Some New Infinite Series of Freeman-Youden Rectangles

B.J. Vowden and D.A.Preece
Institute of Mathematics and Statistics
Cornwallis Building
University of Kent at Canterbury
Canterbury, Kent CT2 7NF, England

ABSTRACT. A Freeman-Youden rectangle (FYR) is a Graeco-Latin row-column design consisting of a balanced superimposition of two Youden squares. There are well known infinite series of FYRs of size $q \times (2q+1)$ and $(q+1) \times (2q+1)$ where $2q+1$ is a prime power congruent to 3 (modulo 4). However, Preece and Cameron [9] additionally gave a single FYR of size 7×15. This isolated example is now shown to belong to one of a set of infinite series of FYRs of size $q \times (2q+1)$ where q, but not necessarily $2q+1$, is a prime power congruent to 3 (modulo 4), $q > 3$; there are associated series of FYRs of size $(q+1) \times (2q+1)$. Both the old and the new methodologies provide FYRs of sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$ where both q and $2q+1$ are congruent to 3 (modulo 4), $q > 3$; we give special attention to the smallest such size, namely 11×23.

1 Introduction with definitions

An $r \times t$ Youden square [7,8] is a rectangular array of t symbols in $r(< t)$ rows and t columns such that each symbol occurs just once in each row and no more than once in each column, the subsets of symbols in the columns being the blocks of a symmetric balanced incomplete block design (SBIBD, often called a symmetric 2-design).

Preece [5,8], Bailey, Preece and Rowley [1] and Preece and Vowden [10] considered what they termed 'balanced superimpositions of one Youden square on another', e.g. the following, where upper and lower case letters are used for, respectively, the symbols of the two Youden squares:

$$\begin{array}{ccccccc} Aa & Bb & Cc & Dd & Ee & Ff & Gg \\ Dg & Ea & Fb & Gc & Ad & Be & Cf \\ Fd & Ge & Af & Bg & Ca & Db & Ec \\ Gf & Ag & Ba & Cb & Dc & Ed & Fe. \end{array} \qquad (1)$$

This 4×7 specimen seems to have been the first published of such balanced superimpositions; it was given, in a different notation, by Clarke [3] (page 99), who obtained it from G.H. Freeman. We therefore, for ease of reference, rename the 'balanced superimpositions ...' as Freeman-Youden rectangles (FYRs).

For a formal definition of an FYR, we describe the superimposition as having 4 factors, namely 'rows', 'columns', 'symbols $S1$ of the first Youden square', and 'symbols $S2$ of the second Youden square'. The rows-factor is orthogonal to each of the others in that each row has exactly one entry from each column, exactly one for each symbol $S1$, and exactly one for each symbol $S2$. The pairwise relationships of the other 3 factors are, however, relationships of balance. Numbering these factors as 0 (for columns), 1 (for $S1$), and 2 (for $S2$), we define an $r \times t$ FYR as an $r \times t$ rectangular array such that

1. each entry is an ordered pair x, y where x is drawn from a set $S1$ of t elements, and y is drawn from a set $S2$ of t elements;

2. if the elements from either $S1$ or $S2$ are disregarded, the array becomes an $r \times t$ Youden square;

3. if n_{21} is the $t \times t$ (0, 1)-matrix whose (i, j)th element $(i, j = 1, 2, \dots, t)$ is the number of times that the ith element of $S2$ is paired with the jth element of $S1$, then n_{21} is the incidence matrix of an SBIBD;

4. if n_{10} is the $t \times t$ (0, 1)-matrix whose (i, j)th element $(i, j = 1, 2, \dots, t)$ is the number of times that the ith element of $S1$ occurs in the jth column, and n_{20} is defined similarly for $S2$, and we write n_{01} for the transpose of n_{10}, etc., then

$$\left.\begin{array}{ll} & n_{01}n_{12}n_{20} + n_{02}n_{21}n_{10} = fI + gJ \\ \text{and therefore} & \\ & n_{12}n_{20}n_{01} + n_{10}n_{02}n_{21} = fI + gJ \\ \text{and} & \\ & n_{20}n_{01}n_{12} + n_{21}n_{10}n_{02} = fI + gJ, \end{array}\right\} \qquad (2)$$

where f and g are integers, I is the $t \times t$ identity matrix, and J is the $t \times t$ matrix whose entries are all 1.

If the elements of $S1$ and $S2$ in (1) are taken in their natural orders, we can write

$$n = n_{21} = n_{10} = n_{20} \tag{3}$$

for (1), so the matrix sum in (2) becomes $2I + 18J$ for Freeman's example.

Using Roman letters for the elements of $S1$, and Greek letters for the elements of $S2$, an FYR can appropriately be classified as a non-orthogonal Graeco-Latin row-column design [6]. Indeed, a $(t-1) \times t$ FYR is merely a $t \times t$ Graeco-Latin square from which a row has been deleted. The notation of Greek and Roman letters is, however, not used in this paper.

Deletion of the first row of (1) leaves an FYR of size 3×7. This smaller design, and design (1) itself, come from well known infinite series of FYRs of sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$ where q is odd, $q > 1$, and $2q+1$ is a prime power congruent to 3 (modulo 4). Such FYRs are of two types [5]. With suitable orderings of columns, of symbols $S1$, and of symbols $S2$, these types can be characterised as follows, where n is the incidence matrix of an SBIBD and satisfies

$$n + n' = J - I \text{ for sizes } q \times (2q+1)$$

or

$$n + n' = J + I \text{ for sizes } (q+1) \times (2q+1):$$

Type 1: $n_{ij} = n_{jk} = n_{ik} = n$ for some assignment of the suffices i, j, k to the factors 0, 1, 2;

Type 2: $n_{ij} = n_{jk} = n_{ki} = n$ for any assignment of the suffices i, j, k to the factors 0, 1, 2.

