The Nonexistence of Quaternary Linear Codes With Parameters [243,5,181], [248,5,185] and [240,5,179]

Noboru Hamada
Department of Applied Mathematics
Osaka Women's University
Daisen-cho, Sakai
Osaka 590
Japan

ABSTRACT. Let $n_4(k,d)$ and $d_4(n,k)$ denote the smallest value of n and the largest value of d, respectively, for which there exists an $[n,k,d]$ code over the Galois field $GF(4)$. It is known (cf. Boukliev [1] and Table B.2 in Hamada [6]) that (1) $n_4(5,179) = 240$ or 241, $n_4(5,181) = 243$ or 244, $n_4(5,182) = 244$ or 245, $n_4(5,185) = 248$ or 249 and (2) $d_4(240,5) = 178$ or 179 and $d_4(244,5) = 181$ or 182. The purpose of this paper is to prove that (1) $n_4(5,179) = 241$, $n_4(5,181) = 244$, $n_4(5,182) = 245$, $n_4(5,185) = 249$ and (2) $d_4(240,5) = 178$ and $d_4(244,5) = 181$.

1 Introduction

Let $V(n,q)$ be an n-dimensional vector space consisting of row vectors over the Galois field $GF(q)$, where $n > 3$ and q is a prime power. If C is a k-dimensional subspace in $V(n,q)$ such that every nonzero vector in C has a Hamming weight of at least d, then C is called an $[n,k,d;q]$-code (or a q-ary linear code with length n, dimension k, and minimum distance d). In the special case $q = 4$, an $[n,k,d;4]$-code is also called a quaternary $[n,k,d]$ code (cf. MacWilliams and Sloane [19]).

Let $n_q(k,d)$ denote the smallest value of n for which there exists an $[n,k,d;q]$-code. An $[n_q(k,d),k,d;q]$ code is therefore optimal in the sense that no shorter code exists with the same k, d and q. In the case $q = 4$ and $k = 4$, the value of $n_4(4,d)$ is known for all $d \leq 4^3$ (cf. Greenough and Hill [3], Hamada [6] and Landgev, Maruta and Hill [18]). But in the case $q = 4$ and $k = 5$, the value of $n_4(5,d)$ is unknown for many integers d and a table of the bounds for $n_4(5,d)$, $1 \leq d \leq 256$, has been given by Hamada [6].

It is known (cf. Table B.2 in Hamada [6]) that (1) $n_4(5,179) = 240$ or 241, $n_4(5,181) = 243$ or 244, $n_4(5,182) = 244$ or 245, $n_4(5,185) = 248$ or 249 and (2) $d_4(240,5) = 178$ or 179 and $d_4(244,5) = 181$ or 182. The purpose of this paper is to prove that (1) $n_4(5,179) = 241$, $n_4(5,181) = 244$, $n_4(5,182) = 245$, $n_4(5,185) = 249$ and (2) $d_4(240,5) = 178$ and $d_4(244,5) = 181$, i.e., to prove the following three theorems using the nonexistence of the corresponding minihypers.

Theorem 1.1. *There is no quaternary* $[243,5,181]$ *code and* $n_4(5,181) = 244$.

It is known that if there exists a quaternary [244,5,182] code, then there exists a quaternary [243,5,181] code. Hence we have

Corollary 1.1. (1) *There is no quaternary* $[244,5,182]$ *code.*
(2) $n_4(5,182) = 245$ *and* $d_4(244,5) = 181$.

Using a method similar to the proof of Theorem 1.1, we can easily prove the following two theorems.

Theorem 1.2. *There is no quaternary* $[248,5,185]$ *code and* $n_4(5,185) = 249$.

Theorem 1.3. *There is no quaternary* $[240,5,179]$ *code and* $n_4(5,179) = 241$.

Remark 1.1. It was shown by Boukliev [1] that there exist a [242,5,180; 4] code and a [247,5,184; 4]-code. Hence it follows that $n_4(5,179) = 240$ or 241, $n_4(5,181) = 243$ or 244 and $n_4(5,182) = 244$ or 245.

Remark 1.2. Recently, it has been shown by Hamada [7] that in the case $k \geq 5$ and $3^{k-1} - (3^{k-3} + 3^{k-2}) < d \leq 3^{k-1} - 3^{k-2}$, there exists a ternary $[n,k,d]$ code meeting the Griesmer bound if and only if $d = 3^{k-1} - 3^{k-2} - \epsilon$ for some integer ϵ in $\{0,1,2\}$. In order to generalize this result for the case $q \geq 4$, it is necessary to show at first that there is no quaternary $[n,5,d]$ code meeting the Griesmer bound for any integer d in $\{177,178,179,181,182,185\}$.

2 Connections between codes and minihypers

Let F be a set of f points in a finite projective geometry $PG(t,q)$ of t dimensions over $GF(q)$, where $f \geq 1$ and $t \geq 2$. If $|F \cap H| \geq m$ for every hyperplane (i.e., $(t-1)$-flat) H in $PG(t,q)$ and $|F \cap H| = m$ for some hyperplane H in $PG(t,q)$, then F is called an $\{f, m; t, q\}$-minihyper, where $m \geq 0$ and $|A|$ denotes the number of elements of the set A. It follows from Theorem A.1 in Appendix A that in order to prove Theorems 1.1 - 1.3, it is sufficient to prove the following Theorems 2.1 - 2.3, respectively. In what follows, let $v_i = (q^i - 1)/(q-1)$ for any integer $i \geq 0$. In the special case

$q = 4$, $v_i = (4^i - 1)/(4-1)$ for any integer $i \geq 0$, i.e., $v_0 = 0$, $v_1 = 1$, $v_2 = 5$, $v_3 = 21$ and $v_4 = 85$.

Theorem 2.1. *There is no $\{3v_1 + 2v_2 + v_4, 2v_1 + v_3; 4, 4\}$-minihyper.*

Theorem 2.2. *There is no $\{3v_1 + v_2 + v_4, v_1 + v_3; 4, 4\}$-minihyper.*

Theorem 2.3. *There is no $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper.*

In order to prove Theorems 2.1, 2.2 and 2.3, we shall use the following three theorems which play an important role in generalizing the result in Remark 1.2 for the case $q \geq 4$. The proof of Theorems 2.4, 2.5 and 2.6 will be given in Sections 5, 6 and 7, respectively.

Theorem 2.4. *In the case $q \geq 3$, $\tau \geq 3$ and $0 \leq \varepsilon \leq q-1$, K is an $\{\varepsilon v_1 + v_\tau, \varepsilon v_0 + v_{\tau-1}; \tau, q\}$-minihyper if and only if K is a disjoint union of ε points and one $(\tau - 1)$-flat in $PG(\tau, q)$.*

Remark 2.1. It is obvious that the if part of Theorem 2.4 holds.

Remark 2.2. Theorem 2.4 is a generalization of the result in Hamada and Deza [8].

Remark 2.3. It follows from Theorem 3.1 in Hamada [5] that Theorem 2.4 holds in the case $\varepsilon = 0$ or 1.

