Some new large sets of KTS(v)

Chang Yanxun *
Department of Mathematics
Northern Jiaotong University
Beijing, 100044
P.R. China

Ge Gennian
Department of Mathematics
Suzhou University
Suzhou, 215006
P.R. China

ABSTRACT. A large set of KTS(v), denoted by LKTS(v), is a collection of ($v-2$) pairwise disjoint KTS(v) on the same set. In this article some new LKTS(v) is constructed.

1 Introduction

A Steiner triple system of order v (briefly STS(v)) is a pair $(X, \mathcal{A})$, where X is a set containing v-elements and $\mathcal{A}$ is a collection of 3-subsets (called *triple*) of X, such that every unordered pair of X appears in exactly one triple. For $\mathcal{A}_1 \subset \mathcal{A}$ and any $x \in X$, if x appears in exactly one triple of $\mathcal{A}_1$, we call $\mathcal{A}_1$ a *parallel class* of the STS(v). If $\mathcal{A}$ can be partitioned into disjoint parallel classes, we call the STS(v) a *Kirkman triple system*, which is denoted by KTS(v).

A large set of KTS(v) (or STS(v)), denoted by LKTS(v) (or LSTS(v)), is a collection of $(v-2)$ pairwise disjoint KTS(v) (or STS(v)) on the same set. The necessary condition for the existence of LKTS(v) is $v \equiv 3 \pmod 6$. So far, knowledge about the existence of LKTS(v) is very limited, see [1], [2], [3], [5], [10]. The known results are summarized as follows.

*Research supported by NSFC grant.

Theorem 1.1. *An LKTS($3^n m$) exists for any positive integer n and $m \in \{1, 5, 11, 17, 25, 35, 43\}$.*

In this article, we give a way to construct LKTS($q+2$), where q is a prime power and $q \equiv 1 \pmod 6$. Using the method, some unknown LKTS(v) is constructed.

2 $Y - Z$ partitions

Let R be a ring containing n elements, where n is not divided by 2 and 3. Let $R^* = R \setminus \{0\}$. A $Y - Z$ partition of R is a partition $R^* = Y \cup Z$ such that

(a) x is never in the same class as $-2x$, and

(b) x is never in the same class as $-x$.

Wilson [9] and Schreiber [4] use the notation of $Y - Z$ partition to construct large set of Steiner triple systems in the following.

Construction. *Let $X = R \cup \{\infty_1, \infty_2\}$, where $R \cap \{\infty_1, \infty_2\} = \emptyset$. $\mathcal{B}_0$ consists of the following triples:*

(1) $\{\infty_1, \infty_2, 0\}$;

(2) $\{\infty_1, x, -2x\}$ *for each x in Y;*

(3) $\{\infty_2, x, -2x\}$ *for each x in Z;*

(4) *all triples $\{x, y, z\}$ such that $x + y + z = 0$ in R.*

Condition (a) ensures that $\mathcal{B}_0$ is a STS(v). If we then add to every element of $\mathcal{B}_0$ a fixed element i, keeping ∞_1 and ∞_2 fixed, we have a 2-design $\mathcal{B}_i$ which is isomorphic with $\mathcal{B}_0$. Condition (b) ensures that the n 2-designs $(R \cup \{\infty_1, \infty_2\}, \mathcal{B}_i)$ $(i = 1, 2, \ldots, n)$ are all disjoint; and this means all n 2-designs $\mathcal{B}_i$ form a large set of Steiner triple system (i.e. LSTS($n+2$)).

Deniston [1] and Schreiber [5] has constructed an LKTS($n+2$) by partitioning $\mathcal{B}_0$ into parallel classes. We state the construction as follows.

Theorem 2.1. *Let R be a ring containing n elements with a $Y - Z$ partition, where n is not divided by 2 and 3. Construct a 2-design $\mathcal{B}_0$ on $X = R \cup \{\infty_1, \infty_2\}$ as the construction listed above. Define $\mathcal{B}_i = \mathcal{B}_0 + i$, for $i \in R$. Then $\{(X, \mathcal{B}_i) : i \in R\}$ forms an LSTS($n+2$). Moreover, if $\mathcal{B}_0$ is also a KTS($n+2$), then the LSTS($n+2$) is also an LKTS($n+2$).*

We give several lemmas on $Y - Z$ partitions.