For these two types, values of the integers f and g from (2) are as in Table 1. Equation (3) shows that Freeman's 4×7 example, given as (1) above, is of type 1; so is the 3×7 example obtained by deleting the first row of (1). The present paper is, however, concerned mainly with FYRs of type 2.

	Size $q \times (2q+1)$		Size $(q+1) \times (2q+1)$	
	f	g	f	g
Type 1	$-(q+1)/2$	$(2q^2-q+1)/2$	$(q+1)/2$	$(2q^2+5q+3)/2$
Type 2	$(3q+1)/2$	$(2q^2-q-1)/2$	$-(3q+1)/2$	$(2q^2+5q+5)/2$

Table 1: Values of f and g for FYRs of types 1 and 2

2 Adjugacy in Freeman-Youden rectangles

The equations (2) imply that, if the factors 0, 1 and 2 (i.e. the columns, symbols $S1$, and symbols $S2$) of an FYR are permuted in any way, then the resulting array is still an FYR. This situation is very similar to that for a Latin square, where any permutation of the factors 'rows', 'columns' and 'symbols' produces an arrangement that is still a Latin square. So we now borrow some terminology from the theory of Latin squares.

We say that two FYRs belong to the same 'transformation set' (alias 'isotopy class') if one can be obtained from the other by a 'transformation', i.e. by a combination of

(i) a permutation of the rows,

(ii) a permutation of the columns,

(iii) a permutation of the symbols $S1$, and

(iv) a permutation of the symbols $S2$

– where one or more of the permutations may be the identity permutation that leaves ordering unchanged. We say that two FYRs from the same transformation set are 'isomorphic' to one another. We say that a transformation of an FYR is an 'automorphism' of the FYR if it maps the FYR exactly on to itself. For example, if we label the rows of (1) as $\alpha, \beta, \gamma, \delta$ and the columns as 1, 2, ... , 7, then automorphisms of (1) include

$$(1234567) \times (ABCDEFG) \times (abcdefg)$$

and

$$(\beta\gamma\delta) \times (235)(476) \times (BCE)(DGF) \times (bce)(dgf);$$

the automorphism group of (1) has order 21.

We say that two FYRs are 'adjugates' of one another if one may be obtained from the other by permuting the factors 0, 1 and 2. An adjugate of an FYR F may or may not be identical to F, and may or may not belong to the same transformation set as F. We say that the complete set of members of a transformation set, together with all their adjugates, constitute a 'species' (alias 'main class') of FYRs.

As with Latin squares, a species of FYRs must contain 1, 2, 3 or 6 transformation sets. But there are now two considerations that do not arise with Latin squares. Firstly, one of the factors 0, 1, 2 in an FYR of type 1 is inevitably distinct from the other two. For example, in each of the type 1 sets of equations

$$n_{ij} = n_{jk} = n_{ik} = n$$

and

$$n_{ji} = n_{kj} = n_{ki} = n'$$

the suffix j appears once as the first of a pair and once as the second of a pair; suffix j is the only suffix to appear in this pattern, so factor j is the 'odd man out'. Secondly, an SBIBD inherent in an FYR may or may not be self-dual. In particular, if the matrix n of a particular type 2 FYR is the incidence matrix of a non-self-dual SBIBD, then the species containing that particular FYR must consist of at least two transformation sets.

Returning to our example (1), which is of type 1, we see that factor 1 (the capital letters) is the 'odd man out'. However, the inherent SBIBD is self-dual. With the columns of (1) again labelled 1, 2, ... , 7, its adjugate obtained by swapping factors 0 and 2 is

a	b	c	d	e	f	g
$A1$	$B2$	$C3$	$D4$	$E5$	$F6$	$G7$
$E2$	$F3$	$G4$	$A5$	$B6$	$C7$	$D1$
$C5$	$D6$	$E7$	$F1$	$G2$	$A3$	$B4$
$B3$	$C4$	$D5$	$E6$	$F7$	$G1$	$A2$

If we now make the symbol-interchanges $(BG), (CF), (DE), (1a), (2g), (3f), (4e), (5d), (6c), (7b)$ and re-order the columns accordingly, we retrieve (1). Thus (1) is isomorphic to one of its adjugates, and belongs to a species of FYRs that contains 3 transformation sets, these having, respectively, factors 1, 2 and 0 as 'odd man out'.

3 The background to the several new infinite series

As stated above, there are well known infinite series of FYRs of types 1 and 2, with sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$ where q is odd, $q > 1$, and $2q+1$ is congruent to 3 (modulo 4). Because their methods of construction require $2q+1$ to be a prime power, which 15 is not, the series have gaps for sizes 7×15 and 8×15. However, Preece and Cameron [9] additionally produced single examples of 7×15 and 8×15 FYRs of type 2, the 7×15 example being obtainable by deleting a row from the 8×15 example, just as the first row of (1) can be deleted to produce a 3×7 FYR. We have now found that Preece and Cameron's 7×15 and 8×15 examples are special cases of general and very fruitful methods of construction for sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$ where $q(> 3)$ is a prime power congruent to 3 (modulo 4). For each such value of q, an FYR of size $(q+1) \times (2q+1)$ can be obtained from a corresponding FYR of size $q \times (2q+1)$ by adding a row in an obvious way, so we restrict most of our exposition to sizes $q \times (2q+1)$.