Theorem 2.5. *Let t, ε_0 and ε_1 be integers such that $t \geq 4$, $0 \leq \varepsilon \leq q-1$ and $0 \leq \varepsilon_1 \leq q-1$. Let G be a $(t-2)$-flat in $PG(t,q)$ and let $H_0, H_1, \ldots, H_{q-1}$ and H_q be $q+1$ $(t-1)$-flats in $PG(t,q)$ which contain G. If there exists an $\{\varepsilon_0 v_1 + \varepsilon_1 v_2 + v_t, \varepsilon_1 v_1 + v_{t-1}; t, q\}$-minihyper F such that $F \cap H_i = A_i \cup S_i$, $i = 0, 1, \ldots, q$, for some $(t-2)$-flat A_i in H_i and some subset S_i of $H_i \setminus G$ such that* (a) *$G \cap A_0 = G \cap A_1 = \cdots = G \cap A_q = B$ for some $(t-3)$-flat B in G and* (b) *$\sum_{i=0}^{q} |S_i| = \varepsilon_0 + \varepsilon_1 v_2$, then F contains a $(t-1)$-flat in $PG(t,q)$.*

Theorem 2.6. *Let t, ε_0 and ε_1 be integers such that $t \geq 4$, $0 \leq \varepsilon_0 \leq 3$, $1 \leq \varepsilon_1 \leq 3$ and $\varepsilon_0 + \varepsilon_1 \geq 4$. Let G be a $(t-2)$-flat in $PG(t,4)$ and let H_0, H_1, H_2, H_3 and H_4 be five $(t-1)$-flats in $PG(t,4)$ which contain G. If there exists an $\{\varepsilon_0 v_1 + \varepsilon_1 v_2 + v_t, \varepsilon_1 v_1 + v_{t-1}; t, 4\}$-minihyper F such that* (i) *$F \cap H_i = A_i \cup S_i$, $i = 0, 1, 2, 3$, for some $(t-2)$-flat A_i in H_i and some subset S_i of $H_i \setminus G$ and* (ii) *$G \cap (F \cap H_4) = B$ and $|F \cap H_4| = v_{t-1} + \delta$ for some integer $\delta \geq 4$, where $G \cap A_0 = G \cap A_1 = G \cap A_2 = G \cap A_3 = B$ for some $(t-3)$-flat B in G and $\sum_{i=0}^{3} |S_i| + \delta = \varepsilon_0 + \varepsilon_1 v_2$, then there exists a $(t-1)$-flat Π in $PG(t,4)$ such that $|F \cap \Pi| = v_t$ or $v_t - 1$.*

Remark 2.4. Let (ω_1), (ω_2), (ζ_1) and (ζ_2) be four linearly independent points in $PG(3,q)$ and let $F = ((\omega_1) \oplus (\zeta_1)) \cup ((\omega_2) \oplus (\zeta_2)) \cup \{\bigcup_{i=0}^{q-2}((\omega_1 + \alpha^i \omega_2) \oplus (\zeta_1 + \alpha^i \zeta_2))\}$, where α is a primitive element of $GF(q)$ and $(\omega) \oplus (\zeta)$ denotes the 1-flat in $PG(3,q)$ passing through two points (ω) and (ζ) in

$PG(3,q)$. Then it is easy to see that F is a $\{qv_1+v_3, qv_0+v_2; 3, q\}$-minihyper such that $|F\cap H| = q+1$ or $2q+1$ for any 2-flat H in $PG(3,q)$ and $(n_{q+1}, n_{2q+1}) = (q^3-q, q^2+2q+1)$, where $v_0=0$, $v_1=1$, $v_2=q+1$ and $v_3=q^2+q+1$. Since F contains no 2-flat in $PG(3,q)$, this shows that Theorem 2.4 does not hold in the case $q\geq 3$, $\tau=3$ and $\varepsilon=q$ (cf. Hamada and Maekawa [13] in the case $q=3$, $\tau=3$ and $\varepsilon=3$).

Remark 2.5. Let F be an $\{f, m; t, q\}$-minihyper and let H and G be a $(t-1)$-flat in $PG(t,q)$ and a $(t-2)$-flat in H, respectively. Then

$$\sum_{i=1}^{q} |F\cap H_i| = |F| - |F\cap H| + q|F\cap G|, \tag{2.1}$$

where $H_1, H_2, \ldots, H_{q-1}$ and H_q denote q $(t-1)$-flats in $PG(t,q)$, except for H, which contain G.

Remark 2.6. If there exists an $\{f, m; 4, 4\}$-minihyper F, then

$$\sum_{i=m}^{85} n_i = v_5,\ \sum_{i=m}^{85} in_i = fv_4 \ \text{ and } \ \sum_{i=m}^{85} \binom{i}{2} n_i = \binom{f}{2} v_3, \tag{2.2}$$

where n_i denotes the number of 3-flats H in $PG(4,4)$ such that $|F\cap H| = i$.

3 The proof of Theorems 2.1 and 2.2

Lemma 3.1. *If there exists a* $\{3v_1+2v_2+v_4, 2v_1+v_3; 4, 4\}$*-minihyper* F*, then* (1) $23\leq |F\cap H|\leq 26$ *or* $30\leq |F\cap H|\leq 85$ *for any 3-flat* H *in* $PG(4,4)$ *and* (2) *there exists a 3-flat* Π *in* $PG(4,4)$ *such that* $|F\cap\Pi| = 85$ *or* 84.

Proof: Let F be a $\{3v_1+2v_2+v_4, 2v_1+v_3; 4, 4\}$-minihyper.

(1) Let H be any 3-flat in $PG(4,4)$ such that $|F\cap H|\leq v_4 = 85$. It follows from Theorem A.2 ($\theta=6$) in Appendix A that $|F\cap H| = \zeta_0 v_1+\zeta_1 v_2+\zeta_2 v_3$ for some ordered set $(\zeta_0, \zeta_1, \zeta_2, 0)$ in $\overline{E}(4,4)$ such that $\zeta_0+\zeta_1+\zeta_2\leq 6$, where $\overline{E}(t,q)$ denotes the set defined in Definition A.1. Since there is no quaternary $[n, 4, d]$ code meeting the Griesmer bound for $d = 41, 42, 43$ (cf. Table B.1 in Hamada [6]), it follows from Theorems A.1, A.2 and Remark A.1 that there is no 3-flat H in $PG(4,4)$ such that $|F\cap H| = \zeta_0 v_1+\zeta_1 v_2+\zeta_2 v_3$ for any ordered set $(\zeta_0, \zeta_1, \zeta_3)$ in $\{(1,1,1), (2,1,1), (3,1,1)\}$. Since $v_1=1$, $v_2=5$, $v_3=21$, $v_4=85$ and $|F\cap H|\leq |H| = v_4$, this implies that (1) holds.

(2) It follows from the definition of a minihyper and Theorem A.2 ($\theta=6$, $q=4$, $\beta=0$) that there exists a 3-flat H_0 in $PG(4,4)$ such that $|F\cap H_0| = 2v_1+v_3$ and $F\cap H_0$ is a $\{2v_1+v_3, 2v_0+v_2; 4, 4\}$-minihyper in H_0. Since H_0 is a 3-flat in $PG(4,4)$, it follows from Remark A.1 and Theorem 2.4 ($q=4$,

$\tau = 3$, $\varepsilon = 2$) that $F \cap H_0 = A_0 \cap \{P_{01}, P_{02}\}$ for some 2-flat A_0 and some points P_{01}, P_{02} in H_0.

