Quasi-Twisted Codes over $\mathbb{F}_{11}$

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Abstract

Let $d_q(n,k)$ be the maximum possible minimum Hamming distance of a linear $[n,k]$ code over $\mathbb{F}_q$. Tables of best known linear codes exist for all fields up to $q = 9$. In this paper, linear codes over $\mathbb{F}_{11}$ are constructed for k up to 7. The codes constructed are from the class of quasi-twisted codes. These results show that there exists a (78,8) arc in PG(2,11). In addition, the minimum distances of the extended quadratic residue codes of lengths 76, 88 and 108 are determined.

1 Introduction

Let $\mathbb{F}_q$ denote the Galois field of q elements, and let $V(n,q)$ denote the vector space of all ordered n-tuples over $\mathbb{F}_q$. A linear $[n,k]$ code C of length n and dimension k over $\mathbb{F}_q$ is a k-dimensional subspace of $V(n,q)$. An $[n,k,d]$ code is an $[n,k]$ code with minimum (Hamming) distance d. Let A_i be the number of codewords of (Hamming) weight (or distance) i in C. Then the numbers $A_0, A_1, \ldots, A_n$ are called the weight distribution of C.

A central problem in coding theory is that of optimising one of the parameters n, k and d for given values of the other two. One version is to find $d_q(n,k)$, the largest value of d for which there exists an $[n,k,d]$ code over $\mathbb{F}_q$. Another is to find $n_q(k,d)$, the smallest value of n for which there exists an $[n,k,d]$ code over $\mathbb{F}_q$. A code which achieves either of these values is called *optimal*. Tables of best known linear codes exist for all fields up to $q = 9$ [8]. In this paper, linear codes over $\mathbb{F}_{11}$ are constructed for k up to 7.

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The Griesmer bound is a well-known lower bound on $n_q(k,d)$

$$n_q(k,d) \geq g_q(k,d) = \sum_{j=0}^{k-1} \left\lceil \frac{d}{q^j} \right\rceil, \tag{1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. For $k \leq 2$, the Griesmer bound is met for all q and d. The Singleton bound [15] is a lower bound on $n_q(k,d)$ and is given by

$$n_q(k,d) \geq d + k - 1 \tag{2}$$

Codes that meet this bound are called maximum distance separable (MDS). MDS codes exist for all values of $n \leq q+1$. Thus for $q = 11$, MDS codes exist for all lengths 12 or less.

For larger lengths and dimensions, far less is known about codes over $\mathbb{F}_{11}$. MDS self-dual codes ($k = n/2$), of lengths 4, 8 and 12 are given in [9]. Self-dual $[16,8,8]$, $[20,10,10]$ and $[24,12,9]$ codes are presented in [4]. A $[40,20,10]$ extended quadratic residue (QR) code is given in [16]. Using Magma [5], we have determined that the next three extended QR codes over $\mathbb{F}_{11}$ have parameters $[76,38,15]$, $[88,44,17]$ and $[108,54,18]$. In this paper we consider codes for dimensions $k = 3 - 7$. These codes establish lower bounds on the minimum distance. Many of these codes meet the Singleton and/or Griesmer bounds, and so are optimal.

A *punctured code* of C is a code obtained by deleting a coordinate from every codeword of C. A *shortened code* of C is a code obtained by taking only those codewords of C having a zero in a given coordinate position and then deleting that coordinate. The following bounds can be established based on these constructions

1) $d_q(n+1,k) \leq d_q(n,k) + 1$,

and

2) $d_q(n+1,k+1) \leq d_q(n,k)$.

Using the codes given in this paper, they provide many additional lower bounds.

The next section presents the class of quasi-twisted codes, and the construction results are given in Section 3.