For ease of exposition, we introduce our account not with Preece and Cameron's 7×15 FYR, but with another similar 7×15 FYR that is obtainable by our methodology and is of type 2. This other example is given in Table 2 where, to reflect our general method of construction, we label the 15 symbols of each of $S1$ and $S2$ as

$$a_0, a_1, \ldots, a_6;\ b_0, b_1, \ldots, b_6; *$$

– in that order. When we come to consider adjugacy properties, we label the 15 columns in the same way.

b_0*	b_5b_4	b_3b_1	a_5a_2	b_6b_2	a_6a_1	a_3a_4	$*a_0$	b_4a_6	b_1a_5	a_4b_5	b_2a_3	a_2b_6	a_1b_3	a_0b_0
a_4a_5	b_1*	b_6b_5	b_4b_2	a_6a_3	b_0b_3	a_0a_2	a_2b_4	$*a_1$	b_5a_0	b_2a_6	a_5b_6	b_3a_4	a_3b_0	a_1b_1
a_1a_3	a_5a_6	b_2*	b_0b_6	b_5b_3	a_0a_4	b_1b_4	a_4b_1	a_3b_5	$*a_2$	b_6a_1	b_3a_0	a_6b_0	b_4a_5	a_2b_2
b_2b_5	a_2a_4	a_6a_0	b_3*	b_1b_0	b_6b_4	a_1a_5	b_5a_6	a_5b_2	a_4b_6	$*a_3$	b_0a_2	b_4a_1	a_0b_1	a_3b_3
a_2a_6	b_3b_6	a_3a_5	a_0a_1	b_4*	b_2b_1	b_0b_5	a_1b_2	b_6a_0	a_6b_3	a_5b_0	$*a_4$	b_1a_3	b_5a_2	a_4b_4
b_1b_6	a_3a_0	b_4b_0	a_4a_6	a_1a_2	b_5*	b_3b_2	b_6a_3	a_2b_3	b_0a_1	a_0b_4	a_6b_1	$*a_5$	b_2a_4	a_5b_5
b_4b_3	b_2b_0	a_4a_1	b_5b_1	a_5a_0	a_2a_3	b_6*	b_3a_5	b_0a_4	a_3b_4	b_1a_2	a_1b_5	a_0b_2	$*a_6$	a_6b_6

Table 2: A 7×15 FYR of type 2

Now writing N (not n, as hitherto) for the incidence matrix of the inherent SBIBD, we have

$$n_{10} = n_{02} = n_{21} = N \tag{4}$$

for the 7×15 FYR in Table 2, with

$$N = \begin{bmatrix} n & n & \underline{1} \\ n+I & n' & \underline{0} \\ \underline{0}' & \underline{1}' & 0 \end{bmatrix} \tag{5}$$

where n, itself still the incidence matrix of an SBIBD, is given by

$$n = \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \end{bmatrix} \tag{6}$$

and where $\underline{1}$ and $\underline{0}$ are column-vectors of, respectively, ones and zeroes; we have $n + n' = J - I$ and $N + N' = J - I$.

There are five mutually non-isomorphic SBIBDs for 15 symbols in blocks of size 7. The matrix N above is the incidence matrix for the non-self-dual SBIBD labelled $[\alpha_1\alpha_1']_1$ by Nandi [4] and C_1 by Bhat and Shrikhande [2]. So the FYR from Table 2 has an adjugate with

$$n_{10} = n_{02} = n_{21} = N'$$

where N' is the incidence matrix of the dual of C_1, this dual being the SBIBD labelled $[\alpha_2\alpha_2']$ and C_3 by the authors already quoted.

In the exposition that follows, most of our description is of the construction of type 2 FYRs of size $q \times (2q+1)$ that satisfy (4) and (5), with $n = J - I - n'$ being an incidence matrix of an SBIBD with q symbols in blocks of size $\frac{1}{2}(q-1)$. But we also indicate that (5) can be replaced by

$$N = \begin{bmatrix} n & n & \underline{1} \\ n'+I & n & \underline{0} \\ \underline{0}' & \underline{1}' & 0 \end{bmatrix}. \tag{7}$$

With n as in (6), the matrix in (7) is the incidence matrix of the self-dual SBIBD labelled $[\gamma\gamma']$ and C_5; we still have $n+n' = J-I$ and $N+N' = J-I$.

4 A general construction for new series

Suppose the positive integer $q(> 3)$ is a prime power congruent to 3 (modulo 4). Let ε represent a primitive element within the finite field $\mathbb{F}_q$ containing q elements. When working with powers of ε it is convenient to employ the notation $\varepsilon^m - 1 = \varepsilon^{\overline{m}}$ in which m is any exponent satisfying $\varepsilon^m \neq 1$ and $\overline{m}$ is taken between 0 and $q-2$, but $\overline{m}$ avoids the value $\frac{1}{2}(q-1)$. Our construction for $q \times (2q+1)$ FYRs extends the scheme described in the previous section. We label the $2q+1$ symbols of each of $S1$ and $S2$ as $a_i (i \in \mathbb{F}_q)$, $b_i (i \in \mathbb{F}_q)$, and $*$. We define a $q \times (2q+1)$ array by specifying separately entries in two successive sets of q columns and in a final further column. Rows are identified by an index i and columns, within each of the two sets of q columns, by an index j; both i and j are drawn from $\mathbb{F}_q$. Entries are ordered pairs of symbols, from $S1$ and $S2$, as follows:-

For the first set of q columns

$a_{\varepsilon^{2r+\alpha_1}+j}$	$a_{\varepsilon^{2r+\alpha_2}+j}$	when $i-j=\varepsilon^{2r}$,
$b_{\varepsilon^{2r-1+\gamma_1}+j}$	$b_{\varepsilon^{2r-1+\gamma_2}+j}$	when $i-j=\varepsilon^{2r-1}$,
b_i	$*$	when $i=j$,

for the second set of columns

$a_{\varepsilon^{2r+\beta_1}+j}$	$b_{\varepsilon^{2r+\beta_2}+j}$	when $i-j=\varepsilon^{2r}$,
$b_{\varepsilon^{2r-1+\delta_1}+j}$	$a_{\varepsilon^{2r-1+\delta_2}+j}$	when $i-j=\varepsilon^{2r-1}$,
$*$	a_i	when $i=j$,

and for the final column

a_i $\qquad\qquad$ b_i

Here $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$ are parameters introduced to aid the construction, each drawn from the integer range $1, \ldots, q-2$. Our construction provides an FYR if these parameters are chosen to satisfy certain requirements which we now present.