Let G be a 2-flat in H_0 such that $G \cap \{P_{01}, P_{02}\} = \emptyset$ and $G \cap A_0$ is a 1-flat (denoted by B) in G. Then $F \cap G = (F \cap H_0) \cap G = A_0 \cap G = B$ and $|F \cap G| = |B| = v_2 = 5$. Let H_1, H_2, H_3, H_4 and H_0 be five 3-fiats in $PG(4,4)$ which contain G, where $|F \cap H_1| \leq |F \cap H_2| \leq |F \cap H_3| \leq |F \cap H_4|$. Since $|F| = 3v_1 + 2v_2 + v_4 = 98$, $|F \cap H_0| = 2v_1 + v_3 = 23$ and $|F \cap G| = 5$, it follows from (2.1) and Lemma 3.1-(1) that

$$\sum_{i=1}^{4} |F \cap H_i| = |F| - |F \cap H_0| + 4|F \cap G| = 95 \tag{3.1}$$

and $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|, |F \cap H_4|) = (23, 24, 24, 24), (23, 23, 24, 25)$ or $(23, 23, 23, 26)$.

Case 1. $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|, |F \cap H_4|) = (23, 24, 24, 24)$. Since $2v_1 + v_3 = 23$ and $3v_1 + v_3 = 24$, it follows from Theorem A.2, Remark A.1 and Theorem 2.4 ($q = 4$, $\tau = 3$, $\varepsilon = 2, 3$) that $F \cap H_1 = A_1 \cup \{P_{11}, P_{12}\}$ for some 2-flat A_1 in H_1 and some points P_{11}, P_{12} in $H_1 \setminus G$ and $F \cap H_i = A_i \cup \{P_{i1}, P_{i2}, P_{i3}\}$, $i = 2, 3, 4$, for some 2-fiat A_i in H_i and some points P_{i1}, P_{i2}, P_{i3} in $H_i \setminus G$, where $G \cap A_1 = G \cap A_2 = G \cap A_3 = G \cap A_4 = B$. Hence it follows from Theorem 2.5 that F contains a 3-flat (denoted by Π) in $PG(4,4)$. Since $|\Pi| = v_4 = 85$, this implies that $|F \cap \Pi| = |\Pi| = 85$.

Case 2. $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|, |F \cap H_4|) = (23, 23, 24, 25)$. Since $4v_1 + v_3 = 25$, it follows from Theorem A.2, Remark A.1, Theorems 2.4 and 2.6 that there exists a 3-flat Π in $PG(4,4)$ such that $|F \cap \Pi| = 85$ or 84.

Case 3. $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|, |F \cap H_4|) = (23, 23, 23, 26)$. Since $5 + v_3 = 26$, it follows from Theorems A.2, 2.4 and 2.6 that there exists a 3-fiat Π in $PG(4,4)$ such that $|F \cap \Pi| = 85$ or 84. This completes the proof.

Remark 3.1. Let F be a $\{3v_1 + 2v_2 + v_4, 2v_1 + v_3; 4, 4\}$-minihyper and let H be a 3-flat in $PG(4,4)$ such that $|F \cap H| = 26$. Since $v_2 + v_3 = 26$, it follows from Theorem A.2 ($\theta = 6$, $q = 4$, $\beta = 1$) and $\theta - 2q < 2 \leq \theta - q$ that either (a) $F \cap H$ is a $\{v_2 + v_3, v_1 + v_2; 4, 4\}$-minihyper in H or (b) $F \cap H$ is a $\{v_2 + v_3, v_2; 4, 4\}$-minihyper in H. Since there exists a $\{v_2 + v_3, v_2; 3, 4\}$-minihyper, it does not follow from the proof of Lemma 3.1 that there is no 3-flat H in $PG(4,4)$ such that $|F \cap H| = 26$.

Lemma 3.2. *There is no $\{3v_1 + 2v_2 + v_4, 2v_1 + v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 85$ for some 3-flat H in $PG(4,4)$.*

Proof: Suppose there exists a $\{3v_1 + 2v_2 + v_4, 2v_1 + v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 85$ for some 3-flat H in $PG(4,4)$. Since $|H| = v_4 = 85$, this implies that $H \subset F$. Let G_i, $i = 1, 2, \ldots, 85$, be v_4 2-flats in H and let H_{i1}, H_{i2}, H_{i3}, H_{i4} and H be five 3-flats in $PG(4,4)$ which contain G_i, where $|F \cap H_{i1}| \leq |F \cap H_{i2}| \leq |F \cap H_{i3}| \leq |F \cap H_{i4}|$. Since $|F| = 98$,

$|F \cap H| = 85$ and $|F \cap G_i| = |G_i| = 21$, it follows from (2.1) and Lemma 3.1-(1) that

$$\sum_{j=1}^{4} |F \cap H_{ij}| = |F| - |F \cap H| + 4|F \cap G_i| = 97 \qquad (3.2)$$

and $(|F\cap H_{i1}|, |F\cap H_{i2}|, |F\cap H_{i3}|, |F\cap H_{i4}|) = (23,23,25,26)$, $(23,24,24,26)$, $(23,24,25,25)$ or $(24,24,24,25)$.

Let x, y and z denote the number of integers i in $\{1,2,\ldots,85\}$ such that $(|F\cap H_{i1}|, |F\cap H_{i2}|, |F\cap H_{i3}|, |F\cap H_{i4}|) = (23,23,25,26)$, $(23,24,24,26)$ and $(23,24,25,25)$, respectively. Since H and H_{ij}'s are v_5 3-flats in $PG(4,4)$, it follows that $n_{23} = 2x+y+z$, $n_{24} = 2y+z+3(85-x-y-z) = 255-3x-y-2z$, $n_{25} = x+2z+(85-x-y-z) = 85-y+z$, $n_{26} = x+y$ and $n_{85} = 1$, where n_k denotes the number of 3-flats Π in $\{H_{ij} \mid i = 1,2,\ldots,85, j = 1,2,3,4\}$ such that $|F\cap\Pi| = k$ for $k = 23,24,25,26$. Hence it follows from (2.2) that

$$\binom{23}{2}(2x+y+z) + \binom{24}{2}(255-3x-y-2z) + \binom{25}{2}(85-y+z) + \binom{26}{2}(x+y) + \binom{85}{2} = \binom{98}{2}v_3.$$

This implies that $3x+2y+z = 363$. On the other hand, it follows from $x+y+z \le 85$ that $3x+2y+z \le 3(x+y+z) \le 255$. This is a contradiction.

Lemma 3.3. *There is no $\{3v_1+2v_2+v_4, 2v_1+v_3; 4,4\}$-minihyper F such that $|F\cap H| = 84$ for some 3-flat H in $PG(4,4)$.*

Proof: Suppose there exists a $\{98,23;4,4\}$-minihyper F such that $|F \cap H| = 84$ for some 3-fiat H in $PG(4,4)$. Since $|H| = 85$, there exists a point Q in H such that $F\cap H = H \setminus \{Q\}$.

