v_4 \dots v_k$.

If k is odd: Let $c''(v_i) = 3$; for $i = 3, 5, 7, \dots, k$, and $c''(v_i) = c'(v_i)$; for $i = k+1, \dots, \nu, 1$. The remaining vertices of C_ν can be colored appropriately.

If k is even: Let $c''(v_3) = 1$, and $c''(v_i) = 3$; for $i = 4, 6, 8, \dots, k$; and $c''(v_1) = c(v_1) = 1$, $c''(v_2) = c(v_2) = 2$. The remaining vertices of C_ν can be colored appropriately.

(ii) K_ν having the property $M(2)$ is M. Hall's result, as was mentioned in Example 1.

(iii) For $K_{m,n}$, let X and Y be two parts of this graph. We prove by contradiction. Suppose that there exists a list of colors each of size 2, for which there is a unique coloring for $K_{m,n}$. Let X_1 be the set of colors used in the vertices of X. For each vertex in X there is a color in the list of colors associated to that vertex which is not used in the coloring. Let X_2 be the set of such colors. We denote similar sets in Y, by Y_1 and Y_2. If there exits an element in X_2 which does not belong to Y_1, by changing the color of corresponding vertices of that element in X, we obtain a different coloring. Thus $X_2 \subset Y_1$. Similarly $Y_2 \subset X_1$. Clearly X_1 and Y_1 are disjoint. Thus X_2 and Y_2 are also disjoint. Now by switching the color of each vertex from used color to the unused color in that vertex, we obtain a new coloring. □

Next we state some lemmas.

Lemma 1. *If an induced subgraph H of a connected graph G is uniquely 2-list colorable, then G is uniquely 2-list colorable also.*

Proof: We prove by induction on $\nu(G) - \nu(H)$. If $\nu(G) - \nu(H) = 0$, then $G = H$, and there is nothing to prove. Now, let $\nu(G) - \nu(H) = k$, where $k > 0$. Since G is connected, there exists a vertex $\nu \in V(G) - V(H)$, which is adjacent to a vertex u in H. We claim that $H_1 = \langle V(H) \cup \{v\} \rangle$ is also uniquely 2-list colorable. The graph H being uniquely 2-list colorable, implies that there exists a list of colors $S_1, S_2, \dots, S_{\nu(H)}$ with $|S_i| = 2$, such that H has a unique coloring $c_1, c_2, \dots, c_{\nu(H)}$ ($c_i \in S_i$) with this list. Now we assign the list $\{c(u), x\}$, where x is a new color, for the vertex v in the induced subgraph H_1, and keep the list of other vertices of H unchanged. It is easy to see that H_1 with the assigned list is uniquely list colorable. But for H_1 we have $\nu(G) - \nu(H_1) = k - 1$, so the result follows by induction. □

Lemma 2. *Let G be a graph which is the union of two graphs G_1 and G_2 which are joined in exactly one vertex, and both have the property $M(2)$. Then G has the property $M(2)$ also.*

Proof: Let G_1 and G_2 be joined in a vertex x. Assume that a list of colors for the vertices of G is given, each of size 2, and there is a coloring c for

G with this list. Suppose that the list given to the vertex x is $\{a, b\}$ and $c(x) = a$. Now with this list of colors, G_1 also has a coloring different from c say c_1. If $c_1(x) = a$, then by changing the colors of vertices of G_1 in G to c_1 coloring, we obtain a different coloring for G. So assume that $c_1(x) = b$. Similarly in c_2, the second coloring of G_2, if $c_2(x) = a$ we are done. Thus $c_2(x) = b$. Then by coloring G_1 with c_1 and G_2 with c_2, which are consistent in x, we obtain a different coloring for G. □

Corollary. *A connected graph G has the property $M(2)$ if and only if each of its blocks has the property $M(2)$.*

Proof: It follows from Lemma 1, Lemma 2, and by induction. □

So we only need to study 2-connected graphs having the property $M(2)$. The following lemma will be useful.

