Radius of (2k-1)-Connected Graphs

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ABSTRACT. We show that if G is a $(2k-1)$-connected graph $(k \geq 2)$ with radius r, then $r \leq \left\lfloor \frac{|V(G)|+2k+9}{2k} \right\rfloor$.

1 Introduction

By a graph, we mean a finite, undirected, simple graph without loops or multiple edges. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v, w \in V(G)$, let $d_G(v, w)$ denote the usual distance between v and w. Set

$$r(G) := \min_{v \in V(G)} \max_{w \in V(G)} d_G(v, w).$$

The number $r(G)$ is called the radius of G. A vertex $z \in V(G)$ is called a central vertex of G if $\max_{w \in V(G)} d_G(z, w) = r(G)$

In [1], Harant and Walther proved that the inequality $r \leq \frac{n}{4} + O(\log n)$ holds for a 3-connected graph with radius r containing precisely n vertices, where O denotes the order as n tends to infinity. The purpose of this paper is to prove the following theorem.

Theorem. *Let $k \geq 2$ be an integer, and let G be a $(2k-1)$-connected graph with radius r containing precisely n vertices. Then the following inequality holds:*

$$r \leq \left\lfloor \frac{n + 2k + 9}{2k} \right\rfloor.$$

2 Preliminary Results

Throughout the rest of the paper, we let G, n, r be as in the Theorem. For a vertex $v \in V(G)$ and a nonnegative integer i, let

$$N_i(v) := \{w \mid w \in V(G), d_G(v, w) = i\}.$$

We write $N(v)$ for $N_1(v)$. Fix a central vertex z, and let

$$X_i := N_i(z) \text{ for } 0 \leq i \leq r.$$

Note that for each i with $1 \leq i \leq r - 1$ and any $x \in X_i$, $N(x) \subset X_{i-1} \cup X_i \cup X_{i+1}$.

Lemma 1. $|X_i| \geq 2k - 1$ *for all* $1 \leq i \leq r - 1$.

Proof: Since $G - X_i$ is disconnected, the desired conclusion immediately follows from the $(2k - 1)$-connectedness of G. □

Lemma 2. $n \geq (2k - 1)r - (2k - 3)$.

Proof: By Lemma 1, $n = \sum_{i=0}^{r} |X_i| \geq 1 + (2k - 1)(r - 1) + 1 = (2k - 1)r - (2k - 3)$. □

3 Proof of the Theorem

We continue with the notation of the preceding section. The bulk of the proof of the Theorem is devoted to the verification of the following proposition, which roughly says that the average of the $|X_i|$ is only slightly less than $2k$, if it is less than $2k$:

Proposition 3. *Let a, b be integers with $a \geq 6$, $a + 2 \leq b \leq r - 5$, and suppose that $|X_a| = |X_b| = 2k - 1$ and $|X_i| > 2k - 1$ for all $a + 1 < i < b$. Then*

$$\sum_{i=a}^{b-1} |X_i| \geq 2k(b - a).$$

To prove Proposition 3, suppose, by way of contradiction, that $\sum_{i=a}^{b-1} |X_i| < 2k(b - a)$. Then one of the following two situations must occur:

(A) $|X_i| = 2k$ for all $a < i < b$; or

(B) $|X_{a+1}| = 2k - 1$, and $|X_i| = 2k$ or $(2k + 1)$ for each $a + 1 < i < b$, and the number of X_i with $|X_i| = 2k + 1$ is at most one.

We now introduce a graph structure $\mathcal{G}$ on X_{a+1} by joining u and v if and only if $d_G(u, v) \leq 2$ and $u \neq v$. Let α denote the independence number of $\mathcal{G}$.

Lemma 4. $\alpha \leq 2$.