Let G_i, $i = 1,2,\ldots,21$, be v_3 2-flats in H such that $Q \in G_i$ and let G_i, $i = 22,23,\ldots,85$, be $v_4 - v_3$ 2-flats in H such that $Q \notin G_i$. Let H_{i1}, H_{i2}, H_{i3}, H_{i4} and H be five 3-flats in $PG(4,4)$ which contain G_i, where $|F\cap H_{i1}| \le |F\cap H_{i2}| \le |F\cap H_{i3}| \le |F\cap H_{i4}|$. Since $|F| = 98$, $|F\cap H| = 84$ and $|F\cap G_i| = 20$ or 21, it follows from (2.1) that

$$\sum_{j=1}^{4} |F \cap H_{ij}| = |F| - |F \cap H| + 4|F \cap G_i| = 94 \text{ or } 98 \qquad (3.3)$$

according as $1 \le i \le 21$ or $22 \le i \le 85$. Hence $(|F\cap H_{i1}|, |F\cap H_{i2}|, |F\cap H_{i3}|, |F\cap H_{i4}|) = (23,23,23,25)$ or $(23,23,24,24)$ for $i = 1,2,\ldots,21$ and $(|F\cap H_{i1}|, |F\cap H_{i2}|, |F\cap H_{i3}|, |F\cap H_{i4}|) = (23,23,26,26)$, $(23,24,25,26)$, $(24,24,24,26)$, $(23,25,25,25)$ or $(24,24,25,25)$ for $i = 22,23,\ldots,85$. Let x, a, b, c and d denote the number of integers i in $\{1,2,\ldots,85\}$ such that

$(|F \cap H_{i1}|, |F \cap H_{i2}|, |F \cap H_{i3}|, |F \cap H_{i4}|) = (23, 23, 23, 25)$, $(23, 23, 26, 26)$, $(23, 24, 25, 26)$, $(24, 24, 24, 26)$ or $(23, 25, 25, 25)$, respectively. Then $n_{23} = 42+x+2a+b+d$, $n_{24} = 170-2x-2a-b+c-2d$, $n_{25} = 128+x-2a-b-2c+d$, $n_{26} = 2a+b+c$ and $n_{84} = 1$. Hence it follows from the third equation of (2.2) that $x+4a+2b+c+d = 381$. On the other hand, it follows from $x \leq 21$ and $a+b+c+d \leq 64$ that $x+4a+2b+c+d \leq x+4(a+b+c+d) \leq 277$. This is a contradiction.

Proof of Theorem 2.1: It follows from Lemmas 3.1, 3.2 and 3.3 that there is no $\{3v_1 + 2v_2 + v_4, 2v_1 + v_3; 4, 4\}$-minihyper. This completes the proof.

Proof of Theorem 2.2: Using a method similar to the proof of Lemmas 3.1-3.3, it can be shown that (1) if there exists a $\{3v_1+v_2+v_4, v_1+v_3; 4, 4\}$-minihyper F, then there exists a 3-fiat Π in $PG(4,4)$ such that $|F \cap H| = 85$ or 84 and (2) there is no $\{3v_1+v_2+v_4, v_1+v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 85$ or 84 for some 3-flat H in $PG(4,4)$. This implies that there is no $\{3v_1 + v_2 + v_4, v_1 + v_3; 4, 4\}$-minihyper.

4 The proof of Theorem 2.3

Lemma 4.1. *If there exists a $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper F, then* (1) $24 \leq |F \cap H| \leq 25$ *or* $30 \leq |F \cap H| \leq 85$ *for any 3-flat H in $PG(4,4)$ and* (2) *there exists a 3-flat Π in $PG(4,4)$ such that $|F \cap \Pi| = 85$ or 84.*

Proof: Let F be a $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper.

(1) Suppose there exists a 3-flat H in $PG(4,4)$ such that $|F \cap H| = v_2 + v_3 = 26$. Then it follows from Theorem A.2 ($\theta = 5$, $q = 4$, $\beta = 0$) and Remark A.1 that $F \cap H$ is a $\{v_2 + v_3, v_1 + v_2; 4, 4\}$-minihyper in H and there exists a $\{v_2 + v_3, v_1 + v_2; 3, 4\}$-minihyper. Since there is no $\{v_2 + v_3, v_1 + v_2; 3, 4\}$-minihyper (cf. Theorem 3.1 in Hamada [5]), this is a contradiction. Hence it follows from the proof of Lemma 3.1-(1) that (1) holds.

(2) It follows from the definition of a minihyper and Theorems A.2 and 2.4 that there exists a 3-flat H_0 in $PG(4,4)$ such that $F \cap H_0 = A_0 \cup \{P_{01}, P_{02}, P_{03}\}$ for some 2-flat A_0 and some points P_{01}, P_{02}, P_{03} in H_0. Let G be a 2-flat in H_0 such that $G \cap \{P_{01}, P_{02}, P_{03}\} = \emptyset$ and $G \cap A_0 = B$ for some 1-flat B in G.

Let H_1, H_2, H_3, H_4 and H_0 be five 3-flats in $PG(4,4)$ which contain G, where $|F \cap H_1| \leq |F \cap H_2| \leq |F \cap H_3| \leq |F \cap H_4|$. Since $\sum_{i=1}^{4} |F \cap H_i| = 97$, it follows from (1) that $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|, |F \cap H_4|) = (24, 24, 24, 25)$.

Since $3v_1 + v_3 = 24$ and $4 + v_3 = 25$, it follows that (i) $F \cap H_i = A_i \cup \{P_{i1}, P_{i2}, P_{i3}\}$, $i = 1, 2, 3$, for some 2-flat A_i in H_i and some points

P_{i1}, P_{i2}, P_{i3} in $H_i \setminus G$ and (ii) $G \cap (F \cap H_4) = B$ and $|F \cap H_4| = v_3 + 4$, where $G \cap A_1 = G \cap A_2 = G \cap A_3 = B$. Hence it follows from Theorem 2.6 that there exists a 3-flat Π in $PG(4,4)$ such that $|F \cap \Pi| = 85$ or 84. This completes the proof.

Lemma 4.2. *There is no $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 85$ for some 3-flat H in $PG(4,4)$.*

Proof: Suppose there exists a $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 85$ for some 3-flat H in $PG(4,4)$. Let G_i and H_{ij} be the 2-flat in H and the 3-flat in $PG(4,4)$, respectively, defined in the proof of Lemma 3.2. Since $\sum_{j=1}^{4} |F \cap H_{ij}| = 100$ for $i = 1, 2, \ldots, 85$, it follows from Lemma 4.1 that $(|F \cap H_{i1}|, |F \cap H_{i2}|, |F \cap H_{i3}|, |F \cap H_{i4}|) = (25, 25, 25, 25)$. Hence $n_{25} = 340$ and $n_{85} = 1$. since

$$\binom{25}{2} \times 340 + \binom{85}{2} \neq \binom{101}{2} v_3, \tag{4.1}$$

this is contradictory to the third equation of (2.2).

