New Simple 3-Designs on 26 and 28 Points

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July 21, 1996

Abstract

In this paper we prove the existence of 22 new 3-designs on 26 and 28 points. The base of the constructions are two designs with a small maximum size of the intersection of any two blocks.

1 Introduction

We assume familiarity with some basic facts from design theory and coding theory.

Let $D = \{B_1, B_2, ..., B_b\}$ be a finite family of k-subsets (called **blocks**) of a v-set $X = X(v) = \{1, 2, ..., v\}$ (with elements called **points**). Then D is a t-(v, k, λ) **design** if every t-subset of X is contained in exactly λ blocks of D. A design without repeated blocks is called a **simple design**. There are tables of the known designs in [4]. We use a method for finding designs [2] based on what follows.

Let $D = \{B_1, B_2, ..., B_b\}$ be a t-(v, k, λ) design. Consider the set $X^{(s)}$ of all s-subsets of $X = \{1, 2, ..., v\}$, where $t + 1 \leq s \leq \lfloor \frac{v}{2} \rfloor$. (It is well-known that if D is a t-(v, k, t) design, then $D_s = \{X(v) \setminus B : B \in D\}$ is a t-$\left(v, v-k, \lambda\binom{v-k}{t} \Big/ \binom{k}{t}\right)$ design, called the **supplemental design** of D. So, we are concerned only with designs with block-size at most $\lfloor \frac{v}{2} \rfloor$.)

The **spectrum of** $A \in X^{(s)}$ **under** D is the set

$$Spec_D(A) = \{\alpha_{t+1}, \alpha_{t+2}, ..., \alpha_m\},$$

where $m = min\{k, s\}$ and

$$\alpha_l = | \{B \in D : |B \cap A| = l\} | .$$

The **spectra set of $X^{(s)}$ under D** is the collection of all possible spectra of the elements of $X^{(s)}$ under D.

The relation $\Re$ on $X^{(s)}$ is defined by $A_i \Re A_j$ if and only if $Spec_D(A_i) = Spec_D(A_j)$. It is obvious that this is an equivalence relation. Therefore $\Re$ partitions $X^{(s)}$ into equivalence classes $X_1^{(s)}, X_2^{(s)}, ..., X_q^{(s)}$ and we write $Spec_D(A) = Spec_D(X_i^{(s)})$ for all $A \in X_i^{(s)} \subset X^{(s)}$. It turns out that some of these classes, or unions of some of these classes, can be t'-designs.

The application of this method requires some computer work.

Let $D = \{B_1, B_2, ..., B_b\}$ be a t-(v, k, λ) design and

$$p = \max_{1 \leq i < j \leq b} |B_i \cap B_j|.$$

We call p the **maximal intersection number of D.**

The following theorem, proved in [5], and corollaries [3], [1], can be used for obtaining new designs from designs with sufficiently small p.

Theorem 1.1 *(DRIESSEN'S THEOREM) If D is a t-(v, k, λ) design with a maximal intersection number $p \leq k - m - l - 1$, for fixed integers $m, l \geq 0$, then*

$$D_{m,l} = \{(B \cup L) \setminus M : B \in D, M \subseteq B, |M| = m, L \subseteq (X \setminus B), |L| = l\}$$

is a

$$t - \left(v, k + l - m, \lambda \binom{v-k}{l} \binom{k+l-m}{t} \binom{k}{m} \Big/ \binom{k}{t}\right)$$

design.

Corollary 1.2 *Designs obtained by Driessen's theorem for pairs m_1, l_1 and m_2, l_2, where $m_1 \neq m_2$, but $m_1 - l_1 = m_2 - l_2$, have the same block-size, and are simple and disjoint.*

Corollary 1.3 *If Driessen's Theorem produces simple nontrivial designs, then $m \leq k - t - 1$, and the initial design is not trivial.*

Given a vector space $V = V_n(K)$ of dimension $n < \infty$ over the field K, with a fixed basis specified for V, a **code** C is a subset of V. The vectors in the code are called **codewords** or simply **words**. The (Hamming) **distance** between two codewords $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ is the number of places where they differ, i.e.,

$$d(\mathbf{x}, \mathbf{y}) = |\{i : 1 \leq i \leq n, x_i \neq y_i\}|.$$

The (Hamming) **weight** of a vector $\mathbf{x} = (x_1, ..., x_n)$ is the number of nonzero coordinates, and is denoted by $wt(\mathbf{x})$, i.e., $wt(\mathbf{x}) = d(\mathbf{x}, \mathbf{o})$. More generally, $wt(\mathbf{x} - \mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. The **minimal distance** of a code is

$$d = \min\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in C, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}.$$

A code is **linear** if it is a subspace of V. The **support** of a codeword is the set of positions of nonzero coordinates.

