What is the size of the smallest latin square for which a weakly completable critical set of cells exists?

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It has been shown by Sittampalam and Keedwell that weak critical sets exist for certain latin squares of order six and that previously claimed examples (for certain latin squares of order 12) are incorrect. This led to the question raised in the title of this paper. Our purpose is to show that five is the smallest order for which weakly completable critical sets exist. In the process of proving this result, we show that, for each of the two types of latin square of order four, all minimal critical sets are of the same type.

1. Introduction.

A *latin square* of order n is an $n \times n$ matrix containing n distinct symbols such that each symbol occurs exactly once in each row and column. We regard the latin square as consisting of n^2 triples (i, j, k) of symbols such that k is the entry in the cell of row i and column j of the square. Usually, we shall use $0, 1, \ldots, n-1$ as our symbols so that rows and columns will be labelled from 0 to $n-1$.

A *uniquely completable set* (*UC set*) U of triples is such that it characterizes only one latin square. That is, there is a unique latin square of assigned order n which has U as a subset of its triples. Such a set U of triples is said to be a *critical set* if no subset of U is uniquely completable. Thus, for example, the sets $U_1 = \{(0, 0, 0), (2, 2, 1)\}$ and $U_2 = \{(0, 0, 0), (0, 1, 1), (1, 0, 1)\}$ are both critical sets for a latin square of order three; each completes uniquely to the Cayley table K of the cyclic group C_3 and no subset of U_1 or U_2 has this property. A *minimal critical set* for a particular latin square is one of smallest cardinal. Thus, for example, U_1 is a minimal critical set for the latin square K.

In the process of completing a UC set to the latin square L which it characterizes, we say that adjunction of a triple $t = (r, c, s)$ is *forced* in the process of completion of a set T of triples ($|T|<n^2$, $U \subseteq T \subset L$) to the complete set of triples which represents L (and which we also write as L), if either

(i) $\forall r' \neq r$, $\exists z \neq c$ such that $(r', z, s) \in T$ or $\exists z \neq s$ such that $(r', c, z) \in T$; or
(ii) $\forall c' \neq c$, $\exists z \neq r$ such that $(z, c', s) \in T$ or $\exists z \neq s$ such that $(r, c', z) \in T$; or
(iii) $\forall s' \neq s$, $\exists z \neq r$ such that $(z, c, s') \in T$ or $\exists z \neq c$ such that $(r, z, s') \in T$.

A UC set is called *strong* if we can define a sequence of sets of triples $U = F_1 \subset F_2 \subset \ldots \subset F_r = L$ such that each triple $t \in F_{v+1} - F_v$ is forced in F_v.

A UC set which is not strong is called *weak*. In particular, a critical set may be weak or strong.

A latin square L' is said to be *isotopic* to the latin square L if L' can be obtained from L by permuting the rows and/or the columns and/or the symbols of L. Then the cell (i, j, k) of L is transformed to the cell $(i\theta, j\varphi, k\psi)$ of L', where θ, φ, ψ, are permutations and (θ, φ, ψ) is the *isotopism*. The whole set of latin squares which can be obtained from L in this way are said to form an *isotopy class*.

Clearly, if the set U of cells of L form a UC set for L then the cells of L' onto which the cells of U are mapped form a set U' which is UC for L' and is of the same type relative to L' as U is relative to L: that is, weak or strong, critical or minimal critical or neither. We shall regard the sets U and U' as equivalent.

The concept of a weak critical set was first discussed in [4]. In that paper, an example of a weakly completable critical set for a latin square of order 12 was offered. However, the square actually exhibited was not a latin square as it contained more than one occurrence of the same symbol in some of its rows and columns. Later, another latin square of order 12 which was claimed to be weakly completable was published in [8]. Subsequently it was shown in [7] that this claim is incorrect. The set is neither critical nor weakly completable. However, it is strongly completable. The authors of [7] also wrote to Colbourn and received in reply (see [3]) a corrected version of the square given in [4]. This turned out to be the same square as that published in [8] but the alleged critical set was slightly different. In fact, this set is not critical but it is uniquely completable (strongly). Very recently (and since the present paper was submitteed), it has been pointed out in [1] that, if the symbols in three of the cells of the square published in [4] are changed, a 12x12 square which is both latin and weakly completable is obtained. In the meanwhile, the authors of [7] had shown that weakly completable critical sets exist for certain latin squares of order six. This led to the question raised in the title of the present paper.