2 Quasi-Twisted Codes

A code C is said to be quasi-twisted (QT) if a constacyclic shift[1] of any codeword by p positions is also a codeword in C [10, 17]. A cyclic code is a QC code with $p = 1$ and $\alpha = 1$. The length of a QT code considered here is $n = mp$. With a suitable permutation of coordinates, many QT codes can be characterized in terms of $(m \times m)$ twistulant matrices. In this case, a QT code can be transformed into an equivalent code with generator matrix

$$G = [R_0 \, R_1 \, R_2 \, ... \, R_{p-1}], \tag{3}$$

where $R_i, i = 0, 1, \ldots, p-1$ is a twistulant matrix of the form

$$R_i = \begin{bmatrix} r_{0,i} & r_{1,i} & r_{2,i} & \cdots & r_{m-1,i} \\ \alpha r_{m-1,i} & r_{0,i} & r_{1,i} & \cdots & r_{m-2,i} \\ \alpha r_{m-2,i} & \alpha r_{m-1,i} & r_{0,i} & \cdots & r_{m-3,i} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha r_{1,i} & \alpha r_{2,i} & \alpha r_{3,i} & \cdots & r_{0,i} \end{bmatrix}. \tag{4}$$

with $\alpha \in \mathbb{F}_{11} \backslash \{0\}$ If $\alpha = 1$, we obtain the class of quasi-cyclic (QC) codes. If $\alpha = -1$, the code is called nega-circulant (NC). For m odd, the QT codes are equivalent to QC codes [1]. In addition, all QT codes over $\mathbb{F}_{11}$ are equivalent to a QC or NC code.

The algebra of $m \times m$ twistulant matrices over $\mathbb{F}_q$ is isomorphic to the algebra of polynomials in the ring $\mathbb{F}_q[x]/(x^m - \alpha)$ if R_i is mapped onto the polynomial $r_i(x) = r_{0,i} + r_{1,i}x + r_{2,i}x^2 + \cdots + r_{m-1,i}x^{m-1}$, formed from the entries in the first row of R_i [15]. The $r_i(x)$ associated with a QT code are called the *defining polynomials* [10]. The set $\{r_0(x), r_1(x), \cdots, r_{p-1}(x)\}$ defines an $[mp, p]$ QT code with $k = m$.

The construction of QT codes requires a representative set of defining polynomials. These are the equivalence class representatives of a partition of the set of polynomials of degree less than m into *cyclic classes*. Two polynomials, $r_j(x)$ and $r_i(x)$ are said to be *equivalent* if they belong to the same class, i.e.

$$r_j(x) = \gamma x^l r_i(x) \bmod (x^m - \alpha),$$

for some integer $l > 0$ and scalar $\gamma \in \mathbb{F}_{11} \backslash \{0\}$. The number of representative defining polynomials, $N(m)$, for QC codes is given below

[1]A constacyclic shift of an m-tuple $(x_0, x_1, \ldots, x_{m-1})$ is the m-tuple $(\alpha x_{m-1}, x_0, \ldots, x_{m-2}), \alpha \in GF(q) \backslash \{0\}$.

m	$N(m)$
2	7
3	45
4	373
5	13225
6	29575

For NC codes, the number of polynomials is the same as for QC codes when m is odd, and $N(2) = 6$ and $N(4) = 366$, so the numbers are similar.

The QT codes presented here were constructed using a stochastic optimization algorithm, tabu search, similar to that in [6] and [13]. By restricting the search to the class of QT codes, and using a stochastic heuristic, codes with high minimum distance can be found with a reasonable amount of computational effort.

3 The Construction Algorithm

Imposing a structure on the codes being considered results in a search space that is smaller than for the original, general problem. The more restrictions on the structure, the smaller the search problem. This results in a tradeoff, since good codes may be missed if too much structure is imposed on the code. However, it is often the case that good codes have significant structure, and this partially explains why the approach presented here works so well.

It is not necessary to check the weight of every codeword in a QT code in order to determine d. Only a subset, $N < M$, of the codewords need be considered since the Hamming weight of $i(x)b_s(x) \bmod (x^m - \alpha)$ is equal to the weight of $i(x)\gamma x^l b_s(x) \bmod (x^m - \alpha)$ for all $l \geq 0$ and $\gamma \in \mathrm{GF}(q) \setminus \{0\}$. Note that this argument also applies to the set of defining polynomials. For example, with $q = 11$ and $m = 3$, from the above table we only have $N = 45$.