To ensure that each symbol of S_1 occurs exactly once in each row we need the pair $\overline{\alpha}_1$ and $\overline{\beta}_1$ to be of opposite parity (i.e. one odd and the other even), and also the pair $\overline{\gamma}_1$ and $\overline{\delta}_1$. Correspondingly for the symbols of S_2 we need the pair $\overline{\alpha}_2$ and $\overline{\delta}_2$ to be of like parity, and also the pair $\overline{\beta}_2$ and $\overline{\gamma}_2$. For the same reasons each of $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$ must be non-zero.

To investigate the relationships of balance between the factors 'columns', 'symbols $S1$' and 'symbols $S2$' we introduce, as in (5) above, matrices n and N where

$$N = \begin{bmatrix} n & n & \underline{1} \\ n+I & n' & \underline{0} \\ \underline{0}' & \underline{1}' & 0 \end{bmatrix} \tag{8}$$

and n is the $q \times q$ matrix of zeroes and ones whose ijth entry equals 1 if $i-j$ is an even power of ε, but is 0 otherwise (where indexing for the rows and columns of n is by means of the elements of $\mathbb{F}_q$). Both n and N are incidence matrices of SBIBDs; we have $n+n' = J-I$ and $N+N' = J-I$. We find

$$n_{10} = \left[\begin{array}{c|c|c} n \text{ or } n' \text{ as } \alpha_1 \text{ is even or odd} & n \text{ or } n' \text{ as } \beta_1 \text{ is even or odd} & \underline{1} \\ \hline n+I \text{ or } n'+I \text{ as } \gamma_1 \text{ is odd or even} & n \text{ or } n' \text{ as } \delta_1 \text{ is odd or even} & \underline{0} \\ \hline \underline{0}' & \underline{1}' & 0 \end{array}\right]$$

So $n_{10} = N$ when $\alpha_1, \beta_1, \delta_1$ are even and γ_1 is odd. In the same manner we find that $n_{02} = N$ when $\beta_2, \gamma_2, \delta_2$ are even and α_2 is odd. The incidence between elements of $S1$ and elements of $S2$ is expressed by

$$n_{21} = \left[\begin{array}{c|c|c} \begin{array}{c} n \text{ or } n' \text{ as } \varepsilon^{\alpha_1} - \varepsilon^{\alpha_2} \\ \text{is an odd or even power of } \varepsilon \end{array} & \begin{array}{c} n \text{ or } n' \text{ as } \varepsilon^{\delta_1} - \varepsilon^{\delta_2} \\ \text{is an even or odd power of } \varepsilon \end{array} & \underline{1} \\ \hline \begin{array}{c} n+I \text{ or } n'+I \text{ as } \varepsilon^{\beta_1} - \varepsilon^{\beta_2} \\ \text{is an odd or even power of } \varepsilon \end{array} & \begin{array}{c} n \text{ or } n' \text{ as } \varepsilon^{\gamma_1} - \varepsilon^{\gamma_2} \\ \text{is an even or odd power of } \varepsilon \end{array} & \underline{0} \\ \hline \underline{0}' & \underline{1}' & 0 \end{array}\right]$$

so that $n_{21} = N$ when $\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2}$, $\varepsilon^{\beta_1} - \varepsilon^{\beta_2}$, $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2}$ are odd powers of ε and $\varepsilon^{\delta_1} - \varepsilon^{\delta_2}$ is an even power of ε. In particular these conditions imply $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\gamma_1 \neq \gamma_2$, $\delta_1 \neq \delta_2$.

Summarising, we can conclude that our construction, which is dependent upon a choice for the parameters $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$, produces a $q \times (2q+1)$ FYR provided these parameters satisfy the conditions:-

1. $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$ are drawn from the range of integers $1, \ldots, q-2$;

2. $\overline{\alpha}_1$ and $\overline{\beta}_1$ are of opposite parity;

3. $\overline{\gamma}_1$ and $\overline{\delta}_1$ are of opposite parity;

4. $\overline{\alpha}_2$ and $\overline{\delta}_2$ are of like parity;

5. $\overline{\beta}_2$ and $\overline{\gamma}_2$ are of like parity;

6. $\alpha_1, \beta_1, \delta_1$ are even and γ_1 is odd;

7. $\beta_2, \gamma_2, \delta_2$ are even and α_2 is odd;

8. $\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2}$, $\varepsilon^{\beta_1} - \varepsilon^{\beta_2}$, $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2}$ are odd powers of ε, whereas $\varepsilon^{\delta_1} - \varepsilon^{\delta_2}$ is an even power of ε.

Consider, by way of example, the case $q = 7$: $\mathbb{F}_7$ identifies with integer arithmetic modulo 7 and $\varepsilon = 3$ is a primitive element. Our final three conditions easily imply that $\beta_1 = 2$, $\beta_2 = 4$, $\delta_1 = 4$ and $\delta_2 = 2$. From the parity conditions we then find that $\alpha_1 = 4$ and $\gamma_2 = 4$. Proceeding in this way we conclude that in this instance there is a unique choice for the 8 parameters, namely $\alpha_1 = 4$, $\beta_1 = 2$, $\gamma_1 = 1$, $\delta_1 = 4$, $\alpha_2 = 5$, $\beta_2 = 4$, $\gamma_2 = 4$, $\delta_2 = 2$, which corresponds to the 7×15 FYR exhibited in Table 2.