It should be recorded here that a formal definition of the concept of strongly completable was first given in [5]. However, the definition given there differs from ours in that adjunction of a particular triple to a UC set is regarded as forced only if property (iii) of our definitioin is satisfied by it. (With this definition, the corresponding critical set in any parastrophe/conjugate of L except its transpose would not be strong.)

Our purpose is to show that five is the smallest order for which weakly completable critical sets exist.

We first remark that a critical set for any 3x3 latin square comprises either two or three cells (examples are given above) and it is easy to check that all are strongly completable.

2.Non-existence of weakly completable critical sets of minimal size for the cyclic group C_4.

The addition table of C_4 will be used in the form shown in Fig. 1a below. To transform this to the more usual form, we first interchange rows 3 and 4, then columns 2 and 3 and finally rows 2 and 3 as shown in Figs. 1b, 1c, 1d respectively.

To cover the four intercalates (2x2 latin subsquares), we need at least four cells in a critical set. Moreover, at least three different symbols must occur in any critical set for a latin square of order four. If, in Fig. 1a, we interchange the symbols 0 and x or y and z then the latin square L' so obtained is a *principal isotope* of L: that is, it can be transformed to L by rearrangement of rows and columns only, the symbols being left fixed. Also, in Fig. 1a, if we interchange the pairs of symbols 0, x and y, z, the latin square L'' so obtained is again a principal isotope of L. Thus, all four of the symbols 0, x, y, z have the same status. It follows that we can suppose without loss of generality that the four symbols of a minimal critical set are $0, x, y, y$ or $0, x, y, z$.

0	x	y	z
x	0	z	y
z	y	0	x
y	z	x	0

Fig. 1a

0	x	y	z
x	0	z	y
y	z	x	0
z	y	0	x

Fig. 1b

0	y	x	z
x	z	0	y
y	x	z	0
z	0	y	x

Fig. 1c

0	y	x	z
y	x	z	0
x	z	0	y
z	0	y	x

Fig. 1d

Since the Cayley table of C_4 has no transversals, two cells must occur in the same row (or column) if the cell entries are all different.

We consider first the possibility of a critical set which has no two of its cells in the same row or column. In that case, we can suppose as above that the cell entries are $0, x, y, y$ and that the critical set includes the cell (0, 0) with entry 0. It must then include also the cell (1, 3) with entry y otherwise the top right intercalate would be uncovered. Since no two cells of the critical set are in the same row or column and since the bottom left and bottom right intercalates must be covered, the remaining two cells of the critical set must be the cells (2, 1) and(3, 2) containing the entries y, x respectively. However, the set of four cells so obtained has two distinct completions (Figs. 1a and 2) so we conclude that no critical set comprising four cells in distinct rows and distinct columns exists.

$\underline{0}$	z	y	x
z	x	0	$\underline{y}$
x	$\underline{y}$	z	0
y	0	$\underline{x}$	z

Fig. 2

$\underline{0}$	•	$\underline{y}$	•
•	•	•	•
•	^	•	•
•	•	$\underline{x}$	•

Fig. 3

$\underline{0}$	z	$\underline{y}$	x
y	x	0	z
x	$\underline{y}$	z	0
z	0	$\underline{x}$	y

Fig. 4a

$\underline{0}$	z	$\underline{y}$	x
y	x	z	0
x	$\underline{y}$	0	z
z	0	$\underline{x}$	y

Fig. 4b

For a critical set of four cells which has two cells in the same row (or column), we may suppose that the rows and symbols are reordered so that these cells are in the first row and contain the entries 0 and y respectively. (They cannot be the cells which contain 0 and x otherwise one intercalate would be covered twice and another left uncovered.)

In that case, the bottom right intercalate must be covered by one of the cells which contains x. If the cell (3, 2) is used, then the fourth cell (one containing y or z) must be the cell (2, 1) otherwise either two rows or two columns would remain uncovered (Fig. 3) or else the bottom left intercalate would remain uncovered. However, in that case, there exist three different completions to a latin square (Figs. 1a, 4a, 4b). We conclude that, if two cells of a critical set of four cells occur in the same row, then the critical set must include the cells (0, 0), (0, 2), (2, 3) containing $0, y, x$ respectively. Then the fourth cell of the critical set is required to cover the bottom left intercalate and must lie in the last row otherwise two rows would be left uncovered. If it is the cell (3, 0) containing y, the critical set has a unique strong completion to the given latin square (Fig. 5). If it is the cell (3, 1) containing z, there exist two different completions to a latin square (Figs. 1a and 6) and so the set is not critical. Hence:

Theorem 1. *All minimal critical sets for a 4×4 latin square based on the cyclic group are equivalent under isotopism and are strongly completable.*

$\underline{0}$	x	$\underline{y}$	z
x	0	z	y
z	y	0	$\underline{x}$
$\underline{y}$	z	x	0

Fig. 5

$\underline{0}$	x	$\underline{y}$	z
z	y	x	0
y	0	z	$\underline{x}$
x	$\underline{z}$	0	y

Fig. 6

3. Non-existence of weakly completable critical sets of minimal size for the elementary abelian group $C_2 \oplus C_2$.

In order to cover all twelve intercalates of the Cayley table L of $C_2 \oplus C_2$ given in Fig, 7, we need at least four cells and these must be those of a transversal as is easily seen from the table of intercalates given in Fig. 8 below. However, the part

0	x	y	z
x	0	z	y
y	z	0	x
z	y	x	0

Fig. 7

${}^{a}{}_{c}{}^{b}\bullet$	${}^{a}{}_{d}{}^{e}\bullet$	${}^{f}{}_{d}{}^{b}\bullet$	${}^{f}{}_{c}{}^{e}\bullet$
${}^{a}{}_{h}{}^{g}\bullet$	${}^{a}{}_{i}{}^{j}\bullet$	${}^{f}{}_{i}{}^{g}\bullet$	${}^{f}{}_{h}{}^{j}\bullet$
${}^{k}{}_{h}{}^{b}\bullet$	${}^{k}{}_{i}{}^{e}\bullet$	${}^{l}{}_{i}{}^{b}\bullet$	${}^{l}{}_{h}{}^{e}\bullet$
${}^{k}{}_{c}{}^{g}\bullet$	${}^{k}{}_{d}{}^{j}\bullet$	${}^{l}{}_{d}{}^{g}\bullet$	${}^{l}{}_{c}{}^{j}\bullet$

Fig. 8

of L which remains when the cells of a transversal are deleted is a CPLS (see Lemma 1 of [2]), so at least one further cell is required to cover this. Thus, a minimal critical set of L contains at least (in fact, exactly) five cells of L. It follows that at least two, and possibly three, of the cells of a critical set must contain the same symbol. Without loss of generality, we may denote this symbol by 0 and suppose that the rows and columns of L have been arranged so that 0 appears on the leading diagonal.

We consider first the possibility of a critical set which includes three cells each of which contains the entry 0. These cells may occur consecutively or non-consecutively along the leading diagonal. However, the second case may be reduced to the first by interchanging the last two rows, then the last two columns and finally re-naming y as z and z as y, as shown in Figs. 9b, 9c, 9d respectively.

$\underline{0}$	x	y	z
x	$\underline{0}$	z	y
y	z	0	x
z	y	x	$\underline{0}$

Fig. 9a

$\underline{0}$	x	y	z
x	$\underline{0}$	z	y
z	y	x	$\underline{0}$
y	z	0	x

Fig. 9b

$\underline{0}$	x	z	y
x	$\underline{0}$	y	z
z	y	$\underline{0}$	x
y	z	x	0

Fig. 9c

$\underline{0}$	x	y	z
x	$\underline{0}$	z	y
y	z	$\underline{0}$	x
z	y	x	0

Fig. 9d

Thus, we need consider only the first case. The three cells which contain 0 cover the intercalates a, b, c, i, j, l of Fig. 8 but leave the intercalates d, e, f, g, h, k uncovered. It is easy to check that no two further cells can cover all six of these uncovered intercalates. Thus, no critical set of five cells can include three cells all of which contain the same symbol.