Proof: Suppose $\alpha \geq 3$. Then there exist $v_1, v_2, v_3 \in X_{a+1}$ such that $d_G(v_i, v_j) \geq 3$ for all $1 \leq i < j \leq 3$. This implies that $(\{v_i\} \cup N(v_i)) \cap (\{v_j\} \cup N(v_j)) = \emptyset$ for all $1 \leq i < j \leq 3$. On the other hand, whether (A) holds or (B) holds, $|X_{a+1}| + |X_{a+2}| \leq 4k$. Consequently, $3(2k-1) \leq \sum_{1 \leq i \leq 3} |N(v_i)| = |\bigcup_{1 \leq i \leq 3} N(v_i)| \leq |(X_a \cup X_{a+1} \cup X_{a+2}) - \{u_1, u_2, u_3\}| \leq (2k-1) + 4k - 3 = 6k - 4$, a contradiction. □

Lemma 5. *If a connected graph H has independence number less than or equal to 2, then $r(H) \leq 2$.*

Proof: We may assume H is not a complete graph. Then there exist $u, v \in V(H)$ with $d_H(u, v) = 2$. Let z' be a vertex adjacent to both u and v. Since the independence number of H is at most 2, each vertex in $V(H) - \{z', u, v\}$ is adjacent to u or v. This means that $d_H(z', x) \leq 2$ for all $x \in V(H)$, and hence $r(H) \leq 2$. □

Now let c' denote the number of components of $\mathcal{G}$. By Lemma 4, $c' = 1$ or 2.

Lemma 6.

(a) *If $c' = 1$, then there exists $u \in X_{a+1}$ such that $d_G(u, u') \leq 4$ for every $u' \in X_{a+1}$.*

(b) *If $c' = 2$, then for some β with $a+1 \leq \beta \leq b-2$, there exists $w \in X_\beta$ such that $d_G(w, w') \leq 6$ for every $w' \in X_\beta$.*

Proof: We first prove (a). Assume that $\mathcal{G}$ is connected, i.e., $c' = 1$. Then since $\alpha \leq 2$ by Lemma 4, $r(\mathcal{G}) \leq 2$ by Lemma 5. Let u be a central vertex of $\mathcal{G}$. Then for any vertex $u' \in X_{a+1}$, $d_{\mathcal{G}}(u, u') \leq 2$, and hence $d_G(u, u') \leq 4$ by the definition of $\mathcal{G}$.

We now prove (b) in a series of claims. Assume $c' = 2$, and let S_{a+1} and T_{a+1} be the vertex sets of the components. For $0 \leq j \leq r$, set

$$S_j = X_j \cap \left(\bigcup_{u \in S_{a+1}} N_{|a+1-j|}(u) \right),$$

$$T_j = X_j \cap \left(\bigcup_{u \in T_{a+1}} N_{|a+1-j|}(u) \right).$$

Since $d_G(v, v') \geq 3$ for any $v \in S_{a+1}$ and $v' \in T_{a+1}$, $S_i \cap T_i = \emptyset$ for each $a \leq i \leq a+2$.

Claim 1: $|S_a| \geq k-1$ and $|T_a| \geq k-1$.

Proof: By way of contradiction, suppose $|S_a| \leq k-2$. Then since $G - (S_a \cup T_{a+2})$ is disconnected, $|T_{a+2}| \geq k+1$ by the $(2k-1)$-connectedness of G, and hence $|S_{a+2}| = |X_{a+2} - T_{a+2}| \leq (2k+1) - (k+1) = k$. But since $G - (S_a \cup S_{a+2})$ is also disconnected, this contradicts the assumption that G is $(2k-1)$-connected. Thus $|S_a| \geq k-1$. We can prove $|T_{\cdot a}| \geq k-1$ in exactly the same way. □

By Claim 1, we may assume $|S_a| = k-1$ and $|T_a| = k$.

Claim 2: Let $a < i \leq r$, and suppose that for each h with $a < h < i$, $d(w_1, w_2) \geq 3$ for any $w_1 \in S_h$ and any $w_2 \in T_h$. Then the following hold.

(1) (a) $|S_i| \geq k-1$. (b) If $i \geq a+2$, $|S_i| \geq k$.