Lemma 3. *Suppose that a graph G has a triangle abc such that, there exists a path $av_1v_2 \ldots v_kb$ from the vertex a to the vertex b and disjoint from c, in which v_k is not adjacent to c. Then G is uniquely 2-list colorable.*

Proof: A subgraph of G is shown in Figure 2. There may be more edges in the induced subgraph on these vertices but c is not adjacent to v_k. We give the following list of colors to the vertices: $\{1, 2\}$ for b and c, $\{2, 3\}$ for a, $\{3, 4\}$ for v_1, $\{4, 5\}$ for $v_2, \ldots, \{k+1, k+2\}$ for v_{k-1}, and $\{k+2, 1\}$ for v_k. This list of colors shows that an induced subgraph of G is uniquely 2-list colorable, therefore by Lemma 1, G is uniquely 2-list colorable. □

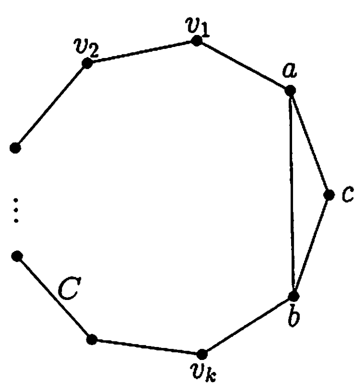

Figure 2. A subgraph of G

Lemma 4. *If a 2-connected graph G which contains a triangle has the property $M(2)$, then G is a complete graph.*

Proof: We prove by contradiction. Assume that G is not a complete graph and let H be a (maximal) clique in G. So by assumption, $V(G) - V(H) \neq \emptyset$, and there exists $v \in V(G) - V(H)$ which is adjacent to a vertex b in H. Since H is a largest complete subgraph of G, there exists a vertex in H, which is not adjacent to v. We consider two cases:

(i) v is adjacent to another vertex a in H. Let c be a vertex in H which is not adjacent to v. Then the subgraph induced on the vertices $\{a, b, c, v\}$ is $K_4 - e$, and as in Example 2, is uniquely 2-list colorable. Thus by Lemma 1, G is uniquely 2-list colorable.

(ii) v is adjacent only to b in H. Then since G is 2-connected, there exists a path from v to a vertex a (different from b) in H. Let c be a third vertex in H different from a and b. By our assumption in this case, c is not adjacent to v. Now applying Lemma 3 follows that G is a uniquely 2-list colorable graph, and this is a contradiction. □

Lemma 5. *If a 2-connected graph G which contains an odd cycle has the property $M(2)$, then G is either a complete graph or an odd cycle.*

Proof: Let C be a smallest odd cycle in G. If C is a triangle, then by Lemma 4, G is a complete graph and the statement follows. So assume that C is not a triangle. Note that C does not have any chord, otherwise we obtain an odd cycle smaller than C. Now if G is not an odd cycle, then there exists a vertex v in $G - C$ which is adjacent to a vertex a in C. There are two cases to be considered:

(i) v is adjacent to at least two vertices of C say a and b. In this case, since C is the smallest odd cycle of G, a and b are at distance two in C and they are the only vertices of C which are adjacent to v. Thus the induced subgraph on $V(C) \cup \{v\}$ in G is a theta graph. It is shown in Figure 3.

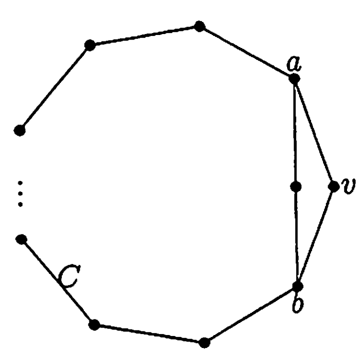

Figure 3. The induced subgraph on $V(C) \cup \{v\}$

If in this subgraph we assign the list of colors as follows: $\{1,3\}$ for both b and v and $\{1,2\}$ for all other vertices, we see that it is uniquely 2-list colorable. So by Lemma 1 this is a contradiction to the assumption.

