Decomposition of Multigraphs into Isomorphic Graphs with Two Edges

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ABSTRACT. We improve upon Caro's general polynomial characterizations, all in terms of modified line graphs, restricted to decomposing a graph into isomorphic subgraphs H with two edges. Firstly, we solve the problem for a multigraph; secondly we decrease polynomial bound on complexity if $H = 2K_2$ and provide original sufficient condition which can be verified in linear time if $H = P_3$.

1 Introduction

All multigraphs M considered in what follows are loopless. Let $V(M)$ and $E(M)$ stand for the vertex set and edge set of M, respectively. Cardinalities of these sets, denoted $v(M)$ and $e(M)$, are called the *order* and *size* of M. Let $k(M)$ stand for the number of connected components of M. As usual $\Delta(M)$ denotes the maximum degree among vertices of M. An edge of M whose removal increases the number of components is called a *cutedge* of M. The union of λ disjoint copies of M is denoted λM. Recall that P_n is a path on n vertices.

In this paper we generalize known characterizations of simple graphs or hypergraphs which are edge-decomposable into isomorphic substructures of size two. The work is prompted by the first author's contribution to solving the third author's problem [10] on P_3-decompositions.

The problem of decompositions into isomorphic parts H with two edges, which possibly satisfy additional conditions, can be reduced to finding a perfect matching in modified line graphs. This idea is presented in Caro's manuscript [2] in case the combinatorial structures to be decomposed are hypergraphs. Examples for possible conditions are e.g.: the two edges are at distance at least k, the two edges are on a cycle of bounded length, the two edges are induced, etc. Caro's idea can be presented in the language of decomposing multihypergraphs M. Given an M, let G be a graph as follows. Edges of M are the only vertices of G. Two vertices of G are made adjacent iff the corresponding edges in M form a copy of H such that all additional conditions are satisfied. Now it is easily seen that M is H-decomposable iff G has a 1-factor. Testing a 1-factor in G can be done in time $O(e(M)^{2.5})$ using the Even-Kariv algorithm. Thus the time complexity of the characterization is polynomial provided that so is the time complexity of constructing the modified line graph G, which depends on the nature of additional conditions. See proof of Theorem 3.2 in [3] for the case of decomposing a graph M into induced copies of a graph H having two edges.

2 $2K_2$-decomposition of a multigraph

Recall that a *cluster* in a multigraph M is defined to be a set of edges which are pairwise adjacent. Therefore a cluster is a subset of edges of a submultigraph induced by vertices of either a star or a triangle. The maximal size among clusters in M is called the *cluster number* and is denoted $\omega_1 = \omega_1(M)$. Hence the cluster number of M is the clique number of the line graph $L(M)$ of M. Moreover,

$$\omega_1(M) = \max\{\Delta(M), \max_{K_3 \subseteq M} e(< K_3 >)\}.$$

In what follows we shall consider multigraphs M with cluster number not exceeding half the number of edges. Then a cluster of size $e(M)/2$ is called a *critical cluster* in M. By a *critical triangle* and a *critical star* we mean a critical cluster induced by vertices of a triangle and a star, respectively. The center of a critical star is called a *critical vertex* of the multigraph.

Our first result generalizes the following result due to Caro [1], cf. [5]. A simple graph G is $2K_2$-decomposable iff $e(G)$ is even, $\Delta(G) \leq e(G)/2$, and $G \neq K_2 \dot{\cup} K_3$.

Theorem 1. (Skupien [10]) *A multigraph M is $2K_2$-decomposable iff its size $e(M)$ is even and its cluster number $\omega_1(M) \leq e(M)/2$.*

Proof: Necessity is clear. To prove sufficiency we proceed by induction on the number $2k$ of edges. Moreover, consider multigraphs without isolated vertices. If $k = 1$ then M is a 2-matching and a decomposition exists. Assume that the result is true for any multigraph with $2k$ edges and consider a multigraph M_1 with $2k+2$ edges and cluster number $\omega_1(M_1) \leq k+1$.

We shall show that there exists a 2-matching $2e$ in M_1 such that $M := M_1 - 2e$ has cluster number $\omega_1(M) \leq k$. Then M satisfies the induction hypothesis and therefore M_1 is $2K_2$-decomposable. Call such a $2e$ to be a *required* 2-matching. Thus a 2-matching $2e$ is a required one if $2e$ covers all critical vertices and each critical triangle contributes one edge to the $2e$.

Consider the following cases.

A: $\omega_1(M_1) \leq k$. Any matching $2e$ is a required 2-matching.
Then $\omega_1(M) \leq k$.

B: $\omega_1(M_1) = k+1$. Consider the following subcases.