Lemma 4.3. *There is no $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 84$ for some 3-flat H in $PG(4,4)$.*

Proof: Suppose there exists a $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper F such that $|F \cap H| = 84$ for some 3-flat H in $PG(4,4)$. Then $F \cap H = H \setminus \{Q\}$ for some point Q in H. Let G be a 2-flat in H such that $Q \notin G$ and let H_1, H_2, H_3, H_4 and H be five 3-flats in $PG(4,4)$ which contain G. Since

$$\sum_{i=1}^{4} |F \cap H_i| = |F| - |F \cap H| + 4|F \cap G| = 101 \tag{4.2}$$

and $|F \cap H_i| = 24, 25, 30, \ldots$ for $i = 1, 2, 3, 4$, there is no solution which satisfies the equation (4.2). This is a contradiction.

Proof of Theorem 2.3: It follows from Lemmas 4.1, 4.2 and 4.3 that there is no $\{v_1 + 3v_2 + v_4, 3v_1 + v_3; 4, 4\}$-minihyper. This completes the proof.

5 The proof of Theorem 2.4

Lemma 5.1. *In the case $t = 3$, $q \geq 3$ and $2 \leq \varepsilon \leq q - 1$, there is no $\{\varepsilon v_1 + v_3, v_2; 3, q\}$-minihyper K such that $|K \cap H| = \zeta_0 v_1 + \zeta_1 v_2$ for some 2-flat H in $PG(3,q)$, where $\zeta_0 \geq 0$, $\zeta_1 \geq 2$ and $\zeta_0 + \zeta_1 \leq \varepsilon + 1$.*

Proof: Suppose there exists an $\{\varepsilon v_1 + v_3, v_2; 3, q\}$-minihyper K such that $|K \cap H| = \zeta_0 v_1 + \zeta_1 v_2$ for some 2-flat H in $PG(3,q)$. Since $\varepsilon + 1 - q < \zeta_0 + \zeta_1 \leq \varepsilon + 1$, it follows from Theorem A.2 ($\theta = \varepsilon + 1$, $\beta = 0$) that

$K \cap H$ is a $\{\zeta_0 v_1 + \zeta_1 v_2, \zeta_1 v_1; 3, q\}$-minihyper in H. Since H is a 2-flat in $PG(3,q)$, there exists a 1-flat L in H such that $|(K \cap H) \cap L| = \zeta_1$. Note that $|K \cap L| = |(K \cap H) \cap L| = \zeta_1$.

Let $H_1, H_2, \ldots, H_{q-1}, H_q$ and H be $q+1$ 2-flats in $PG(3,q)$ which contain L, where $|K \cap H_1| \le |K \cap H_2| \le \cdots \le |K \cap H_q|$. Since $|K| = \varepsilon + (q^2+q+1)$, $|K \cap H| = \zeta_0 + \zeta_1(q+1)$ and $|K \cap L| = \zeta_1$, it follows from (2.1) that

$$\sum_{i=1}^{q} |K \cap H_i| = |K| - |K \cap H| + q|K \cap L| = qv_2 + (q - \zeta_0 - \zeta_1). \quad (5.1)$$

Hence it follows from $|K \cap H_i| \ge v_2$ that there exists an integer δ_i such that $|K \cap H_i| = v_2 + \delta_i$, $\delta_1 + \delta_2 + \cdots + \delta_q = q - \zeta_0 - \zeta_1$ and $0 \le \delta_1 \le \delta_2 \le \cdots \le \delta_q$. Since $\zeta_0 + \zeta_1 \ge 2$, this implies that $\delta_1 = 0$ and $|K \cap H_1| = v_2$.

Since $\varepsilon + 1 - q < 1$, it follows from Theorem A.2 ($\theta = \varepsilon + 1$, $\beta = 0$) and Theorem 3.1 in Hamada [5] that $K \cap H_1$ is a $\{v_2, v_1; 3, q\}$-minihyper in H_1 and $K \cap H_1$ is a 1-flat (denoted by L_1) in H_1. Since $2 \le \zeta_1 \le \varepsilon + 1 \le q$ and $|K \cap L| = |(K \cap H_1) \cap L| = |L_1 \cap L| = 1$ or $q+1$, this is contradictory to $|K \cap L| = \zeta_1$.

Lemma 5.2. *If K is an $\{\varepsilon v_1 + v_3, v_2; 3, q\}$-minihyper for some integers q and ε such that $q \ge 3$ and $2 \le \varepsilon \le q-1$, then K is a disjoint union of ε points and one 2-flat in $PG(3,q)$.*

Proof: Let K be an $\{\varepsilon v_1 + v_3, v_2; 3, q\}$-minihyper and let H be any 2-flat in $PG(3,q)$. Since $|H| = v_3$ and $|F \cap H| \ge v_2$, it follows from Theorem A.2 ($\theta = \varepsilon + 1$) that $|K \cap H| = v_3$ or $\zeta_0 v_1 + \zeta_1 v_2$ for some integers ζ_0 and ζ_1 such that $\zeta_0 \ge 0$, $\zeta_1 \ge 1$ and $\zeta_0 + \zeta_1 \le \varepsilon + 1$. Hence it follows from Lemma 6.1 that $|K \cap H| = v_3$ or $\zeta_0 + v_2$ for some integer ζ_0 such that $0 \le \zeta_0 \le \varepsilon$.

Case 1. $|K \cap H| = v_3$ for some 2-flat H in $PG(3,q)$. It follows from $|H| = v_3$ that $H \subset K$. Since $|K| = \varepsilon + v_3$, this implies that F is a disjoint union of ε points and the 2-flat H in $PG(3,q)$.

Case 2. $v_2 \le |K \cap \Pi| \le v_2 + \varepsilon$ for any 2-flat Π in $PG(3,q)$. It follows from the definition of a minihyper that there exists a 2-flat H in $PG(3,q)$ such that $|K \cap H| = v_2$. Since $K \cap H$ is a $\{v_2, v_1; 3, q\}$-minihyper, it follows from Theorem 3.1 in Hamada [5] that $K \cap H$ is a 1-flat (denoted by L) in H.

Let $H_1, H_2, \ldots, H_{q-1}, H_q$ and H be $q+1$ 2-flats in $PG(3,q)$ which contain L. Since $|K| = \varepsilon + (q^2+q+1)$, $|K \cap H| = q+1$ and $|K \cap L| = q+1$, it follows that

$$\sum_{i=1}^{q} |K \cap H_i| = |K| - |K \cap H| + q|K \cap L| = 2q^2 + q + \varepsilon. \quad (5.2)$$

Since $|K \cap H_i| \le (q+1)+\varepsilon$ for $i = 1, 2, \ldots, q$, it follows from $\varepsilon < q$ that

$$\sum_{i=1}^{q} |K \cap H_i| \le q(q+1+\varepsilon) < 2q^2 + q + \varepsilon, \tag{5.3}$$

a contradiction. This completes the proof.

Lemma 5.3. *If K is an $\{\varepsilon v_1 + v_t, v_{t-1}; t, q\}$-minihyper for some integers t, q and ε such that $t \ge 3$, $q \ge 3$ and $2 \le \varepsilon \le q-1$, then K is a disjoint union of ε points and one $(t-1)$-flat in $PG(t,q)$.*

Proof: We shall prove Lemma 5.3 by induction on t.

Case 1. $t = 3$. It follows from Lemma 5.2 that Lemma 5.3 holds.