5 The selection of parameters for one of the new infinite series

For selecting values of the parameters in our general construction of $q \times (2q+1)$ FYRs, we need to classify each parameter value m in the integer range $1, \ldots, q-2$ according to the parity of both m and $\overline{m}$. Vowden [11] showed that, if m is even, then $\overline{m}$ and $\overline{(q-1)-m}$ have opposite parity, but that, if m is odd , $\overline{m}$ and $\overline{(q-1)-m}$ agree in parity. As the prime power $q(> 3)$ is congruent to 3 (modulo 4), we may write $q = 4\lambda + 3$ for some positive integer λ. The counts of the 4 possible parity combinations for m and $\overline{m}$ are displayed in Table 3.

m	$\overline{m}$	Number of possibilities
Even	Even	λ
Even	Odd	λ
Odd	Even	$\lambda + 1$
Odd	Odd	λ

Table 3: Counts of integers satisfying the different parity combinations

We choose for β_2 any even value for which $\overline{\beta_2}$ is odd. We next allow r to range through all even values with $\overline{r}$ odd, and consider the λ values of $m = \beta_2 + r$, with reduction modulo $q-1$ when this is necessary to bring

m into our range. Because the choice $r = (q-1) - \beta_2$ is excluded, we obtain values with $1 \le m < q-1$ and which are all different, all even, and all distinct from β_2. For at least one choice of r we have $\overline{m}$ even. We set $\beta_1 = m$, so that β_1 and $\overline{\beta}_1$ are even, and $\varepsilon^{\beta_1} - \varepsilon^{\beta_2} = \varepsilon^{\overline{r}+\beta_2}$, an odd power of ε.

We let $\delta_1 = \beta_2$ and $\delta_2 = \beta_1$; this gives $\varepsilon^{\delta_1} - \varepsilon^{\delta_2} = -(\varepsilon^{\beta_1} - \varepsilon^{\beta_2}) = \varepsilon^{2\lambda+1}(\varepsilon^{\beta_1} - \varepsilon^{\beta_2})$, and so $\varepsilon^{\delta_1} - \varepsilon^{\delta_2}$ is an even power of ε.

Next we choose an integer $s < 2\lambda+1$ with s odd but $\overline{s}$ even. When $\overline{2s}$ is odd we take $\alpha_1 = 2s$ and $\alpha_2 = s$, so that $\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2} = \varepsilon^{s+\overline{s}}$, an odd power of ε. In the contrary case, when $\overline{2s}$ is even, we take $\alpha_1 = (q-1) - 2s$ and $\alpha_2 = (q-1) - s$, so that $\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2} = \varepsilon^{(q-1)-2s}(1-\varepsilon^{s}) = \varepsilon^{(q-1)-2s+(2\lambda+1)+\overline{s}}$, which is again an odd power of ε.

Finally we choose an integer t with both t and $\overline{t}$ odd. We need γ_1 to be odd but $\overline{\gamma}_1$ even. There are $\lambda+1$ possibilities for γ_1. Considering any one of these, we take $\gamma_2 = \gamma_1 - t$, and as before we reduce modulo $q-1$ where necessary. Whatever selection we make for γ_1, the parameter γ_2 is even and the $\lambda+1$ values are all different. Since $\overline{\gamma_2}$ can be even for at most λ of them, we can choose γ_1 so that $\overline{\gamma_2}$ is odd. Finally, $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2} = \varepsilon^{\gamma_2+\overline{t}}$, which is an odd power of ε.

This selection scheme for the 8 parameters associated with the construction described in §4, enables us to exhibit a $q \times (2q+1)$ FYR for each prime power $q (> 3)$ that is congruent to 3 (modulo 4). When $q = 7$ we obtain the unique parameter set that generates a 7×15 FYR. For the next larger size $q = 11$ and, by considering the primitive element $\varepsilon = 2$ of $\mathbb{F}_{11}$, we are led to the choice $\beta_2 = 6$, then $\beta_1 = 2$, $\delta_1 = 6$, $\delta_2 = 2$, $\alpha_1 = 8$, $\alpha_2 = 9$, $\gamma_1 = 1$ and $\gamma_2 = 8$. The 11×23 FYR which this parameter set produces via our construction in §4 is displayed in Table 4, and has an automorphism group of order 55. It is, therefore, distinct from any 11×23 FYR of type 2 that arises in the earlier $q \times (2q+1)$ construction where $2q+1$, but not q, is required to be a prime power congruent to 3 (modulo 4). The earlier construction provided the 11×23 type 2 example given by Preece [5] (see also Bailey, Preece and Rowley [1]), and, for $2q+1 = 23$, produces only type 2 FYRs for which the order of the automorphism group is 253.

6 Variants of the general method

The allocation to rows and columns of ordered pairs of symbols drawn from S_1 and S_2 which §4 specifies for our general construction of $q \times (2q+1)$ FYRs is not the only feasible arrangement. We may, for example, adapt the scheme from §4 by reassigning the entries corresponding to the second set of columns as follows:-

$$\begin{array}{lll} b_{\varepsilon^{2r+\beta_1}+j} & a_{\varepsilon^{2r+\beta_2}+j} & \text{when } i-j=\varepsilon^{2r}, \\ a_{\varepsilon^{2r-1+\delta_1}+j} & b_{\varepsilon^{2r-1+\delta_2}+j} & \text{when } i-j=\varepsilon^{2r-1}, \end{array}$$