We consider the remaining case in which two cells of a critical set of five cells each contain the symbol 0. Again, it is easy to see that we can assume without loss of generality that these cells are the first two on the leading diagonal. (An isomorphism involving interchange of rows, columns and symbols will put any two cells containing 0 into these positions.) Because the bottom right 2×2 subsquare in Fig. 7 is an intercalate and must be covered, there must be a cell containing x and it must be either the cell (2, 3) or else the cell (3, 2). The symmetry of the square shows that these are equivalent. We can assume, therefore, that any critical set of five cells includes the cells (0, 0), (1, 1) and (2, 3). Jointly, these leave the intercalates d, f, g, k uncovered. No cell covers more than two of these four intercalates. They can be covered by either of the pairs of cells (0, 2) and (3, 0) or (3, 1) and (1, 2). When each of these pairs of cells in turn is adjoined to the previous three, the second of the two sets of five cells thus produced is isotopic to the transpose of the first. (This is illustrated in Fig. 10, where we first interchange the first and last pairs of columns and then interchange the first and last pairs of rows. Fig. 10d is the transpose of Fig. 10c and contains the first of the two critical sets.).

$\underline{0}$	•	•	•
•	$\underline{0}$	z	•
•	•	•	$\underline{x}$
•	y	•	•

Fig. 10a

•	$\underline{0}$	•	•
$\underline{0}$	•	•	z
•	•	$\underline{x}$	•
y	•	•	•

Fig. 10b

$\underline{0}$	•	•	z
•	$\underline{0}$	•	•
y	•	•	•
•	•	$\underline{x}$	•

Fig. 10c

$\underline{0}$	•	y	•
•	$\underline{0}$	•	•
•	•	•	$\underline{x}$
z	•	•	•

Fig. 10d

Since either set is strongly completable, we have:

Theorem 2. *All minimal critical sets for a 4×4 latin square based on the elementary abelian 2-group $C_2 \oplus C_2$ are equivalent under the operations of isotopism and/or transposition and are strongly completable.*

It is not too difficult to check that all uniquely completable sets of five or more cells for a latin square based on the the cyclic group C_4 and all uniquely completable sets of six or more cells for a latin square based on the group $C_2 \oplus C_2$ are strongly completable. Consider first the latter. Let us suppose that a stage has been reached at which there are no forced triples. Then, by hypothesis, at least six cells have been filled. If any three of these filled cells contain the same symbol, then the fourth cell to contain that symbol is forced: a contradiction. We may therefore assume that no symbol occurs more than twice unless it occurs four times. Also, no three of the filled cells can be in the same row or column otherwise the entry of the fourth cell of that row or column would be forced: another contradiction. We may therefore assume that any row or column which is not completely filled contains at most two filled cells.

We consider first the case when no symbol occurs four times and no row or column is completely filled. Since there are at least six filled cells, there must exist two rows (and two columns) each of which has two filled cells. As in the argument used to establish Theorem 2, we can suppose without loss of generality that the cells (0, 0) and (1, 1) both contain the entry 0 and that the cell (2, 3) contains x (Fig. 11). The remaining cells of our supposedly weakly completable set must include cells containing either x, y, z or x, y, y (up to isomorphism the same as x, z, z) or y, y, z (up to isomorphism the same as z, z, y). Moreover, the top right and bottom left intercalates must be covered. It is then very easy to check that any such set is in fact strongly completable (or else that some intercalate is left uncovered).

$\underline{0}$	•	•	•
•	$\underline{0}$	•	•
•	•	•	$\underline{x}$
•	•	•	•

Fig. 11

0	1	2	•	•
1	0	•	•	2
•	•	•	•	•
3	2	•	•	1
•	•	•	•	0

Fig. 12

Suppose now that some symbol occurs four times in our supposedly weakly completable set. Again without loss of generality, we can assume that it is the symbol 0. Also, because the top right intercalate must be covered, we can assume that our critical set includes the cell (0, 2) containing y. (If it contains one of the other cells of this intercalate, an interchange of symbols and/or rows and columns will replace that cell by the one proposed.) Since
(i) no three cells of our weakly completable set can be in the same row or column, (ii) the cells of a uniquely completable set must jointly contain at least three distinct symbols, and (iii) the intercalates e, g, h, k shown in Fig. 8 must be covered, one of the following situations must occur:
Case I. Our set includes the cell (1, 0) containing x and cell (2, 1) containing z.
Case II. Our set includes the cell (3, 0) containing z and cell (2, 3) containing x.

In either case, it is easy to check that the resulting set of cells is strongly completable.

The cases in which a row (or column) is completely filled is equivalent to the case when some symbol occurs four times because, in the consideration of whether a particular set of triples does or does not form a critical set, the constraints row/column/symbol are interchangeable and so the roles of row/column/symbol can be interchanged without changing the nature (weak or strong) of a critical set..