Case 2. $t = \tau + 1$ and $\tau \ge 3$. Suppose Lemma 5.3 holds in the case $t = \tau$. Let K be an $\{\varepsilon v_1 + v_{\tau+1}, v_\tau; \tau+1, q\}$-minihyper. There exists a τ-flat H_0 in $PG(\tau+1, q)$ such that $|K \cap H_0| = v_\tau$. Since $K \cap H_0$ is a $\{v_\tau, v_{\tau-1}; \tau+1, q\}$-minihyper in H_0, it follows from Remarks A.1 and 2.3 that $K \cap H_0$ is a $(\tau-1)$-flat (denoted by A_0) in H_0.

Let G be a $(\tau-1)$-flat in H_0 such that $G \cap A_0 = B$ for some $(\tau-2)$-flat B in A_0. Let $H_1, H_2, \ldots, H_{q-1}, H_q$ and H_0 be $q+1$ τ-flats in $PG(\tau+1,q)$ which contain G. Since $|K| = \varepsilon + v_{\tau+1}$, $|K \cap H_0| = v_\tau$ and $|K \cap G| = v_{\tau-1}$, it follows from (2.1) that $\sum_{i=1}^{q} |K \cap H_i| = qv_\tau + \varepsilon$. Note that $v_{i+1} = qv_i + 1$.

Since $|K \cap H_i| \ge v_\tau$ for $i = 1, 2, \ldots, q$, there exists a nonnegative integer δ_i such that $|K \cap H_i| = v_\tau + \delta_i$ and $\delta_1 + \delta_2 + \cdots + \delta_q = \varepsilon$. Since $K \cap H_i$ is a $\{\delta_i + v_\tau, v_{\tau-1}; \tau+1, q\}$-minihyper in H_i and H_i is a τ-flat, it follows by induction on t that $K \cap H_i = A_i \cup S_i$ for some $(\tau-1)$-flat A_i and some subset S_i of $H_i \setminus G$ such that $G \cap A_i = B$ and $|S_i| = \delta_i$. Hence it follows from Theorem 2.5 ($\varepsilon_0 = \varepsilon$, $\varepsilon_1 = 0$, $t = \tau+1$) that K contains a τ-flat in $PG(\tau+1, q)$. This completes the proof.

Proof of Theorem 2.4: It follows from Remarks 2.1, 2.3 and Lemma 5.3 that Theorem 2.4 holds. This completes the proof.

6 The proof of Theorem 2.5

Let F be an $\{\varepsilon_0 v_1 + \varepsilon_1 v_2 + v_t, \varepsilon_1 v_1 + v_{t-1}; t, q\}$-minihyper which satisfies the condition in Theorem 2.5. Then

$$F = \left(\bigcup_{i=0}^{q} A_i\right) \cup \left(\bigcup_{i=0}^{q} S_i\right). \tag{6.1}$$

Let $\overline{A}_i = H_i \cap (A_0 \oplus A_1)$ for $i = 0, 1, \ldots, q$, where $A_0 \oplus A_1$ denotes the $(t-1)$-flat in $PG(t,q)$ which contains two $(t-2)$-flats A_0 and A_1. Then $\overline{A}_i$ is a $(t-2)$-flat in H_i such that $\overline{A}_0 = A_0$, $\overline{A}_1 = A_1$ and either $\overline{A}_j = A_j$ or $A_j \cap \overline{A}_j = B$ for $j = 2, 3, \ldots, q$.

If $A_j = \overline{A}_j$ for $j = 2, 3, \ldots, q$, then it follows that $\bigcup_{i=0}^{q} A_i = \bigcup_{i=0}^{q} \overline{A}_i = A_0 \oplus A_1$. Since $A_0 \oplus A_1$ is a $(t-1)$-flat in $PG(t, q)$, it follows from (7.1) that F contains the $(t-1)$-flat $A_0 \oplus A_1$ in $PG(t, q)$. Hence, in order to prove Theorem 2.5, it is sufficient to prove that $A_j = \overline{A}_j$ for $j = 2, 3, \ldots, q$.

Suppose there exists an integer j in $\{2, 3, \ldots, q\}$ such that $A_j \neq \overline{A}_j$, i.e., $A_j \cap \overline{A}_j = B$. Without loss of generality, we can assume that $j = q$, i.e., $A_q \cap \overline{A}_q = B$. Let $\Pi_1, \Pi_2, \ldots, \Pi_{q-1}, \Pi_q$ and H_q be $q+1$ $(t-1)$-flats in $PG(t, q)$ which contain $\overline{A}_q$. Since $A_0 \oplus \overline{A}_q = A_0 \oplus A_1$ and $A_0 \oplus \overline{A}_q$ is a $(t-1)$-flat in $PG(t, q)$ which contains $\overline{A}_q$, we can assume without loss of generality that $\Pi_q = A_0 \oplus A_1$.

Since $A_i \subset \bigcup_{j=1}^{q} \Pi_j$ for $i = 2, 3, \ldots, q-1$, there exists an integer m_i in $\{1, 2, \ldots, q\}$ such that $A_i \subset \Pi_{m_i}$ for each integer i in $\{2, 3, \ldots, q-1\}$. Since $\{1, 2, \ldots, q-1\} \setminus \{m_2, m_3, \ldots, m_{q-1}\} \neq \emptyset$, there exists at least one integer θ in $\{1, 2, \ldots, q-1\}$ such that $\theta \notin \{m_2, m_3, \ldots, m_{q-1}\}$. This implies that A_i does not contain in Π_θ, i.e., $A_i \cap \Pi_\theta = B$ for $i = 2, 3, \ldots, q-1$.

Since $A_q \cap \Pi_\theta = A_q \cap \overline{A}_q = B$ and $A_i \subset \Pi_q$, i.e., $A_i \cap \Pi_\theta = B$ for $i = 0, 1$, it follows from (7.1) that $|F \cap \Pi_\theta| = |(A_0 \cup A_1 \cup \cdots \cup A_q) \cap \Pi_\theta| + |(S_0 \cup S_1 \cup \cdots \cup S_q) \cap \Pi_\theta| \leq |B| + \sum_{i=0}^{q} |S_i| = v_{t-2} + \varepsilon_0 + \varepsilon_1(q+1) < \varepsilon_1 + v_{t-1}$ in the case $t \geq 4$. Since Π_θ is a $(t-1)$-flat in $PG(t, q)$, this is contradiction.

7 The proof of Theorem 2.6

Let F be an $\{\varepsilon_0 v_1 + \varepsilon_1 v_2 + v_t, \varepsilon_1 v_1 + v_{t-1}; t, 4\}$-minihyper which satisfies the condition in Theorem 2.6. Then

$$F = (\bigcup_{i=0}^{3} A_i) \cup (\bigcup_{i=0}^{3} S_i) \cup (F \cap H_4). \tag{7.1}$$

Let $\Pi = A_0 \oplus A_1$ and $\overline{A}_i = H_i \cap (A_0 \oplus A_1)$ for $i = 0, 1, 2, 3, 4$. Then Π is a $(t-1)$-flat in $PG(t, 4)$ such that $\Pi = \bigcup_{i=0}^{4} \overline{A}_i$ and $\overline{A}_i$ is a $(t-2)$-flat in H_i such that $\overline{A}_0 = A_0$, $\overline{A}_1 = A_1$ and either $\overline{A}_j = A_j$ or $A_j \cap \overline{A}_j = B$ for $j = 2, 3$.