(2) $|T_i| \geq k$.

Proof: From the assumptions of the lemma, it follows that $G - (S_i \cup T_a)$ is disconnected, and hence (1)(a) follows from the assumption that G is $(2k-1)$-connected. Similarly, $G - (S_a \cup T_i)$ is disconnected and, in the case where $i \geq a+2$, $G - (S_a \cup S_i)$ is also disconnected, and hence (1)(b) and (2) also follow from the $(2k-1)$-connectedness of G. □

We define an integer C as follows. Set

$$Q := \left\{ i \,\middle|\, a < i < b, \begin{array}{c} \text{there exists } w_1 \in S_i \text{ and there exists} \\ \text{w}_2 \in T_i \text{ such that } d(w_1, w_2) \leq 2 \end{array} \right\}.$$

We have $Q \neq \emptyset$ because if $Q = \emptyset$, then $|X_b| = |S_b| + |T_b| \geq k + k = 2k$ by Claim 2, which contradicts the assumption that $|X_b| = 2k-1$. Now set

$$C = \min Q.$$

Note that $a + 2 \leq C \leq \max Q \leq b - 1$ by the definition of S_{a+1} and T_{a+1}. The following remarks immediately follow from the definition of C.

Remark. For each $a \leq i \leq C$, we have $X_i - S_i = T_i$.

Remark. For $x \in S_i$ and $y \in T_i$ $(a+1 \leq i \leq C-1)$, $N(x) \subset S_{i-1} \cup S_i \cup S_{i+1}$ and $N(y) \subset T_{i-1} \cup T_i \cup T_{i+1}$.

The following two claims also immediately follow from Claim 2.

Claim 3:

(1) If (A) holds, then $|S_{a+1}| = k-1$ or k, and $|S_i| = k$ for each $a+2 \leq i \leq C$.

(2) If (B) holds, then $|S_{a+1}| = k-1$, $|S_i| = k$ or $k+1$ for each $a+2 \leq i \leq C$, and the number of those indices i with $a+2 \leq i \leq C$ for which $|S_i| = k+1$ is at most one.

Claim 4:

(1) If (A) holds, then $|T_{a+1}| = k$ or $k+1$, and $|T_i| = k$ for each $a+2 \le i \le C$.

(2) If (B) holds, then $|T_i| = k$ or $k+1$ for each $a+1 \le i \le C$, and the number of those indices i with $a+1 \le i \le C$ for which $|T_i| = k+1$ is at most one.

Claim 5: $|S_{i-1} \cup S_i \cup S_{i+1}| \le 3k+1$ for each $a+1 \le i \le C-1$.

Proof: Since Claim 3 implies that $|S_i| \le k+1$ for each $a+1 \le i \le C$, and that the number of indices i with $a+1 \le i \le C$ such that $|S_i| = k+1$ is at most one, the desired inequality follows immediately. □

Claim 6: $|T_{i-1} \cup T_i \cup T_{i+1}| \le 3k+1$ for each $a+1 \le i \le C-1$.

Proof: Since Claim 4 implies that $|T_i| \le k+1$ for each $a+1 \le i \le C$, and that the number of indices i with $a+1 \le i \le C$ such that $|T_i| = k+1$ is at most one, the desired inequality follows immediately. □

Claim 7: Let $a+1 \le i \le C-1$.

(a) For any $x, x' \in S_i$, $d(x, x') \le 2$.

(b) For any $y, y' \in T_i$, $d(y, y') \le 2$.