(ii) a is the only vertex in C which is adjacent to v. Since G is a 2-connected graph, there exists a path from v to a vertex in $C - a$. Let $bv_1v_2\dots v_kv$ ($k \geq 1$), be a shortest path from v to a vertex b in $C - a$. We show that the induced subgraph $G_1 = \langle V(C) \cup \{v, v_1, v_2, \dots, v_k\}\rangle$ is uniquely 2-list colorable, which by Lemma 1 is a contradiction to the assumption. In Figure 4, a spanning subgraph of G_1 is shown. G_1 may also contain other edges than shown in the figure, but v is not adjacent to any vertex in C except to a. Assign the following lists to the vertices of G_1: $\{1,3\}$ for b, $\{1,2\}$ for all other vertices of C, and $\{3,4\}$ for v_1, $\{4,5\}$ for $v_2, \dots, \{k+2, k+3\}$ for v_k, $\{1, k+3\}$ for v. Now we prove that with this list, G_1 is uniquely list colorable. First, we show that if there *exists* a coloring for G_1 it is unique. Assume that there is a coloring c for G_1. In any coloring of the odd cycle C with the given list, the color of b must be 3. This forces the following colors for other vertices: $c(v_1) = 4$, $c(v_2) = 5, \dots, c(v_k) = k+3$, $c(v) = 1$, and $c(a) = 2$. Then the other vertices of $C - b$ are *forced* to be either 1 or 2. So the coloring is unique. But it is interesting to note that the above coloring in fact, is a proper coloring for G_1. □

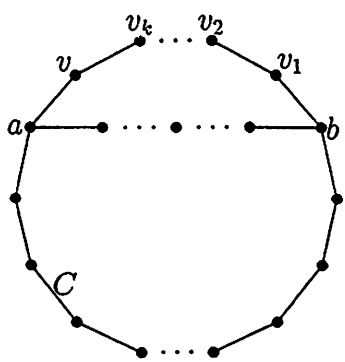

Figure 4. A spanning subgraph of $G_1 = \langle V(C) \cup \{v, v_1, v_2, \dots, v_k\}\rangle$

Now we turn our attention to the bipartite graphs.

Lemma 6. *If a bipartite 2-connected graph G which contains a square has the property $M(2)$, then G is a complete bipartite graph.*

Proof: Let H be a maximal complete bipartite subgraph of G, which contains a square. If G is not a complete bipartite graph, then $V(G) - V(H) \neq \emptyset$. So there exists a vertex $v \in V(G) - V(H)$, which is adjacent to a vertex in H. Let $V(H) = X \cup Y$, where X and Y are independent sets. Assume that v is adjacent to a vertex a in X. There exists a vertex b in X which is not adjacent to v, otherwise it is contrary to H being a maximal complete bipartite subgraph. There exists a path in $G - a$ from v to any vertex of Y. Let P be a shortest path in $G-a$ joining v to a vertex c in Y ($k \geq 1$). Since H contains a square, $Y-c$ is non-empty. Let d be a vertex in $Y-c$. Note that since P is a shortest path, it does not contain d. Depending on whether P contains b or not, we have two cases:

(i) $b \notin P$. Let $P = vv_k \ldots v_2v_1c$, and consider the induced subgraph $G_1 = \langle\{a,b,c,d,v,v_1,v_2,\ldots, v_k\}\rangle$. The graph shown in Figure 5, is a spanning subgraph of G_1. There may be more edges in G_1 than shown in the figure, but b is not adjacent to v. G_1 is a uniquely 2-list colorable graph, as can be seen by the following list of colors: $\{1,2\}$ for a, $\{1,3\}$ for both b and c, $\{2,3\}$ for d, $\{3,4\}$ for v_1, $\{4,5\}$ for $v_2,\ldots,\{k+2,k+3\}$ for v_k, and $\{k+3,1\}$ for v.