Lemma 2.2. *If R_1 has a $Y_1 - Z_1$ partition $R_1 \setminus \{0\} = Y_1 \cup Z_1$ and R_2 has a $Y_2 - Z_2$ partition $R_2 \setminus \{0\} = Y_2 \cup Z_2$, then $R_1 \times R_2$ has a $Y - Z$ partition.*

Proof: Let

$$Y = (Y_1 \times R_2) \cup (\{0\} \times Y_2),$$
$$Z = (Z_1 \times R_2) \cup (\{0\} \times Z_2).$$

It is readily checked that $R_1 \times R_2$ has a $Y - Z$ partition. □

Lemma 2.3. *Let GF(q) be a finite field and t be the multiplicative order of −2 in GF(q)*. Then GF(q) has a $Y - Z$ partition if and only if $t \equiv 2 \pmod 4$.*

Proof: Suppose that GF(q) has a $Y - Z$ partition. Without loss of the generality, let $1 \in Y$. By the condition (a) of $Y - Z$ partition, we have $(-2)^1 = -2 \in Z, (-2)^2 \in Y, \ldots, (-2)^{2i-1} \in Z, (-2)^{2i} \in Y, \ldots$.

Since $1 = (-2)^t \in Y$, it implies that t is even. Let $t = 2s$. Note that $(-2)^s = -1$. By the condition (b) of $Y - Z$ partition, $(-2)^s \in Z$, which implies s odd. Thus, $t \equiv 2 \pmod 4$.

If $t \equiv 2 \pmod 4$, let $\langle -2 \rangle$ be the multiplicative subgroup of GF$(q)^*$ generated by -2, and let $h_0, h_1, \ldots, h_{\frac{q-1}{t}-1}$ be all the representative elements of coset classes. Define

$$Y = \{h_j(-2)^{2i-1} : i = 0, 1, \ldots, \frac{t}{2}; j = 0, 1, \frac{q-1}{t} - 1\};$$
$$Z = \{h_j(-2)^{2i} : i = 0, 1, \ldots, \frac{t}{2}; j = 0, 1, \ldots, \frac{q-1}{t} - 1\}.$$

It is readily checked that GF(q) has a $Y - Z$ partition. □

Corollary 2.4. *Let GF(q) be a finite field with $q \equiv 7 \pmod 8$. Then GF(q) has a $Y - Z$ partition .*

Proof: Let $q = p^n$, where p is a prime. It is easy to see that $q \equiv 7 \pmod 8$ implies $p \equiv 7 \pmod 8$. Note that $(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}} = 1$ and $(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}} = -1$. Then

$$(\frac{-2}{p}) = -1,$$

which implies -2 is a quadratic non-residue. Let t be the multiplicative order of -2, i.e., $(-2)^t = 1$. So, t is even. As t divides $q-1$ and $q - 7 \pmod 8$, we have $t \equiv 2 \pmod 4$. The result follows from Lemma 2.3. □

3 A way to construct LKTS(v)

Let GF(q) be a finite field containing q elements, where q is a prime power and $q \equiv 1 \pmod 6$. Let g be a primitive element of GF(q).

For any $\lambda \neq \mu \in Z_{q-1}^*$, if $g^\lambda + g^\mu = -1$, then we call the unordered pair $\{\lambda, \mu\}$ a couple. Let $-2 = g^\theta$. If $\{\lambda, \mu\}$ is a couple, then $\{\lambda, \mu\} \subset Z_{q-1}^* \setminus \{\theta, q-1-\theta, \frac{q-1}{2}\}$. It is not difficult to see that there are $\frac{q-5}{2}$ couples for given q. All couples form a partition of $Z_{q-1}^* \setminus \{\theta, q-1-\theta, \frac{q-1}{2}\}$.

Given a couple $\{\lambda, \mu\}$, the collection of triples

$$\{\{g^x, g^{x+\lambda}, g^{x+\mu}\} : x \in Z_{q-1}\}$$

are called a g-orbit defined by $\{\lambda, \mu\}$. It is easy to see that the g-orbits defined by $\{\lambda, \mu\}$, $\{-\mu, \lambda-\mu\}$, or $\{\mu-\lambda, -\lambda\}$ are the same thing. In general, if $\mu \neq 2\lambda$, or $\frac{1}{2}\lambda$, the three couples $\{\lambda, \mu\}$, $\{-\mu, \lambda-\mu\}$ and $\{\mu-\lambda, -\lambda\}$ are different. Thus the corresponding g-orbit (called *normal g-orbit*) contains $q-1$ disjoint triples. If $\lambda = 2\mu$, i.e., $\lambda = \frac{q-1}{2}$, $\mu = \frac{2(q-1)}{3}$, the three couples $\{\lambda, \mu\}$, $\{-\mu, \lambda-\mu\}$ and $\{\mu-\lambda, \lambda\}$ are the same. The g-orbit defined by $\{\frac{q-1}{3}, \frac{2(q-1)}{3}\}$ (called *short g-orbit*) contains $\frac{q-1}{3}$ disjoint triples. We summarize as follows.