B1: There are two edge-disjoint critical triangles. If they are vertex-disjoint too, a required 2-matching has one edge in each of these triangles. Otherwise, the common vertex, x, of the triangles exists and can be the only critical vertex in M. To form a required 2-matching, choose an edge incident to x in one of triangles and an edge non-incident to x in the other one.

B2: There are two critical triangles with two vertices and all connecting them edges in common. Then each critical vertex belongs to the union of the two triangles because they together include more than half of the edges of M_1. In particular, both common vertices are critical. Moreover, if there is a third critical triangle, the order of M_1 is four (because isolated vertices are excluded). Therefore a required 2-matching exists.

B3: There is exactly one critical triangle. Note that each critical vertex of the critical triangle is adjacent to a vertex outside the critical triangle. On the other hand, there is none or one critical vertex outside the critical triangle. Moreover, different critical vertices are adjacent. Therefore a required 2-matching exists.

B4: No critical triangle exists. A required 2-matching is one that covers all critical vertices in this subcase. Considering the sum of degrees, note that the number of critical vertices is four at most. If this number is exactly four, four it is the order of M_1 and then any 2-matching is a required one. In this case the underlying graph of M_1 is either C_4 or $2K_2$. Otherwise, assume that there are two nonadjacent critical vertices. Then each edge of M_1 belongs to the union of stars at those two vertices and therefore a required 2-matching exists. So is the case when the number of critical vertices is at most two. The only remaining case is that there are three mutually adjacent critical vertices. Then one can see that each of critical vertices has a non-critical neighbour. Therefore required 2-matchings clearly exist. □

Remark: The existence of $2K_2$-decomposition of a multigraph M can be verified in polynomial time. The time bound is actually linear, $O(v(M))$, which was kindly remarked by Dr Z. Lonc. It is so because M has at most seven vertices of degree greater than $e(M)/4$ and each triangle thicker than critical one clearly contains no less than two such vertices.

3 P_3-decomposition of a multigraph

Given a multigraph M, define *$*$-line graph* of M, denoted $L^*(M)$, to be a graph with the vertex set $V(L^*(M))=E(M)$ and the edge set $E(L^*(M))=\{w_1w_2: \ w_1, w_2 \in E(M), |w_1 \cap w_2| = 1\}$. Evidently, $L^*(M)$ is obtainable from the ordinary line graph $L(M)$ by removal of all edges which represent multiple adjacency of edges in the root multigraph M. In other words, the operator L^* represents doubly adjacent edges in M as if they were nonadjacent in M.

Given a connected multigraph M, let

$$\eta(M) = \begin{cases} 0 & \text{if } v(M) \neq 2 \text{ and } e(M) \text{ is even,} \\ 1 & \text{if } v(M) \neq 2 \text{ and } e(M) \text{ is odd,} \\ e(M) & \text{if } v(M) = 2. \end{cases}$$

If M is a disconnected multigraph with components $M_1, \ldots, M_k$, define $\eta(M) = \eta(M_1) + \ldots + \eta(M_k)$, $k = k(M)$. Note that $\eta(M)$ is the number of odd (odd order) components in $L^*(M)$.

In what follows we present a polynomial time characterization of P_3-decomposable multigraphs M. The proof is based on the famous Tutte's 1-factor theorem.

Theorem 2 (Tutte) *A graph G has a 1-factor iff the order $v(G)$ of G is even and there is no set S, $S \subset V(G)$, such that the number of odd components of $G - S$ exceeds $|S|$.* □

Theorem 3 *Given a multigraph M, the following statements are equivalent:*
(i) M is P_3-decomposable,
(ii) $L^*(M)$ has a 1-factor,
(iii) $\eta(M - S) \leq |S|$ for all $S \subset E(M)$.

Proof: The neighbouring properties are clearly mutually equivalent. □

Remark: The notion of $*$-line graph and the equivalence $(i) \Leftrightarrow (ii)$ in Theorem 3 are implicitly included in Caro's manuscript [2]. Note that Caro applies his constructions only to hypergraphs.

Corollary 4 *Parity of the size $e(M)$ is a necessary condition for a multigraph M to be P_3-decomposable.* □

As follows from what is in Introduction, the time complexity of the characterization in Theorem 3 above is $O(e(M)^{5/2})$ or in fact, $O((v(G)+e(G))v(G)^{1/2})$ where $G = L^*(M)$ and $v(G) = e(M)$. We are going to present a sharp sufficient condition for M to be P_3-decomposable, which can be verified in the time $O(e(N))$ where N is the underlying graph of a multigraph M. Such is the time complexity of finding cutedges and leaves (i.e., maximal submultigraphs without cutedges).

Given a cutedge e and its endvertex x in a multigraph M, call x to be an *even vertex* of e if the component of $M - e$ containing x has even size. For any adjacent vertices x and y of M, let $p(x,y)$ denote the number of edges joining x and y, that is, $p(x,y)$ is the multiplicity of x–y adjacency. Write $x \diamond y$ if $p(x,y) \geq 2$.