To simplify the process of searching for good codes, the weights of the subset of codewords can be stored in an array, and a matrix, D, can be formed from the arrays for the defining polynomials to be considered

$$D = \begin{array}{c|ccccccc} & b_1(x) & b_2(x) & \cdots & b_s(x) & \cdots & b_y(x) \\ \hline i_1(x) & w_{11} & w_{12} & \cdots & w_{1s} & \cdots & w_{1y} \\ i_2(x) & w_{21} & w_{22} & \cdots & w_{2s} & \cdots & w_{2y} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ i_t(x) & w_{t1} & w_{t2} & \cdots & w_{ts} & \cdots & w_{ty} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ i_z(x) & w_{z1} & w_{z2} & \cdots & w_{zs} & \cdots & w_{zy}, \end{array}$$

where $i_t(x)$ is the tth information polynomial, $b_s(x)$ is the sth defining polynomial, and w_{ts} is the Hamming weight of $i_t(x)b_s(x) \bmod (x^m - 1)$. Since $i_t(x)$ and $b_s(x)$ correspond to the same polynomials, D is a square ($y = z = N$), symmetric (by letting $i_t(x) = b_t(x)$ for all $1 \leq t \leq N$) matrix.

The complete weight distribution for a QT code composed of any set of $b_s(x)$ can be constructed from D. The search for a good code consists of finding p columns of D with a large minimum row sum, since the weight of a minimum distance codeword must be contained in these sums.

Having decided on the values of m and p (and thus also $n = mp$), the entries of the integer matrix D can be calculated and the problem formulated as a combinatorial optimization problem. Namely, we want to find

$$\max_{S} \min_{1 \leq j \leq N} \sum_{s \in S} w_{j,s}, \tag{5}$$

where $S \subseteq \{1, 2, \ldots, N\}$ and $|S| = p$. In general, one can take a multiset S with p elements, but it was found in past studies that for the new codes obtained, no defining polynomial occurs more than once, so (also because this made the optimization procedure perform better), S is here required to be a set.

The optimization method used in this work is *tabu search* [7]. This method can produce good near-optimal (optimal in some cases) solutions to difficult optimization problems with a reasonable amount of computational effort. It cannot, however, be used to prove or disprove the optimality of solutions found. For an extensive survey of optimization methods in coding theory, particularly stochastic procedures, see [14].

Tabu search is a local search algorithm, which means that starting from an initial solution, a series of solutions is obtained so that every new solution only differs slightly from the previous one. A potential new solution is called a *neighbor* of the old solution, and all neighbors of a given solution constitute the *neighborhood* of that solution. To evaluate the quality of solutions, a *cost function* is needed. Tabu search always proceeds to a best possible solution in the neighborhood of the current solution. In a simple version, if there are several equally good neighbors with the best cost, a random choice is made (note that it is possible that the best neighbor has a worse cost than the current solution has). To ensure that the search does not loop on a subset of moves or solutions, attributes of recent solutions are stored in a so-called tabu list; new moves or solutions with these attributes are then not allowed for a certain period of time (here, for a predefined number of moves, L).

Tabu search is applied here to the problem of finding QT codes, defined as a minimization problem, in the following way. First, the problem is not formulated as generally as in (5), as the desired minimum distance, d, of the code is fixed. A solution is any set $S \subseteq \{1, 2, \ldots, N\}$ of p columns of D, the neighborhood of a solution is the set of solutions obtained by replacing one column with a column that is not in the code, and the cost function is of the form

$$C = \sum_{j=1}^{N} \max(0, d - \sum_{s \in S} w_{j,s})$$

A solution with cost 0 now corresponds to a code with minimum distance at least d. If we find such a solution we know that we have reached a global optimum, and we can quit the search and save the code. Otherwise, the search is continued (and possibly restarted occasionally), until a given time or iteration limit is reached. The tabu list is simply the indexes of the new columns. Thus, if a column is replaced by another, the new column must not be replaced during the next L moves.