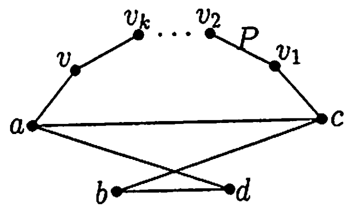

Figure 5. A spanning subgraph of $G_1 = \langle\{a,b,c,d,v,v_1,v_2,\ldots,v_k\}\rangle$

(ii) $b \in P$. Let $P = vv_k \ldots v_2v_1bc$, and consider the induced subgraph $G_2 = \langle\{a,b,c,d,v,v_1,v_2,\ldots, v_k\}\rangle$. The graph shown in Figure 6, is a spanning subgraph of G_2. Also there may be more edges in G_2 than shown in the figure, but b is not adjacent to v. G_2 is a uniquely 2-list colorable graph, as can be seen by the following list of colors: $\{2,3\}$ for a, $\{1,3\}$ for both b and c, $\{1,2\}$ for d, $\{3,4\}$ for v_1, $\{4,5\}$ for $v_2,\ldots,\{k+2,k+3\}$ for v_k, and $\{k+3,3\}$ for v.

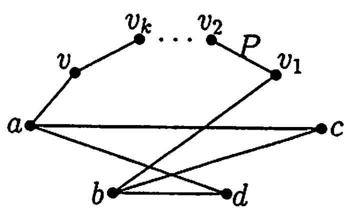

Figure 6. A spanning subgraph of $G_2 = \langle\{a,b,c,d,v,v_1,v_2,\ldots,v_k\}\rangle$

In either case, by Lemma 1, there is a contradiction to G having the property $M(2)$. □

Lemma 7. *If a bipartite 2-connected graph G has the property $M(2)$, then G is either an even cycle or a complete bipartite graph.*

Proof: Let C be a smallest cycle in G. If C is a square, then by the previous lemma we are done. So the size of C is at least 6. Also there is no chord in C. If G is not a cycle, then $V(G)-V(C)$ is non-empty. Let $v \in V(G)-V(C)$ be a vertex adjacent to a vertex a in C. v is not adjacent to any other vertex of C; for, then we obtain a cycle smaller than C. But there exists a shortest path, $vv_k \dots v_2v_1b$, from v to some vertex $b \in C - a$ ($k \geq 1$). Now, consider the induced subgraph $G_1 = \langle V(C) \cup \{v, v_1, v_2, \dots, v_k\} \rangle$. The graph shown in Figure 7, is a spanning subgraph of G_1. There may be more edges in G_1 than shown in the figure, but there is no chord in C and v is not adjacent to any vertex of C except to a.

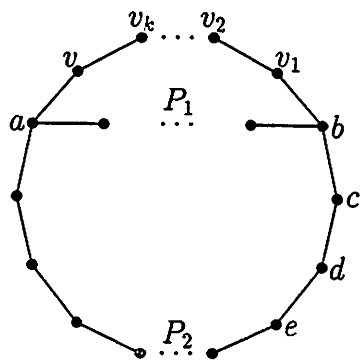

Figure 7. A spanning subgraph of $G_1 = \langle V(C) \cup \{v, v_1, v_2, \dots, v_k\} \rangle$

The cycle C consists of two paths P_1 and P_2 each joining b to a. Note that the length of at least one of these paths, say P_2, is greater than or equal to three. So, let $P_2 = bcde \dots a$. We do not exclude the case that e may be equal to a; i.e. the lengths of P_2 and P_1 both are equal to three. The graph G_1 is uniquely 2-list colorable, as can be seen by the following list of colors: $\{2,3\}$ for both b and c, $\{1,3\}$ for d, $\{1,2\}$ for all other vertices of C including a and e, $\{3,4\}$ for v_1, $\{4,5\}$ for $v_2, \dots, \{k+2, k+3\}$ for v_k, and $\{k+3, j\}$ for v. Here j is a fixed element of the set $\{1,2\}$, and we will give the exact value of it shortly. Note that since C is an even cycle, in any proper coloring with the above list the color of b must be 3. Which in turn forces the color of the vertex c to be 2. Now if we fix the color of e to be 1, we obtain a unique coloring for C. Let i be the color of a in this coloring ($i \in \{1,2\}$). In the list of colors for v we put $j = \{1,2\} - \{i\}$. The uniqueness of coloring can be checked easily. Thus by Lemma 1, it is a contradiction to G having the property M(2). □