Proof: Take $x, x' \in S_i$. If $x = x'$ or $xx' \in E(G)$, then we clearly have $d(x, x') \le 2$. Thus assume $x \ne x'$ and $xx' \notin E(G)$. Then

$$N(x) \cup N(x') \subset (S_{i-1} \cup S_i \cup S_{i+1}) - \{x, x'\}.$$

Since $|S_{i-1} \cup S_i \cup S_{i+1}| \le 3k+1$ by Claim 5, this implies

$$|N(x) \cup N(x')| \le |S_{i-1} \cup S_i \cup S_{i+1} - \{x, x'\}| \le 3k+1-2 = 3k-1.$$

On the other hand, since G is $(2k-1)$-connected, $|N(x)| \ge 2k-1$ and $|N(x')| \ge 2k-1$. Consequently, $N(x) \cap N(x') \ne \emptyset$ because otherwise we would get $k \le 1$ from $4k-2 \le |N(x) \cap N(x')| \le 3k-1$. Hence $d(x, x') \le 2$. Thus (a) is proved. We can prove (b) in exactly the same way by using Claim 6 in place of Claim 5. □

We now establish Lemma 6(b) by proving the following statement:

there exists $w \in X_{C-1}$ such that $d_G(w, w') \le 6$ for every $w' \in X_{C-1}$.

By the definition of C, there exist $w_1 \in S_C$ and $w_2 \in T_C$ such that $d_G(w_1, w_2) \le 2$. Let w be a vertex in X_{C-1} which is on a shortest $z - w_1$ path. Then $w \in S_{C-1}$. Now let $w' \in X_{C-1}$. We show that $d_G(w, w') \le 6$.

If $w' \in S_{C-1}$, Claim 7 implies that $d_G(w, w') \leq 2 \leq 6$. Thus we may assume $w' \in T_{C-1}$. Let w'' be a vertex in X_{C-1} which is on a shortest $z - w_2$ path. We clearly have $d_G(w, w_1) = 1$ and $d_G(w_2, w'') = 1$, and $d_G(w'', w') \leq 2$ by Claim 7. Since $d_G(w_1, w_2) \leq 2$, we obtain $d_G(w, w') \leq d_G(w, w_1) + d_G(w_1, w_2) + d_G(w_2, w'') + d_G(w'', w') \leq 1 + 2 + 1 + 2 = 6$, as desired. □

For convenience, we restate Lemma 6 in the following form:

Lemma 7. *There exists m with $a+1 \leq m \leq b-1$ and there exists $v \in X_m$ such that $d_G(v, v') \leq 6$ for every $v' \in X_m$, and such that in the case where $m = b-1$, $d_G(v, v') \leq 4$ for every $v' \in X_m$.*

We now complete the proof of Proposition 3.

Proof of Proposition 3: Let m and v be as in Lemma 7. Observe that $6 \leq a$, $a+1 \leq m$, $m \leq b-1$ and $b \leq r-5$.

Case 1. $r - m \leq m$.

Let z' be a vertex in X_{r-m} which is on a shortest $z - v$ path. Then $d_G(z', z) = r - m$ and $d_G(z', v) = m - (r - m) = 2m - r$. Take $x \in V(G)$, and let $x \in X_p$. First assume that $0 \leq p < m$. Then

$$d_G(z', x) \leq d_G(z', z) + d_G(z, x) = r - m + p < r - m + m = r.$$

Next assume that $m \leq p \leq r$. Let v' be a vertex in X_m which is on a shortest $z - x$ path. Then $d_G(v', x) = p - m \leq r - m$, and hence $d_G(z', x) \leq d_G(z', v) + d_G(v, v') + d_G(v', x) \leq (2m - r) + d_G(v, v') + (r - m) = m + d_G(v, v')$. On the other hand, it follows from Lemma 7 that if $m \leq b-2$, then $m + d_G(v, v') \leq (b-2) + 6$; and if $m = b-1$, then $m + d_G(v, v') \leq (b-1) + 4$. Since $b \leq r - 5$, we now get $d_G(z', x) \leq m + d_G(v, v') < r$. Thus in either case, $d_G(z', x) < r$. Since x was arbitrary, this contradicts the fact that r is the radius of G.

Case 2. $r - m > m$.