The values of L used were in the range $p/10 \leq L \leq p/5$. If a code was not found within 1000–2000 iterations, the search was restarted from a new random initial solution. As many as 1000 restarts were performed for given values of m and p. The total number of iterations to find a code varied between about one hundred and a few million.

The best QC codes found are given in Tables 1 to 5. In only 5 cases, a NC code was found which had a higher minimum distance. These codes are shown in Tables 6 and 7. The defining polynomials are listed with the lowest degree coefficient on the left, i.e., 7321 corresponds to the polynomial

$x^3 + 2x^2 + 3x + 7$, with leading zeroes left out for brevity. The digit 10 is denoted by (10). As an example, consider the [30,3] code in Table 1 with $m = 3$ and $p = 10$ defining polynomials. Using these polynomials gives the following generator matrix

$$G = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 019 & 12(10) & 012 & 175 & 152 & 165 & 016 & 127 & 01(10) & 129 \\ 901 & (10)12 & 201 & 517 & 215 & 516 & 601 & 712 & (10)01 & 912 \\ 190 & 2(10)1 & 120 & 751 & 521 & 651 & 160 & 271 & 1(10)0 & 291 \end{array} \right]$$

with weight distribution

i	A_i
0	1
26	570
27	220
28	270
29	120
30	150

This code is optimal since it meets the Griesmer bound (1), and so establishes that $d_{11}(30, 3) = 26$.

For $m = 4$ and $p = 11$, the best code found is NC (a QT code with $\alpha = -1$). This code has generator matrix

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c} 1125 & 0198 & 1291 & 0015 & 1217 & 0132 & 1127 & 1786 & 1529 & 1282 & 13(10)2 \\ 6112 & 3019 & (10)129 & 6001 & 4121 & 9013 & 4112 & 5178 & 2152 & 9128 & 913(10) \\ 9611 & 2301 & 2(10)12 & (10)600 & (10)412 & 8901 & 9411 & 3517 & 9216 & 3912 & 1913 \\ (10)961 & (10)230 & 92(10)1 & 0(10)60 & 9(10)41 & (10)890 & (10)941 & 4351 & 6921 & 9391 & 8191 \end{array} \right]$$

with weight distribution

i	A_i
0	1
37	1440
38	2280
39	2360
40	2720
41	2400
42	1960
43	960
44	520

This code is optimal since it meets the Griesmer bound (1), and so establishes that $d_{11}(44, 4) = 37$.

Note that all codes with $n \leq 12$ given in the tables are MDS (this includes two codes with $\alpha = -1$). In addition, the codes for $m = 3$ with $p = 5, 6, 7, 9, 10, 13, 14, 17, 18, 21, 22$ meet the Griesmer bound. For larger lengths, numberous codes were found which meet this bound. Of note is the $[78, 3, 70]$ code which establishes that $d_{11}(78, 3) = 70$. This code has generator matrix

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|}
016 & 118 & 114 & 152 & 001 & 129 & 117 & 011 & 013 & 13(10) & 125 & 157 & 124 \\
601 & 811 & 411 & 215 & 100 & 912 & 711 & 101 & 301 & (10)13 & 512 & 715 & 412 \\
160 & 181 & 141 & 521 & 010 & 291 & 171 & 110 & 130 & 3(10)1 & 251 & 571 & 241
\end{array}\right.$$

$$\left.\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
119 & 134 & 142 & 175 & 012 & 135 & 16(10) & 116 & 018 & 11(10) & 01(10) & 12(10) & 158 \\
911 & 413 & 214 & 517 & 201 & 513 & (10)16 & 611 & 801 & (10)11 & (10)01 & (10)12 & 815 \\
191 & 341 & 421 & 751 & 120 & 351 & 6(10)1 & 161 & 180 & 1(10)1 & 1(10)0 & 2(10)1 & 581
\end{array}\right]$$

with weight distribution

i	A_i
0	1
70	810
71	180
72	160
73	90
74	30
75	30
78	30

This improves on the known bounds and shows that there exists a (78,8) arc in PG(2,11) [2]. The order of the automorphism group of this code is 30.