Theorem 5 *Assume that M is a connected multigraph of even size $e(M)$ without any cutedge whose even vertex is incident to multiple edges of M. Moreover, for any pair, denoted $x \diamond y$, of multiply adjacent vertices x and y,*

$$deg(x) + deg(y) \geq 4p(x,y) - \varepsilon(x,y) \tag{1}$$

where $\varepsilon(x,y) \in \{0, 1, 2, 3\}$ and $\sum_{x \diamond y} \varepsilon(x,y) = 3$. Then M is P_3-decomposable.

Proof: Let S be any subset of $E(M)$. Define recursively M' and S'. To this end, let $M' = M$ and $S' = S$. Find, if possible, an odd size component, Q, of $M' - S'$ such that $E(Q)$ has exactly one neighbour, e_Q, in S' and perform updating:

$$M' \leftarrow M' - V(Q), \quad S' \leftarrow S' - \{e_Q\}.$$

Continue this procedure as long as it is possible. Eventually we get fixed objects M' and S' with property that the number of edges in S' which are incident to any odd size component of M' is at least two. Because at each step an even number of edges are removed, the property of being an even vertex of a cutedge is invariant under passing to a new M'. Therefore the degree conditions for multiply adjacent vertices in M' coincide with their original versions in M. Our assumption on degrees implies that among removed components of $M - S$ none is of order two and size bigger then one.

If S' is empty then $k(M') = 1$ and M' has even size. Hence

$$|S| = k(M - S) - 1 = \eta(M - S).$$

Therefore we consider the remaining case $S' \neq \emptyset$. Let k' be the number of all specialized components $M_1, \ldots, M_{k'}$ of $M' - S'$ which are ordered so that $v(M_i) = 2$ and $e(M_i) > 1$ for $i = 1, \ldots, t$ where t is an integer, $0 \leq t \leq k'$, else $e(M_j)$ is odd for $j = t+1, \ldots, k'$.

If $i \leq t$ then $V(M_i) = \{x, y\}$ for some vertices x, y and our assumption implies that $E(M_i)$ has at least $2p(x,y) - \varepsilon(x,y)$ ($\geq 2e(M_i) - \varepsilon(x,y)$) neighbours in S'. If $t < j \leq k'$ then M_j is an odd size component whose edge set has two or more neighbours in S'. Moreover, each edge in S' is clearly incident to at most two components of $M' - S'$. Therefore bounds on the sum of numbers of those neighbours are

$$2|S'| \geq 2e(M_1) + \ldots + 2e(M_t) - \sum_{x \diamond y} \varepsilon(x,y) + 2(k' - t),$$

whence

$$|S'| + 3/2 \geq \eta(M' - S').$$

Therefore

$$|S'| \geq \eta(M' - S')$$

because both sides are of the same parity due to even size of M'. Hence

$$|S| \geq \eta(M - S).$$

By Theorem 3, this completes the proof. □

Easy examples show that Theorem 5 is sharp, that is, right-hand sides of (1) cannot be essentially smaller, see Figure 1. New examples are obtainable by adding a number of edge-disjoint copies of a Hamiltonian cycle to any multigraph M_i in Figure 1.

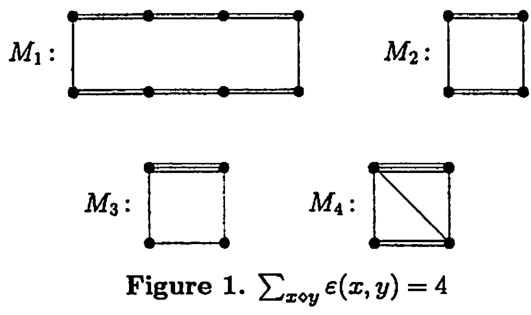

Figure 1. $\sum_{x \diamond y} \varepsilon(x,y) = 4$

Corollary 6 *Let M be a multigraph of even size $e(M)$ and let an integer $p \geq 2$ be the maximum multiplicity of edges in M. Then, if M is $(p+1)$-connected, M is P_3-decomposable.* □

Corollary 7 ([6, 4]) *A simple graph G is P_3-decomposable iff each component of G is of even size.* □

This result can also be deduced from Theorem 3 above and the following result due to Sumner [11] and Las Vergnas [7]. Recall that a claw-free graph is a graph without any induced subgraph isomorphic to the star $K_{1,3}$.

Theorem 8. *Every connected claw-free graph of even order has a 1-factor.*

It is known that a line graph of any graph (general or simple) is claw-free, cf. [8,9]. So is a *-line graph of a simple graph but not necessarily that of a multigraph.

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