b_0*	$b_{10}b_9$	a_7a_1	b_8b_5	b_7b_3	b_6b_1	$a_{10}a_3$	a_8a_9	a_6a_4	b_2b_4	a_2a_5	$*a_0$	b_3a_7	a_3b_6	b_9a_{10}	b_1a_6	b_4a_2	a_9b_7	a_5b_{10}	a_1b_2	b_5a_8	a_4b_8	a_0b_0
a_3a_6	b_1*	b_0b_{10}	a_8a_2	b_9b_6	b_8b_4	b_7b_2	a_0a_4	a_9a_{10}	a_7a_5	b_3b_5	a_5b_9	$*a_1$	b_4a_8	a_4b_7	$b_{10}a_0$	b_2a_7	b_5a_3	$a_{10}b_8$	a_6b_0	a_2b_3	b_6a_9	a_1b_1
b_4b_6	a_4a_7	b_2*	b_1b_0	a_9a_3	$b_{10}b_7$	b_9b_5	b_8b_3	a_1a_5	$a_{10}a_0$	a_8a_6	b_7a_{10}	a_6b_{10}	$*a_2$	b_5a_9	a_5b_8	b_0a_1	b_3a_8	b_6a_4	a_0b_9	a_7b_1	a_3b_4	a_2b_2
a_9a_7	b_5b_7	a_5a_8	b_3*	b_2b_1	$a_{10}a_4$	b_0b_8	$b_{10}b_6$	b_9b_4	a_2a_6	a_0a_1	a_4b_5	b_8a_0	a_7b_0	$*a_3$	b_6a_{10}	a_6b_9	b_1a_2	b_4a_9	b_7a_5	a_1b_{10}	a_8b_2	a_3b_3
a_1a_2	$a_{10}a_8$	b_6b_8	a_6a_9	b_4*	b_3b_2	a_0a_5	b_1b_9	b_0b_7	$b_{10}b_5$	a_3a_7	a_9b_3	a_5b_6	b_9a_1	a_8b_1	$*a_4$	b_7a_0	a_7b_{10}	b_2a_3	b_5a_{10}	b_8a_6	a_2b_0	a_4b_4
a_4a_8	a_2a_3	a_0a_9	b_7b_9	a_7a_{10}	b_5*	b_4b_3	a_1a_6	b_2b_{10}	b_1b_8	b_0b_6	a_3b_1	$a_{10}b_4$	a_6b_7	$b_{10}a_2$	a_9b_2	$*a_5$	b_8a_1	a_8b_0	b_3a_4	b_6a_0	b_9a_7	a_5b_5
b_1b_7	a_5a_9	a_3a_4	a_1a_{10}	b_8b_{10}	a_8a_0	b_6*	b_5b_4	a_2a_7	b_3b_0	b_2b_9	$b_{10}a_8$	a_4b_2	a_0b_5	a_7b_8	b_0a_3	$a_{10}b_3$	$*a_6$	b_9a_2	a_9b_1	b_4a_5	b_7a_1	a_6b_6
b_3b_{10}	b_2b_8	a_6a_{10}	a_4a_5	a_2a_0	b_9b_0	a_9a_1	b_7*	b_6b_5	a_3a_8	b_4b_1	b_8a_2	b_0a_9	a_5b_3	a_1b_6	a_8b_9	b_1a_4	a_0b_4	$*a_7$	$b_{10}a_3$	$a_{10}b_2$	b_5a_6	a_7b_7
b_5b_2	b_4b_0	b_3b_9	a_7a_0	a_5a_6	a_3a_1	$b_{10}b_1$	$a_{10}a_2$	b_8*	b_7b_6	a_4a_9	b_6a_7	b_9a_3	b_1a_{10}	a_6b_4	a_2b_7	a_9b_{10}	b_2a_5	a_1b_5	$*a_8$	b_0a_4	a_0b_3	a_8b_8
a_5a_{10}	b_6b_3	b_5b_1	b_4b_{10}	a_8a_1	a_6a_7	a_4a_2	b_0b_2	a_0a_3	b_9*	b_8b_7	a_1b_4	b_7a_8	$b_{10}a_4$	b_2a_0	a_7b_5	a_3b_8	$a_{10}b_0$	b_3a_6	a_2b_6	$*a_9$	b_1a_5	a_9b_9
b_9b_8	a_6a_0	b_7b_4	b_6b_2	b_5b_0	a_9a_2	a_7a_8	a_5a_3	b_1b_3	a_1a_4	$b_{10}*$	b_2a_6	a_2b_5	b_8a_9	b_0a_5	b_3a_1	a_8b_6	a_4b_9	a_0b_1	b_4a_7	a_3b_7	$*a_{10}$	$a_{10}b_{10}$

Table 4: An 11 × 23 FYR of type 2

all other entries remaining unchanged. This necessitates a modification to the constraints imposed on the parameters $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$, and we must now require that:-

$\overline{\alpha}_1$ and $\overline{\delta}_1$ have the same parity, likewise for $\overline{\beta}_1$ and $\overline{\gamma}_1$;

$\overline{\alpha}_2$ and $\overline{\beta}_2$ have opposite parity, likewise for $\overline{\gamma}_2$ and $\overline{\delta}_2$;

α_1 is even and β_1, γ_1, δ_1 are odd; γ_2 is even and α_2, β_2, δ_2 are odd;

$\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2}$, $\varepsilon^{\beta_1} - \varepsilon^{\beta_2}$, $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2}$ are odd powers of ε, and $\varepsilon^{\delta_1} - \varepsilon^{\delta_2}$ is an even power of ε.