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Table 1: Best QC Codes over $\mathbb{F}_{11}$ with $p = 3$

code	d	$r_i(x)$
[6,3]	4	19, 11
[9,3]	7	116, 11, 14
[12,3]	10	14, 13, 124, 142
[15,3]	12	19, 11, 114, 14, 175
[18,3]	15	19, 11, 117, 14, 175, 16(10)
[21,3]	18	18, 165, 157, 115, 143, 113, 11(10)
[24,3]	20	19, 11, 116, 14, 175, 16(10), 165, 15
[27,3]	23	19, 158, 112, 14, 175, 16(10), 165, 15, 12
[30,3]	26	19, 12(10), 12, 175, 152, 165, 16, 127, 1(10), 129
[33,3]	28	19, 11, 112, 14, 175, 16(10), 165, 113, 12, 115, 17
[36,3]	31	11, 158, 132, 14, 17, 118, 149, 12, 157, 142, 123, 152
[39,3]	34	149, 115, 118, 13(10), 1, 125, 142, 16, 117, 147, 152, 135, 114
[42,3]	37	158, 11, 129, 119, 118, 13, 123, 157, 152, 19, 143, 117, 14(10), 14
[45,3]	39	12, 14, 118, 149, 175, 157, 142, 123, 152, 137, 16, 19, 15, 16(10), 128
[48,3]	42	1, 12, 158, 112, 175, 149, 124, 113, 134, 115, 127, 16, 19, 117, 132, 11
[51,3]	45	123, 13, 129, 18, 17, 11, 149, 12, 116, 152, 112, 137, 132, 165, 119, 125, 117
[54,3]	48	125, 112, 12(10), 11(10), 18, 157, 152, 165, 16, 12, 119, 175, 14(10), 15, 117, 19, 1(10), 116
[57,3]	50	13, 11(10), 112, 1, 123, 138, 116, 16, 143, 13(10), 18, 11, 158, 124, 132, 152, 149, 113, 14
[60,3]	53	113, 127, 1, 18, 124, 165, 17, 112, 11(10), 11, 12(10), 143, 137, 138, 149, 13(10), 128, 158, 13, 118
[63,3]	56	117, 1, 112, 134, 19, 115, 11, 18, 175, 16(10), 149, 113, 12, 125, 129, 165, 157, 124, 138, 14, 118
[66,3]	59	123, 114, 19, 18, 12, 16, 14, 17, 11(10), 132, 116, 147, 16(10), 134, 142, 11, 117, 13(10), 113, 157, 158, 118
[69,3]	61	157, 11, 112, 1, 13(10), 125, 175, 149, 18, 114, 117, 16, 13, 12(10), 135, 129, 16(10), 123, 116, 17, 118, 113, 19
[72,3]	64	142, 113, 15, 14(10), 1, 147, 138, 124, 12(10), 158, 12, 19, 16(10), 165, 157, 11, 117, 152, 127, 123, 118, 13, 112, 14