In this case, $2m < r$. Let $z' = v \in X_m$. Then $d_G(z', z) = m$. Take $x \in V(G)$, and let $x \in X_p$. First assume that $0 \leq p < m$. Then

$$d_G(z', x) \leq d_G(z', z) + d_G(z, x) = m + p < 2m < r.$$

Next assume that $m \leq p \leq r$. Let v' be a vertex in X_m which is on a shortest $z - x$ path. Then $d_G(v', z) = p - m$. Since $d_G(z', v') = d_G(v, v') \leq 6$ by Lemma 7 and since $m \geq a + 1 \geq 7$, we get

$$d_G(z', x) \leq d_G(z', v') + d_G(v', x) \leq 6 + (p - m) \leq r + (6 - m) < r.$$

Thus in either case, $d_G(z', x) < r$. Since x was arbitrary, this contradicts the fact that r is the radius of G.

This completes the proof of Proposition 3. □

Proposition 8. *Suppose that $r \geq 12$. Then $\sum_{i=6}^{r-5} |X_i| \geq 2k(r-10) - 2$.*

Proof: Let $I := \{i \mid 6 \leq i \leq r-5, |X_i| = 2k-1\}$. We may assume $|I| \geq 3$. Let $I = \{i_1, i_2, \ldots, i_{|I|}\}$ with $i_1 < i_2 < \cdots < i_{|I|}$. From I, we define a new sequence $j_1 < j_2 < \cdots < j_s$ inductively as follows. Set $j_1 = i_1$. For $l \geq 2$, set $j_l = \min\{i \mid i \in I, i \geq j_{l-1} + 2\}$ (if $\{i \mid i \in I, i \geq j_{l-1} + 2\} = \emptyset$, then we set $s = l - 1$ and terminate this procedure). We have $j_s = i_{|I|}$ or $i_{|I|-1}$ by the definition.

By Proposition 3, $\sum_{i=j_{h-1}}^{j_h - 1} |X_i| \geq 2k(j_h - j_{h-1})$ for all $2 \leq h \leq s$. Taking the sum of these inequalities, we get $\sum_{i=j_1}^{j_s-1} |X_i| = \sum_{h=2}^{s} \sum_{i=j_{h-1}}^{j_h-1} |X_i| \geq 2k(j_s - j_1)$. Consequently, $\sum_{i=6}^{r-5} |X_i| = \sum_{i=6}^{j_1-1} |X_i| + \sum_{i=j_1}^{j_s-1} |X_i| + \sum_{i=j_s}^{r-5} |X_i| \geq 2k(j_1 - 6) + 2k(j_s - j_1) + 2k(r - 4 - j_s) - 2 = 2k(r-10) - 2$, as desired. □

We are now in a position to complete the proof of the Theorem. If $r \leq 11$, the conclusion follows from Lemma 2. Thus we may assume $r \geq 12$. We clearly have $|X_0| = 1$ and $|X_r| \geq 1$ and, by Lemma 1, $|X_i| \geq 2k-1$ for all $1 \leq i \leq 5$ and all $r-4 \leq i \leq r-1$. By Proposition 8, $\sum_{i=6}^{r-5} |X_i| \geq 2k(r-10) - 2$. Adding all $|X_i|$, we obtain

$$n = \sum_{i=0}^{r} |X_i| \geq 1 + (2k-1) \times 5 + \{2k(r-10) - 2\} + (2k-1) \times 4 + 1 = 2kr - 2k - 9.$$

This completes the proof of the Theorem. □

4 Remarks

If $k \geq 6$, there are infinitely many graphs which attain equality in the Theorem. To see this, let t be a positive integer, and let C be a cycle with length $2kt + 2$. Define a graph G with $V(G). = V(C)$ by letting $E(G) = \{uv \mid d_C(u, v) \leq k\}$. Then G is $2k$-connected (so $(2k-1)$-connected), and $r(G) = t + 1 = \left\lfloor \frac{(2kt+2)+2k+9}{2k} \right\rfloor$.

References

[1] J. Harant and H. Walther, On the Radius of Graphs, *J. Combin. Theory Ser. B* **30** (1981), 113–117.