If we consider again the case $q = 7$, there are six parameter sets that satisfy these new conditions and so generate 7×15 FYRs. Of these parameter sets, one gives rise to an FYR that is an adjugate of the 7×15 FYR displayed in Table 2 and whose construction we described in §4. One other parameter set, namely $\alpha_1 = 4$, $\beta_1 = 1$, $\gamma_1 = 1$, $\delta_1 = 3$, $\alpha_2 = 3$, $\beta_2 = 5$, $\gamma_2 = 4$, $\delta_2 = 5$ is especially interesting from the adjugacy point of view: the associated FYR is identical to each of its adjugates got from even permutations of factors 0, 1 and 2.

Besides the two schemes that we have now described, there are a further six admissible ways of distributing the pairs of a's and b's within the first and second column sets, and for the case $q = 7$ we find a total of 78 parameter sets that generate FYRs. Of course there are many adjugacy relations between these FYRs, though we also find further instances of FYRs self-adjugate with respect to even permutations of the factors. We classify these 78 parameter sets presently.

We may further vary the construction described in §4 by relacing the matrix N in (8) that characterises the incidence between factors. Retaining our earlier definition of n, the matrix

$$N_1 = \begin{bmatrix} n & n' & \underline{1} \\ n' + I & n' & \underline{0} \\ \underline{0}' & \underline{1}' & 0 \end{bmatrix} \tag{9}$$

is also the incidence matrix of a SBIBD, in fact of the design dual to that which N describes (and this may be verified by noting that N_1 is obtained from N' as the result of a simple reordering of rows and columns). These two SBIBDs can be shown to be non-isomorphic. [2] The eight parameters α_1, β_1, γ_1, δ_1, α_2, β_2, γ_2, δ_2 are introduced exactly as in §4, and to ensure that $n_{10} = n_{02} = n_{21} = N_1$ the first five of the conditions 1 to 8 appearing there remain unchanged; but in place of the final three we must now require that:-

$\alpha_1, \gamma_1, \delta_1$ are even and β_1 is odd;

β_2 is even and $\alpha_2, \gamma_2, \delta_2$ are odd;

$\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2}$, $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2}$, $\varepsilon^{\delta_1} - \varepsilon^{\delta_2}$ are odd powers of ε, and $\varepsilon^{\beta_1} - \varepsilon^{\beta_2}$ is an even power.

Where $q = 7$ a valid choice of parameter set is $\alpha_1 = 4$, $\beta_1 = 5$, $\gamma_1 = 2$, $\delta_1 = 4$, $\alpha_2 = 3$, $\beta_2 = 4$, $\gamma_2 = 3$, $\delta_2 = 3$. The corresponding 7×15 FYR is that previously exhibited by Preece and Cameron [9]. We can describe an infinite series of FYRs that starts with this example. As before, suppose $q(> 3)$ is a prime power that is congruent to 3 (modulo 4) and write $q = 4\lambda+3$. Choose some odd integer r in the range $1, \ldots, q-2$ for which $\overline{r}$ is even. Set $\beta_1 = 2\lambda+1$. If $\overline{\beta_1}$ and $\overline{\beta_1 + r}$ have opposite parity, set $\alpha_1 = \beta_1 + r$ and $\beta_2 = (q-1) - \alpha_1$; otherwise set $\beta_2 = \beta_1 + r$ and $\alpha_1 = (q-1) - \beta_2$. Finally set $\gamma_1 = \alpha_1$, $\delta_1 = \beta_2$, $\alpha_2 = \gamma_2 = \delta_2 = \beta_1$. Of course any FYR whose construction is based on the incidence matrix N_1 is necessarily an adjugate (obtained by swapping a pair of factors) of one derived from the incidence matrix N, so, working with N_1, there are again 78 parameter sets that generate FYRs.

Towards the end of §3 we gave in (7) a further replacement for the matrix N that appears in our general construction, namely

$$N_2 = \begin{bmatrix} n & n & \underline{1} \\ n' + I & n & \underline{0} \\ \underline{0}' & \underline{1}' & 0 \end{bmatrix} \tag{10}$$

which is the incidence matrix of a self-dual SBIBD when n is the matrix employed previously in (8) and (9). The schemes we have described for creating FYRs function as before, again working via selections of the parameters α_1, β_1, γ_1, δ_1, α_2, β_2, γ_2, δ_2 employed to control the assignment of ordered pairs of symbols (each such symbol being one of the a_i, the b_i or $*$) to locations within the $q \times (2q+1)$ row and column array. Working with N_2, there are 18 parameter sets that generate FYRs.

It is now convenient to introduce a system for referring to any of the eight ways of distributing the pairs of a's and b's within and between the first and second column sets. We assign the code 000 to our original arrangement from §4, and the code 001 to the arrangement described at the beginning of this section, in which, within the second set of columns, the allocation to rows of the a,b and b,a pairs is reversed when rows are distinguished according to whether a row index attaches to an even or odd power of the primitive element ε. Likewise the second binary digit of our code is used to specify the alternative row allocations of a,a and b,b pairs within the first set of columns. Finally, the a,a and b,b pairs may be allocated to the second set of columns instead of to the first, where the a,b and b,a pairs are then placed. We employ the left-most binary digit of our code to express this additional means of adapting the original scheme. The numbers of parameter sets that generate 7×15 FYRs for these eight arrangements and the three underlying incidence matrices N, N_1 and N_2 are as in Table 5.

code	incidence matrix N	N_1	N_2
000	1	16	4
001	6	16	0
010	1	16	4
011	6	16	1
100	16	6	0
101	16	1	4
110	16	6	1
111	16	1	4
total	78	78	18

Table 5: Numbers of 7×15 FYRs of type 2 for the various construction schemes

Six of the eighteen 7×15 FYRs based on the self-dual N_2 are self-adjugate with respect to even permutations of factors 0, 1 and 2. In particular this is so for the FYR generated from scheme 011, where $\alpha_1 = 3$, $\beta_1 = 2$, $\gamma_1 = 3$, $\delta_1 = 5$, $\alpha_2 = 1$, $\beta_2 = 4$, $\gamma_2 = 4$, $\delta_2 = 2$. For this arrangement of the a and b pairs the parameter choice must satisfy:-