Lemma 3.1. *All $\frac{q-5}{2}$ couples form one short g-orbit and $\frac{q-7}{6}$ normal g-orbit.*

In what follows, let $\{\lambda_i, \mu_i\}$ $(i = 1, 2, \ldots, \frac{q-7}{6})$ denote all normal g-orbits. Take $R = \mathrm{GF}(q)$ in Theorem 2.1. We obtain an LSTS$(q+2) = \{(GF(q) \cup \{\infty_1, \infty_2\}, \mathcal{B}_i) : i \in GF(q)\}$ where $\mathcal{B}_i = \mathcal{B}_0 + i$. If there exist $\frac{q-7}{6}$ elements x_i and an element y such that

$$\begin{aligned}
&\cup_{i=1}^{\frac{q-7}{6}} \{x_i, x_i+\lambda_i, x_i+\mu_i\} \cup \\
&\quad (\cup_{i=1}^{\frac{q-7}{6}} \{x_i + \frac{q-1}{2}, x_i+\lambda_i+\frac{q-1}{2}, x_i+\mu_i+\frac{q-1}{2}\}) \\
=&Z_{q-1}^* \setminus \{\frac{q-1}{2}, \theta, y, \theta+\frac{q-1}{2}, y+\frac{q-1}{2}\}. \qquad (1)
\end{aligned}$$

Then $\mathcal{B}_0$ can be partitioned into parallel classes as follows:

$\mathcal{P}_0$: $\{\infty_1, \infty_2, 0\}, \{g^j, g^{\frac{q-1}{3}+j}, g^{\frac{2(q-1)}{3}+j}\}, j = 0, 1, \ldots, \frac{q-1}{3} - 1.$

$\mathcal{P}_1$: $\{\infty_1, g^0, g^\theta\}, \{\infty_2, g^{\frac{q-1}{2}}, g^{\theta+\frac{q-1}{2}}\}, \{0, g^y, g^{y+\frac{q-1}{2}}\}, \{g^{x_i}, g^{x_i+\lambda_i}, g^{x_i+\mu_i}\},$
$\{-g^{x_i}, -g^{x_i+\lambda_i}, -g^{x_i+\mu_i}\}, i = 1, 2, \ldots, \frac{q-7}{6}.$

Let

$$\mathcal{P}_i = \{\{g^{a+i}, g^{b+i}, g^{c+i}\} : \{g^a, g^b, g^c\} \in \mathcal{P}_1\}$$

for $i = 1, 2, \ldots, \frac{q-1}{2}$.

It is easy to check that $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{\frac{q-1}{2}}$ are pairwise disjoint and $\mathcal{P}_i \subseteq \mathcal{B}_0$. So,

$$\mathcal{B}_0 = \cup_{i=1}^{\frac{q-1}{2}} \mathcal{P}_i$$

Hence $\mathcal{B}_0$ is a KTS$(q+2)$. By Theorem 2.1 the LSTS$(q+2)$ is also an LKTS$(q+2)$. We state as follows.

Theorem 3.2. *Let GF(q) be a finite field with a $Y-Z$ partition, where $q \equiv 1 \pmod 6$. If there exist $\frac{q-7}{6}$ elements x_i and an element y in $Z_{q-1}\setminus\{0\}$ satisfying the condition (1), then there exists an LKTS$(q+2)$.*

By Corollary 2.4 and Theorem 3.2, we obtain

Corollary 3.3. *Let GF(q) be a finite field and $q \equiv 7 \pmod{24}$. If there exist $\frac{q-7}{6}$ elements x_i and an element y satisfying the condition (1), then there exists an LKTS$(q+2)$.*

Lemma 3.4. *There exists an LKTS$(q+2)$ for $q = 199, 367$.*

Proof: For $q = 199$ and 367, there exist $\frac{q-7}{6}$ elements x_i and an element y listed as follows satisfying the condition (1). By Corollary 3.3 there exists an LKTS$(q+2)$.