If $A_2 = \overline{A}_2$, $A_3 = \overline{A}_3$ and $F \cap \overline{A}_4 = \overline{A}_4$ or $\overline{A}_4 \setminus \{Q\}$ for some point Q in $\overline{A}_4$, then $(\bigcup_{i=0}^{3} A_i) \cap \Pi = \bigcup_{i=0}^{3} \overline{A}_i, (\bigcup_{i=0}^{3} S_i), \cap \Pi = \emptyset$, $(F \cap H_4) \cap \Pi = (F \cap H_4) \cap \overline{A}_4 = F \cap \overline{A}_4 = \overline{A}_4$ or $\overline{A}_4 \setminus \{Q\}$ and $F \cap \Pi = \Pi$ or $\Pi \setminus \{Q\}$. Hence in order to prove Theorem 2.7, it is sufficient to prove that (i) $A_2 = \overline{A}_2$, $A_3 = \overline{A}_3$ and (ii) $|F \cap \overline{A}_4| = v_{t-1}$ or $v_{t-1} - 1$.

Case 1. $A_2 = \overline{A}_2$ and $A_3 = \overline{A}_3$. Let Π_1, Π_2, Π_3, Π and H_4 be five $(t-1)$-flats in $PG(t, 4)$ which contain $\overline{A}_4$, where $|F \cap \Pi_1| \leq |F \cap \Pi_2| \leq |F \cap \Pi_3|$. Then $(\bigcup_{i=0}^{3} A_i) \cup \Pi_j = (\bigcup_{i=0}^{3} \overline{A}_i) \cup \overline{A}_4 = B$, $(F \cap H_4) \cap H_j = F \cap \overline{A}_4$ for $j = 1, 2, 3$ and $\sum_{j=1}^{3} |(\bigcup_{i=0}^{3} S_i) \cup \Pi_j| = |(\bigcup_{i=0}^{3} S_i) \cap (\bigcup_{j=1}^{3} \Pi_j)| = |\bigcup_{i=0}^{3} S_i| = \sum_{i=0}^{3} |S_i| = \varepsilon_0 + 5\varepsilon_1 - \delta$.

Since $|F\cap\Pi_1| \le |F\cap\Pi_2| \le |F\cap\Pi_3|$, it follows from (8.1) that $|F\cap\Pi_1| = |B| + |(\bigcup_{i=0}^{3} S_i)\cap\Pi_1| + (|F\cap\overline{A}_4| - |B|) \le |F\cap\overline{A}_4| + \{\varepsilon_0 + 5\varepsilon_1 - \delta\}/3$. Since $|F\cap\Pi_1| \ge v_{t-1} + \varepsilon_1$, $\varepsilon_0 \le 3$, $\varepsilon_1 \le 3$ and $\delta \ge 4$, it follows that $|F\cap\overline{A}_4| \ge v_{t-1} - (\varepsilon_0 + 2\varepsilon_1 - \delta)/3 \ge v_{t-1} - 5/3$. Since $|F\cap\overline{A}_4|$ is an integer such that $|F\cap\overline{A}_4| \le |\overline{A}_4| = v_{t-1}$, this implies that $|F\cap\overline{A}_4| = v_{t-1}$ or $v_{t-1} - 1$.

Case 2. $A_2 = \overline{A}_2$ and $A_3 \cap \overline{A}_3 = B$. Let Π_1, Π_2, Π_3, Π and H_4 be five $(t-1)$-flats in $PG(t,4)$ which contain $\overline{A}_4$. Since $A_3 \subset \Pi_i$ for some integer i in $\{1,2,3\}$, we can assume without loss of generality that $A_3 \not\subset \Pi_1$, $A_3 \not\subset \Pi_2$, $A_3 \subset \Pi_3$ and $|F\cap\Pi_1| \le |F\cap\Pi_2|$. Since $(\bigcup_{i=0}^{3} A_i)\cap\Pi_1 = B$ and $|(\bigcup_{i=0}^{3} S_i)\cap\Pi_1| + |(\bigcup_{i=0}^{3} S_i)\cap\Pi_2| = |(\bigcup_{i=0}^{3} S_i)\cap(\Pi_1\cup\Pi_2)| \le |\bigcup_{i=0}^{3} S_i| = \sum_{i=0}^{3}|S_i| = \varepsilon_0 + 5\varepsilon_1 - \delta$, it follows that $|F\cap\Pi_1| = |(\bigcup_{i=0}^{3} A_i)\cap\Pi_1| + \{|F\cap\overline{A}_4| - |B|\} + |(\bigcup_{i=0}^{3} S_i)\cap\Pi_1| \le |F\cap\overline{A}_4| + (\varepsilon_0 + 5\varepsilon_1 - \delta)/2$. Since $|F\cap\Pi_1| \ge v_{t-1} + \varepsilon_1$, this implies that $|F\cap\overline{A}_4| \ge v_{t-1} - (\varepsilon_0 + 3\varepsilon_1 - \delta)/2$.

Let A_{41}, A_{42}, A_{43}, $\overline{A}_4$ and G be five $(t-2)$-flats in H_4 which contain B. Let x_i denote the number of points in $F\cap(A_{4i}\setminus B)$ for $i = 1,2,3$. Since $|F\cap H_4| = |F\cap\overline{A}_4| + x_1 + x_2 + x_3$ and $|F\cap\overline{A}_4| \ge v_{t-1} - (\varepsilon_0 + 3\varepsilon_1 - \delta)/2$, it follows from $|F\cap H_4| = v_{t-1} + \delta$ that $x_1 + x_2 + x_3 \le (\varepsilon_0 + 3\varepsilon_1 + \delta)/2$.

Let $\Delta_i = \overline{A}_3 \oplus A_{4i}$ for $i = 1,2,3$, where $|F\cap\Delta_1| \le |F\cap\Delta_2| \le |F\cap\Delta_3|$. Then Δ_i's are $(t-1)$-flats in $PG(t,4)$ such that $|F\cap\Delta_1| + |F\cap\Delta_2| + |F\cap\Delta_3| = 3|F\cap\overline{A}_3| + |(\bigcup_{i=0}^{2} S_i)\cap(\Delta_1\cup\Delta_2\cup\Delta_3)| + x_1 + x_2 + x_3 \le 3|F\cap\overline{A}_3| + \sum_{i=0}^{2}|S_i| + (\varepsilon_0 + 3\varepsilon_1 + \delta)/2$. Since $|F\cap\overline{A}_3| \le v_{t-2} + |S_3|$, $\sum_{i=0}^{2}|S_i| = (\varepsilon_0 + 5\varepsilon_1) - \delta - |S_3|$ and $|S_3| \le \varepsilon_0 + \varepsilon_1$, it follows that $|F\cap\Delta_1| \le v_{t-2} + (3\varepsilon_0 + 13\varepsilon_1 - \delta + 4|S_3|)/6 \le v_{t-2} + (7\varepsilon_0 + 17\varepsilon_1 - 4)/6 < v_{t-1} + \varepsilon_1$. Since Δ_1 is a $(t-1)$-flat in $PG(t,4)$, this is a contradiction.