2 New $3-(26,k,\lambda)$ designs

Let D be the unique 3-(26,6,1) design. We will prove the existence of a 3-(26,8,14) design D' with $p = 5$. We use the method described in the previous section. The size of the spectra set of $X^{(8)}$ under D is 13. One of the equivalence classes is D' with $|D'| = 650$ and $Spec_D(D') = \{12, 0, 0\}$. It is a 3-(26,8,14) design with $p = 5$. Thus we have the following

Theorem 2.1 *The 8-subsets of $X(26)$ that intersect each of the blocks of the unique $3-(26,6,1)$ design D in at most 4 points and have a common 4-set with exactly 12 blocks of D form a $3-(26,8,14)$ design with $p = 5$.*

The application of Driessen's theorem and corollaries on D and D' produces the following designs (there are more, but we do not need them).

$D_{1,1} : 3-(26,6,120)$	$D'_{2,0} : 3-(26,6,140)$
$D_{0,1} : 3-(26,7,35)$	$D'_{1,0} : 3-(26,7,70)$
$D_{1,2} : 3-(26,7,1995)$	$D'_{1,1} : 3-(26,8,2016)$
$D_{0,2} : 3-(26,8,532)$	$D'_{0,0} \cup D'_{1,1} : 3-(26,8,2030)$
$D_{0,3} : 3-(26,9,4788)$	$D'_{0,1} : 3-(26,9,378)$
	$D'_{0,2} : 3-(26,10,4590)$

The designs from the first column are mentioned in [3]. The designs from the second column are new (with the exception of $D'_{2,0}$).

Theorem 2.2 *The following are sets of pairwise disjoint designs: D, $D_{1,1}$ and $D'_{2,0}$; $D_{0,1}$, $D_{1,2}$ and $D'_{1,0}$; $D_{0,2}$, D' and $D'_{1,1}$; $D_{0,3}$ and $D'_{0,1}$.*

Proof. We investigate the intersections of the designs with the initial design D. In what follows, we essentially use the content of the Driessen's theorem.

Consider the designs D, $D_{1,1}$ and $D'_{2,0}$. The maximal intersection number of D is 2. Any block B of $D_{1,1}$ has an intersection of size 5 with a block B^* of D and an intersection of size less than 5 with each of the remaining blocks of D (because B is obtained from B^* by removing a point and adding a new point from the set $X(26) \setminus B^*$). Any block of $D'_{2,0}$ is a 6-subset of a block of D' and any block of D' has at most 4 points in common with a block of D. Therefore, any block of $D'_{2,0}$ has at most 4 points in common with a block of D. Consequently, the designs D, $D_{1,1}$ and $D'_{2,0}$ are pairwise disjoint.

Consider the designs $D_{0,1}$, $D_{1,2}$ and $D'_{1,0}$. Any block of $D_{0,1}$ contains a block of D. Any block of $D_{1,2}$ has an intersection of size 5 with a block of D and an intersection of size less than 5 with each of the remaining blocks of D. The blocks of $D'_{1,0}$ are the 7-subsets of the blocks of D'. Therefore, any block of $D'_{1,0}$ has an intersection of size at most 4 with each of the blocks of D. Consequently, the designs $D_{0,1}$, $D_{1,2}$ and $D'_{1,0}$ are pairwise disjoint.

Consider $D_{0,2}$, D' and $D'_{1,1}$. Any block of $D_{0,2}$ contains a block of D. Any block of D' has at most 4 points in common with each of the blocks of D. Any block of $D'_{1,1}$ is obtained by removing a point from a block of D and adding a point from the supplement of the same block. Therefore, a block of $D'_{1,1}$ cannot have more than 5 points in common with a block of D. The designs $D' = D'_{0,0}$ and $D'_{1,1}$ are disjoint by corollary 1.2. Consequently, the three designs $D_{0,2}$, D' and $D'_{1,1}$ are pairwise disjoint.