Finally, in the cases when the set is uniquely completable and two symbols each occur four times or a symbol occurs four times and a row or column is completely filled, it is trivial to check that the completion is strong.

The corresponding argument for the case of the cyclic group is similar but rather more lengthy. We may outline it as follows:

As before, let us suppose that a stage has been reached at which there are no forced triples. Since we are assuming that at least five cells are already filled, at least two cells of our supposedly weakly completable critical set contain the same symbol. Since, in Fig. 1a, there exist isotopisms which interchange the symbols 0 and x and/or y and z and also an isotopism which exchanges the pair 0, x for the pair y, z, we can suppose that the repeated symbol is 0. 0 cannot appear thrice (unless it occurs four times) otherwise the fourth occurrence of 0 would be forced, a contradiction. There are two main cases according as the two occurrences of 0 are in the same or different intercalates.

Case I. The cells (0, 0) and (1, 1) contain the symbol 0. At least two symbols other than 0 must occur. Since (in Fig. 1a) the bottom right intercalate must be covered, x must occur either in cell (2, 3) or else in cell (3, 2). Since y and z are exchangeable and since the top left intercalate must be covered, we can suppose without loss of generality that y occurs in cell (0, 2). Also, for unique completion, one of the cells of the bottom left intercalate must be in our critical set. It is easy to check that, whichever one is included, the square completes uniquely to that of Fig. 1a.

Case II. The cells (0, 0) and (2, 2) contain the symbol 0. One of the symbols y or z must occur. We can suppose without loss of generality that y occurs. There are two subcases according as y occurs in cell (0, 2) or in cell (1, 3) of the top right intercalate.

In the first subcase, there can be no further cells in the zeroth row or second column otherwise the fourth entry of that row or column would be forced. Also, the bottom left intercalate must be covered. Suppose, for example, that z occurs in cell (2, 0). For weak completion, no row or column must have three of its cells filled so the only other cells which can be filled are cell (3, 1) with z and/or cell (1, 3) with y. In any event, the square completes strongly and uniquely to Fig. 1a. The cases z occurs in cell (3, 1) or y occurs in cell (2, 1) or in cell (3, 0) are similar.

In the second subcase, y occurs in cell (1, 3). There are then a large number of situations to consider according to which cell or cells of the bottom left intercalate are filled. If cell (2, 0) is filled with z, completion to Fig. 1a is unique and strong. The remaining possibilities are considerably more tedious to check but in all cases completion is unique and strong.

From this and evidence obtained elsewhere, we make the conjecture that, *for a latin square of any order based on a cyclic group, no weakly completable critical sets exist.*

Since every latin square of order four is isotopic either to C_4 or else to $C_2 \oplus C_2$ we can conclude that:

Theorem 3. *There are no weakly completable critical sets for any latin square of order four.*

Next, we note that the partial latin square W of order five exhibited in Fig. 12 has no forced triples but that it is UC: that is, it is weakly completable. It is due to D. R. Burgess and was obtained with the aid of a weakly completable edge-colouring scheme for a certain directed graph (see [2]). In fact, the triples of W form a critical set but this is less immediate to show. Note that it is not a minimal critical set. The details are in [2]. Hence, we have:

Theorem 4. *The smallest order for which there exists a latin square which has a weakly critical set is order five.*

References

[1] J. A. Bate and G. H. J. van Rees, The size of the smallest strong critical set in a latin square, Submitted.

[2] D. R. Burgess, Ph.D. Thesis, University of Surrey, 1996, In preparation.

[3] C. J. Colbourn, Private communication to A. D. Keedwell (January, 1995).

[4] C. J. Colbourn, M. J. Colbourn and D. R. Stinson, The computational complexity of recognizing critical sets. In Proc. First SE Asian Graph Theory Colloq., *Lecture Notes in Math. No.* 1073, pp. 248-253, Springer-Verlag, 1984.

[5] J. Cooper, D. Donovan and J. Seberry, Latin squares and critical sets of minimal size, *Austral. J. Combin.* 4 (1991), 113-120.

[6] A. D. Keedwell, A note on critical sets for the elementary abelian 2-group. *Utilitas Math.*, to appear.

[7] A. G. Sittampalam (with A. D. Keedwell), Critical sets for dihedral groups. *Discrete Math.*, to appear.

[8] A. P. Street, Defining sets for *t*-designs and critical sets for latin squares, *New Zealand J. Math.* 21(1992), 133-144.