When $q = 199$, $g = 3$, $\theta = 7$, $y = 87$

i	x_i	λ_i	μ_i	i	x_i	λ_i	μ_i
1	1	1	113	17	42	18	151
2	3	2	145	18	72	19	97
3	6	3	57	19	55	20	137
4	4	4	174	20	45	21	142
5	11	5	41	21	54	22	118
6	13	6	98	22	34	23	49
7	14	8	138	23	31	27	153
8	17	9	123	24	38	30	109
9	10	10	158	25	46	32	135
10	18	11	62	26	44	33	116
11	21	12	71	27	62	34	69
12	23	13	42	28	47	37	111
13	25	14	39	29	89	38	105
14	56	15	73	30	43	43	107
15	24	16	110	31	37	44	152
16	50	17	48	32	74	52	122

When $q = 367$, $g = 6$, $\theta = 109$, $y = 106$

i	x_i	λ_i	μ_i	i	x_i	λ_i	μ_i
1	134	1	195	31	21	33	177
2	27	2	65	32	85	34	108
3	70	3	169	33	173	35	75
4	126	4	59	34	127	36	296
5	153	5	22	35	100	37	191
6	110	6	328	36	168	41	86
7	32	7	305	37	140	42	320
8	91	8	135	38	19	43	308
9	80	9	217	39	3	47	225
10	115	10	150	40	48	48	129
11	76	11	210	41	132	49	115
12	4	12	327	42	33	50	180
13	105	13	152	43	13	53	256
14	49	14	276	44	14	54	224
15	128	15	234	45	145	56	299
16	59	16	290	46	102	57	261
17	104	18	248	47	107	60	174
18	165	19	314	48	133	62	221
19	22	20	213	49	101	69	271
20	58	21	287	50	11	73	284
21	88	23	246	51	17	78	190
22	136	24	277	52	138	80	281
23	139	25	294	53	41	83	182
24	152	26	205	54	176	84	218
25	129	27	91	55	34	87	272
26	69	28	121	56	61	96	215
27	9	29	146	57	131	98	255
28	120	30	107	58	141	102	233
29	5	31	229	59	161	103	228
30	117	32	238	60	31	116	242

□

4 Conclusion

A Kirkman triple system $(X, \mathcal{B})$ of order v is called transitive, denoted by TKTS(v), if there exists a transitive automorphism group G of order v of $(X, \mathcal{B})$.

Lei and Chang [7] and Lei [8] investigate the existence of TKTS(v). We summarize the result as follows.

Lemma 4.1. *There exists a TKTS*($3^k 5^l 11^m 17^n q_1 q_2 \dots q_t$), *where* $k \geq 1$, $l, m, n \in \{0, 1\}$ *and* q_i *are prime power and* $q_i \equiv 1 \pmod 6$ *for* $1 \leq i \leq t$.

With the notation of TKTS(v), Denniston [3] obtain a recursive construction of LKTS($3v$) from an LKTS(v), i.e.,

Lemma 4.2. *If an LKTS(v) and a TKTS(v) both exist, then there exists an LKTS(3v).*

Corollary 4.3. *A TKTS($3^n \cdot 67$) exists for any $n \geq 1$.*

Proof: It follows by Lemma 4.1. □

Theorem 4.4. *An LKTS($3^n \cdot 67$) and an LKTS(369) exist for any $n \geq 1$.*

Proof: It follows immediately from Lemma 3.4, Lemma 4.2 and Corollary 4.3. □

Acknowledgment. The authors would like to thank Professor L. Zhu for his advice and useful suggestions.

References

[1] R.H.F. Denniston, Four doubly resolvable complete 3-designs, *Ars Combinatoria* **7** (1979), 265–272.

[2] R.H.F. Denniston, Doubly resolvability of some complete 3-designs, *Manuscripta Math.* **12** (1974), 105–112.

[3] R.H.F. Denniston, Further case of double resolvability, *J. Combin. Theory (A)* **26** (1979), 298–303.

[4] S. Schreiber, Covering all triples on n marks by disjoint Steiner systems, *J. Comb. Theory (A)* **15** (1973), 347–350.

[5] S. Schreiber, quoted by H. Hanani, Resolvable designs, Colloquio Internazional Sulle Teorie Combinatorie (*Atti dei Convegni Lincei*, **17**), Rome (1976), Tomo II, 249–252.

[6] Kang Qingde, Partitions of triples into resolvable MTS and DTS for order $p^n + 2$, to appear.

[7] Lei Jianguo and Chang Yanxun, Transitive Kirkman triple systems, *Journal of Hebei Science Academy*, **No. 2** (1990), 1–7.

[8] Lei Jianguo, Further results on transitive Kirkman triple systems, *J. Stat. Plann. and Infer.* **51** (1996).

[9] R.M. Wilson, Some partitions of all triples into Steiner triple systems, *Lecture Notes in Math.* **411** (1974), 267–277.

[10] Wu Lisheng, On large set of KTS(v), in: W.D. Wallis *etal.* eds., *Combinatorial Designs and Applications*, Marcel Dekker, New York, (1990), 175–178.