Now we are ready to prove the main theorem.

Theorem. (Main) *A connected graph has the property $M(2)$ if and only if every block of which is either a cycle, a complete graph, or a complete bipartite graph.*

Proof: The result follows by Proposition 1, Proposition 2, Lemma 2, Lemma 5, and Lemma 7. □

3 Addendum

The definition of having the property $M(2)$ can be generalized naturally. Let G be a graph with ν vertices. We say that G *has the property* $M(k)$, if for any list of colors $S_1, S_2, \ldots, S_\nu$, with $|S_i| \geq k$; having a coloring for G implies that there exists also a different coloring for G. Conversely, let G be a graph with ν vertices. If there exists a list of colors $S_1, S_2, \ldots, S_\nu$ on its vertices each of size k such that there exists a unique coloring for G from this list of colors, then G is called a *uniquely k-list colorable graph.* So G is uniquely k-list colorable if and only if it does not have the property $M(k)$. Similar questions such as Question 1 can be asked again.

Question 2. *For each k, characterize all uniquely k-list colorable graphs.*

It is clear that:

If G is a uniquely $(k+1)$-list colorable graph, then G is also a uniquely k-list colorable graph. In the following example we introduce a uniquely k-list colorable graph for each k. Example 2, given earlier, is a special case of the following when $k = 2$.

Example 3. *Let $G = K_{X_1, X_2, \ldots, X_{2k-1}}$ be a complete $(2k-1)$-partite graph with the parts $X_1, X_2, \ldots, X_{2k-1}$; such that $|X_1| = |X_2| = \cdots = |X_{k-1}| = 2$ and $|X_k| = |X_{k+1}| = \cdots = |X_{2k-1}| = 1$. We assign a list of colors to vertices of G as follows. Each list consists of k colors and we have:*
$\{\{1,2,3,\ldots,k\}, \{1,k+1,k+2,\ldots,2k-1\}\}$ for the vertices in X_1,
$\{\{1,2,3,\ldots,k\}, \{1,2,k+1,\ldots,2k-2\}\}$ for the vertices in X_2,
$\{\{1,2,3,\ldots,k\}, \{1,2,3,k+1,\ldots,2k-3\}\}$ for the vertices in $X_3, \ldots,$
$\{\{1,2,3,\ldots,k\}, \{1,2,3,\ldots,k-1,k+1\}\}$ for the vertices in X_{k-1}, and
$\{1,2,3,\ldots,k\}, \{1,2,3,\ldots,k-1,k+1\}, \ldots, \{1,2,3,\ldots,k-1,2k-1\}$ for the vertices in $X_k, X_{k+1}, \ldots, X_{2k-1}$ respectively. G is a uniquely k-list colorable graph.

Proof: Since G is complete $(2k-1)$-partite graph, at least $(2k-1)$ different colors are needed to color its vertices. But the union of all the lists given above, has $(2k-1)$ colors in total. So vertices in each part X_i, $i = 1, \ldots, k-1$ must have the same color. This forces the color 1 for the vertices of X_1, 2 for the vertices of $X_2, \ldots,$ and $k-1$ for the vertices of X_{k-1}. Thus, colors of the remaining vertices are determined uniquely. □

Graphs having the property $M(k)$ have applications in the discussion of defining sets in graph colorings and in the critical sets in latin squares. See [5] and [6].

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