Finally, consider $D_{0,3}$ and $D'_{0,1}$. Any block of $D_{0,3}$ contains a block of D. Any block of $D'_{0,1}$ contains a block of D' and one more element. Consequently, a block of $D'_{0,1}$ cannot have more than 5 points in common with a block of D. Therefore, the designs $D_{0,3}$ and $D'_{0,1}$ are disjoint. □.

The observations made so far lead to the following (cf. table 1).

Corollary 2.3 *There exist designs*

$$\begin{array}{lll} 3-(26,6,m) & for & m = 141, 280, 281; \\ 3-(26,7,m35) & for & m = 2, 3, 59, 60; \\ 3-(26,8,m7) & for & m = 2, 78, 288, 290, 364, 368; \\ 3-(26,9,m21) & for & m = 18, 246; \\ 3-(26,10,m3) & for & m = 1530. \end{array}$$

3 New $3-(28,k,\lambda)$ designs

Van Lint and MacWilliams [6] have constructed a 3-(28, 9, 28) design D'' from the subsets of coordinate places holding codewords of weight 9 in a linear code over $GF(4)$. The code has minimal distance 9 (equal to the minimal weight of a codeword). We prove that the design D'' has $p \leq 6$. There are three codewords for each support. Any other word of the code must be at a distance at least 9 from each of these three. Suppose $p > 6$. Then there must be a codeword which has at least 7 nonzero elements on positions occupied by the nonzero elements of the support. At least three of these elements must be identical. This gives two codewords at a distance at most 8, which is a contradiction. Thus $p \leq 6$ for the 3-$(28, 9, 28)$ design D'' obtained in [6].

The application of the Driessen's theorem and corollaries now proves the following (cf. table 1).

Theorem 3.1 *There exist designs with the parameters* $3-(28,7,420)$; $3-(28,8,168)$; $3-(28,9,m28)$, $m = 171, 172$; $3-(28,10,760)$ *and* $3-(28,11,9405)$.

4 Conclusion

We have constructed a 3-(26,8,14) design D' with $p = 5$ and proved the existence of a 3-(28,9,28) design D'' with $p = 6$. The first of these two designs together with the 3-(26,6,1) design with $p = 2$ were the base of the construction of 16 new 3-designs on 26 points via Driessen's theorem and corollaries. The second design produces 6 other new 3-designs on 28 points. Two of the new designs, the 3-(26,8,14) design and the 3-(26,9,378) design have *the smallest known* λ for fixed values of the other parameters. The results are summarized in table 1. At this point, it seems that the study of designs with small intersection number p could be a good source for finding new designs.

References

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[4] **Y.M. Chee, C.J. Colbourn, D.L. Kreher**, Simple t-designs with $v \leq 30$, Ars Combinatoria, 29 (1990), 193-258.

[5] **L.H.M.E. Driessen**, t-designs, $t \geq 3$, Doctoral Dissertation, Department of Mathematics, Eindhoven University of Technology, Holland, 1978.

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Table 1

New design		Construction
Parameters	m	
3-(26,6,m)	141	$D \cup D'_{2,0}$
	280	$D_{1,1} \cup D'_{2,0}$
	281	$D \cup D_{1,1} \cup D'_{2,0}$
3-(26,7,m35)	2	$D'_{1,0}$
	3	$D_{0,1} \cup D'_{1,0}$
	59	$D_{1,2} \cup D'_{1,0}$
	60	$D_{0,1} \cup D_{1,2} \cup D'_{1,0}$
3-(26,8,m7)	2	D'
	78	$D' \cup D_{0,2}$
	288	$D'_{1,1}$
	290	$D' \cup D'_{1,1}$
	364	$D_{0,2} \cup D'_{1,1}$
	366	$D' \cup D_{0,2} \cup D'_{1,1}$
3-(26,9,m21)	18	$D'_{0,1}$
	246	$D_{0,3} \cup D'_{0,1}$
3-(26,10,m3)	1530	$D'_{0,2}$
3-(28,7,m5)	84	$D''_{2,0}$
3-(28,8,m42)	4	$D''_{1,0}$
3-(28,9,m28)	171	$D''_{1,1}$
	172	$D''_{0,0} \cup D''_{1,1}$
3-(28,10,m20)	38	$D''_{0,1}$
3-(28,11,m495)	19	$D''_{0,2}$