Case 3. $A_2 \cap \overline{A}_2 = B$ and $A_3 \cap \overline{A}_3 = B$. Since $A_0 = \overline{A}_0$, $A_1 = \overline{A}_1$ and $(A_0\oplus A_1)\cap(A_2\oplus A_3)$ is a $(t-2)$-flat in the $(t-1)$-flat $A_0\oplus A_1$ which contains B, it follows from $A_0\oplus A_1 = \bigcup_{i=0}^{4}\overline{A}_i$ that $(A_0\oplus A_1)\cap(A_2\oplus A_3) = A_0$, A_1 or $\overline{A}_4$.

(A) In the case $(A_0\oplus A_1)\cap(A_2\oplus A_3) = A_0$, it follows that $H_0\cap(A_2\oplus A_3) = A_0$ and $H_1\cap(A_2\oplus A_3) \ne A_1$. Hence we have a contradiction from Case 2.

Similarly, it follows from Case 2 that $(A_0\oplus A_1)\cap(A_2\oplus A_3) \ne A_1$.

(B) In the case $(A_0\oplus A_1)\cap(A_2\oplus A_3) = \overline{A}_4$, let Π_1, Π_2, Π_3, Π and H_4 be five $(t-1)$-flats in $PG(t,4)$ which contain $\overline{A}_4$. Since $H_4\cap(A_2\oplus A_3) = \overline{A}_4$, we can assume without loss of generality that $\Pi_3 = A_3\oplus\overline{A}_4 = A_2\oplus A_3$. This implies that $A_2 \not\subset \Pi_1$, $A_3 \not\subset \Pi_1$, $A_2 \not\subset \Pi_2$, $A_3 \not\subset \Pi_2$, $A_2 \subset \Pi_3$ and $A_3 \subset \Pi_3$. Hence it follows from Case 2 that $|F\cap\overline{A}_4| \ge v_{t-1} - (\varepsilon_0 + 3\varepsilon_1 - \delta)/2$ and $x_1 + x_2 + x_3 \le (\varepsilon_0 + 3\varepsilon_1 + \delta)/2$.

Let $\Delta_i = \overline{A}_3 \oplus A_{4i}$ for $i = 1,2,3$. Since $A_2 \subset \Delta_i$ for some integer i in $\{1,2,3\}$, we can assume without loss of generality that $A_2 \not\subset \Delta_1$, $A_2 \not\subset \Delta_2$ and $A_2 \subset \Delta_3$. Hence $|F\cap\Delta_1| + |F\cap\Delta_2| = 2|F\cap\overline{A}_3| + |(\bigcup_{i=0}^{2} S_i)\cap$

$(\Delta_1 \cup \Delta_2)| + x_1 + x_2 \le 2(v_{t-2} + |S_3|) + \sum_{i=0}^{2} |S_i| + (\varepsilon_0 + 3\varepsilon_1 + \delta)/2 \le 2v_{t-2} + (7\varepsilon_0 + 13\varepsilon_1 + \delta)/2$ and $|F \cap \Delta_1| \le v_{t-2} + (4\varepsilon_0 + 7\varepsilon_1)/2 \le v_{t-1} + \varepsilon_1$, where $|F \cap \Delta_1| \le |F \cap \Delta_2|$ and $4 \le \delta \le \varepsilon_0 + \varepsilon_1$. Since Δ_1 is a $(t-1)$-flat in $PG(t, 4)$, this is a contradiction. Hence Theorem 2:6 holds.

Appendix A. Preliminary results

Let $E(k-1, q)$ denote the set of all ordered sets $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{k-2})$ such that $\varepsilon_i \in \{0, 1, \ldots, q-1\}$ for $i = 0, 1, \ldots, k-2$ and $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{k-2}) \ne (0, 0, \ldots, 0)$. The following theorem due to Hamada [5] plays an important role in proving Theorems 1.1 - 1.3.

Theorem A.1. *In the case $d = q^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i$ and $n = v_k - \sum_{i=0}^{k-2} \varepsilon_i v_{i+1}$ for some ordered set $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{k-2})$ in $E(k-1, q)$, there is a one-to-one correspondence between the set of all nonequivalent $[n, k, d; q]$-codes meeting the Griesmer bound and the set of all $\{\sum_{i=0}^{k-2} \varepsilon_1 v_{i+1}, \sum_{i=0}^{k-2} \varepsilon_i v_i; k-1, q\}$-minihypers, where $v_i = (q^i - 1)/(q-1)$ for any integer $i \ge 0$.*

Definition A.1. Let $\overline{E}(t, q)$ denote the set of all ordered sets $(\zeta_0, \zeta_1, \ldots, \zeta_{t-1})$ of integers ζ_i such that (a) $(\zeta_0, \zeta_1, \ldots, \zeta_{t-1}) \in E(t, q)$, or (b) $\zeta_0 = q$, $0 \le \zeta_1 \le q-1, \ldots, 0 \le \zeta_{t-1} \le q-1$, or (c) $\zeta_0 = \zeta_1 = \cdots = \zeta_{\lambda-1} = 0$, $\zeta_\lambda = q$, $0 \le \zeta_{\lambda+1} \le q-1, \ldots, 0 \le \zeta_{t-1} \le q-1$ for some integer λ in $\{1, 2, \ldots, t-1\}$.

The following theorem due to Hamada and Helleseth [9], [10] plays an important role in characterizing some minihypers and in proving the nonexistence of some minihypers.

Theorem A.2. *If there exists a $\{\sum_{i=0}^{t-1} \varepsilon_1 v_{i+1}, \sum_{i=0}^{t-1} \varepsilon_i v_i; t, q\}$-minihyper F for some ordered set $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{t-1})$ in $\overline{E}(t, q)$ and H is a $(t-1)$-flat in $PG(t, q)$ such that $|F \cap H| = \sum_{i=0}^{t-1} \zeta_i v_{i+1}$ for some ordered set $(\zeta_0, \zeta_1, \ldots, \zeta_{t-1})$ in $\overline{E}(t, q)$, then:*

(1) $\sum_{i=0}^{t-1} \zeta_i \le \theta$, where $\theta = \sum_{i=0}^{t-1} \varepsilon_i$.

(2) In the case $\theta - (\beta+1)q < \sum_{i=0}^{t-1} \zeta_i \le \theta - \beta q$ for some integer $\beta \ge 0$, $F \cap H$ is a $\{\sum_{i=0}^{t-1} \zeta_i v_{i+1}, \sum_{i=0}^{t-1} \zeta_i v_i - \gamma; t, q\}$-minihyper in H for some integer γ in $\{0, 1, \ldots, \beta\}$.

(3) If there is no $(t-1)$-flat Π in $PG(t, q)$ such that $\sum_{i=1}^{t-1} \varepsilon_i v_i < |F \cap \Pi| \le s + \sum_{i=1}^{t-1} \varepsilon_i v_i$ for some positive integer $s < q$, then $\sum_{i=0}^{t-1} \zeta_i = \theta$ or $\sum_{i=0}^{t-1} \zeta_i < \theta - s$.

Remark A.1. There exists an $\{f, m; t, q\}$-minihyper F such that $F \subset H$ for some $(t-1)$-flat H in $PG(t, q)$ if and only if there exists an $\{f, m; t-1, q\}$-minihyper, where $0 \le m < f < v_t$.

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