$\overline{\alpha}_1$ and $\overline{\beta}_1$ have opposite parity, likewise for $\overline{\gamma}_1$ and $\overline{\delta}_1$;

$\overline{\alpha}_2$ and $\overline{\delta}_2$ have the same parity, likewise for $\overline{\beta}_2$ and $\overline{\gamma}_2$;

$\alpha_1, \gamma_1, \delta_1$ are odd and β_1 is even; $\beta_2, \gamma_2, \delta_2$ are even and α_2 is odd;

$\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2}$, $\varepsilon^{\beta_1} - \varepsilon^{\beta_2}$, $\varepsilon^{\delta_1} - \varepsilon^{\delta_2}$ are odd powers of ε, and $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2}$ is an even power of ε.

We conclude by describing an infinite series of FYRs based on the self-dual N_2 that starts with the 7×15 example just given. As previously $q(>3)$ is a prime power congruent to 3 (modulo 4), and $q = 4\lambda + 3$. Choose any odd α_1 with $\overline{\alpha}_1$ odd. Consider the $\lambda + 1$ distinct values that we obtain from $u = x - \alpha_1$ when x varies through those odd values for which $\overline{x}$ is even. Again arithmetic for parameter values is performed modulo $(q-1)$ in order to bring the result into the range 0 to $q-2$. Table 3 shows that $\overline{u}$ is odd for at least one such value of x. We set $\alpha_2 = \alpha_1 + u$ and note that $\varepsilon^{\alpha_1} - \varepsilon^{\alpha_2} = \varepsilon^{\alpha_1 + \overline{u} + (2\lambda+1)}$. Similarly, take an even β_1 with $\overline{\beta}_1$ even also, and consider the λ values of $\beta_1 + v$ when v and $\overline{v}$ are even. Because $v \neq (q-1) - \beta_1$, the value of $\beta_1 + v$ is never zero and since it also avoids the value β_1 there is a choice for v so that $\overline{\beta_1 + v}$ is odd. We set $\beta_2 = \beta_1 + v$ so that $\varepsilon^{\beta_1} - \varepsilon^{\beta_2} = \varepsilon^{\beta_1 + \overline{v} + (2\lambda+1)}$. Consider any odd y with $1 \leq y < 2\lambda + 1$ such that $\overline{y}$ is even. Set $z = (q-1) - y$ and note that, as for y, z is odd but $\overline{z}$ is even. Then $2z = 2(q-1) - 2y$ which is congruent to $(q-1)-2y$ modulo $(q-1)$. Hence either $\overline{2y}$ or $\overline{2z}$ is even. We may thus choose an odd γ_1 for which $\overline{\gamma}_1$ is even and $\overline{2\gamma_1}$ is odd. We set $\gamma_2 = 2\gamma_1$, and find $\varepsilon^{\gamma_1} - \varepsilon^{\gamma_2} = \varepsilon^{\gamma_1+\overline{\gamma}_1+(2\lambda+1)}$. Likewise by considering an odd y with $1 \le y < 2\lambda+1$ such that $\overline{y}$ is odd, together with $z = (q-1)-y$, we may choose an odd δ_1 for which $\overline{\delta}_1$ is odd and $\overline{2\delta_1}$ is even. We set $\delta_2 = 2\delta_1$, so that $\varepsilon^{\delta_1} - \varepsilon^{\delta_2} = \varepsilon^{\delta_1+\overline{\delta}_1+(2\lambda+1)}$ and this is an odd power of ε as we require.

Just as with our main construction, so all of these variants produce FYRs of type 2; as yet this whole approach will not go forward for type 1.

References

[1] R.A. Bailey, D.A. Preece and C.A. Rowley, Randomization for a balanced superimposition of one Youden square on another, *J.R. Statist.Soc.* **B57** (1995), 459–469.

[2] V.N. Bhat and S.S. Shrikhande, Non-isomorphic solutions of some balanced incomplete block designs I, *J. Combin.Theory* **9** (1970), 174–191.

[3] G.M. Clarke, A second set of treatments in a Youden square design, *Biometrics* **19** (1963), 98–104.

[4] H.K. Nandi, A further note on non-isomorphic solutions of incomplete block designs, *Sankhyā* **7** (1946), 313–316.

[5] D.A. Preece, Some row and column designs for two sets of treatments, *Biometrics* **22** (1966), 1–25.

[6] D.A. Preece, Non-orthogonal Graeco-Latin designs, in *Combinatorial Mathematics IV*, Adelaide 1975 (ed. L.R.A. Casse and W.D. Wallis), *Lecture Notes in Mathematics* **569** (ed. A. Dold and B. Eckmann), pp.7–26, Berlin: Springer Verlag (1976).

[7] D.A. Preece, Fifty years of Youden squares: a review, *Bull.Inst.Maths.Applics.* **26** (1990), 65–75.

[8] D.A. Preece, Youden squares, in *The CRC Handbook of Combinatorial Designs* (ed. C.J. Colbourn and J.H. Dinitz), pp.511–514, Boca Raton: CRC Press (1996).

[9] D.A. Preece and P.J. Cameron, Some new fully-balanced Graeco-Latin Youden squares, *Utilitas Mathematica* **8** (1975), 193–204.

[10] D.A. Preece and B.J. Vowden, Graeco-Latin squares with embedded superimpositions of Youden squares, *Discrete Mathematics* **138** (1995), 353–363.

[11] B.J. Vowden, Infinite series of double Youden rectangles, *Discrete Mathematics* **125** (1994), 385–391.