Table 2: Best QC Codes over $\mathbb{F}_{11}$ with $p = 4$

code	d	$r_i(x)$
[8,4]	5	1, 1168
[12,4]	8	14, 1147, 174
[16,4]	12	12, 1133, 11(10), 124(10)
[20,4]	16	1498, 1317, 1112, 1423, 154
[24,4]	19	188, 143, 1163, 1438, 1(10)6, 177
[28,4]	22	128(10), 134, 124, 1252, 158, 129, 11
[32,4]	26	12, 1835, 12(10)2, 123, 103, 186(10), 125, 1176
[36,4]	30	1116, 1273, 198, 116, 1146, 166, 153, 156(10), 175
[40,4]	33	114, 136, 145, 184, 1232, 1(10)5, 1457, 13(10), 1647, 11(10)4
[44,4]	37	11, 146, 111, 186, 1468, 15, 1273, 1142, 1835, 1718, 139
[48,4]	40	122, 1328, 113, 1298, 1318, 156(10), 133, 127(10), 1143, 1259, 1245, 1294
[52,4]	44	1354, 15, 1387, 1327, 12, 171, 163, 126(10), 128(10), 1148, 1145, 1152, 1279
[56,4]	47	116, 1114, 1263, 139, 111, 17(10), 166, 1(10)6, 1268, 132, 1427, 1176, 142, 141
[60,4]	51	1135, 157, 17(10), 155, 1317, 1492, 1747, 127(10), 14, 1157, 1264, 1186, 1133, 1149, 179
[64,4]	54	185, 16, 1114, 14, 193, 1155, 1254, 16(10), 1378, 125(10), 18, 1173, 1195, 1468, 1268, 129(10)
[68,4]	58	117, 11(10)4, 1249, 1265, 13, 1123, 162, 182, 11, 121(10), 1143, 1154, 11(10)8, 138, 1435, 1128, 186
[72,4]	61	14, 139, 143(10), 1159, 112, 113, 1(10)9, 1747, 157, 16(10), 1257, 125(10), 1119, 1492, 164(10), 1575, 156(10), 1164
[76,4]	65	1232, 129, 1525, 1169, 1(10), 139, 1149, 11, 1217, 11(10), 1747, 123(10), 1263, 12, 152, 1116, 1187, 1347, 1(10)3
[80,4]	68	199, 1275, 11(10)6, 1(10)1, 14, 18, 1(10)8, 1389, 1184, 1, 115(10), 1154, 1163, 152, 1313, 1123, 1126, 174(10), 175, 1469
[84,4]	72	1135, 1, 1174, 18(10), 1159, 1219, 1352, 111, 171(10), 1273, 1254, 1137, 1432, 19, 132, 198, 1118, 197, 142, 116, 12(10)4
[88,4]	75	18, 11(10)9, 13, 111(10), 1295, 103, 12(10)2, 1265, 1112, 1518, 1175, 125(10), 1384, 129, 1698, 1179, 137, 118(10), 135, 1147, 1532, 1(10)4
[92,4]	79	154, 114, 1517, 1475, 1137, 1215, 1565, 1479, 1253, 118(10), 11, 166, 1384, 1184, 184, 1167, 121, 1139, 1139, 1115, 1189, 1352, 1263
[96,4]	82	114, 1252, 1(10)8, 12, 141, 1253, 1518, 16(10), 174, 149, 1164, 138(10), 103, 1379, 1429, 129, 1435, 176, 1319, 125(10), 1479, 1342, 107, 11(10)6

Table 3: Best QC Codes over $\mathbb{F}_{11}$ with $p = 5$

code	d	$r_i(x)$
[10,5]	6	11, 1158
[15,5]	10	1096, 132(10)7, 12719
[20,5]	14	1093, 128(10)4, 18325, 11(10)1
[25,5]	18	1483, 1(10)6, 12(10), 11935, 168(10)
[30,5]	23	12, 11225, 12192, 1097, 1023, 11714
[35,5]	27	1084, 12179, 1032, 12782, 1445, 121(10)5, 1519
[40,5]	31	14(10), 14, 14715, 13568, 10(10)9, 11382, 114(10)7, 1182(10)
[45,5]	35	107, 127, 131(10)4, 17(10)4, 1547, 1699, 1696(10), 11858, 1996
[50,5]	40	1072, 12638, 1733, 12(10)62, 1867, 1258(10), 1894, 16, 14918, 1828
[55,5]	44	107, 1(10)67, 13457, 11527, 14253, 12(10)8(10), 14925, 15, 1194, 11414, 1934
[60,5]	48	13, 10(10), 11547, 1097, 11976, 111(10), 12687, 13792, 16(10)17, 11(10)95, 1074, 1266
[65,5]	53	197, 125, 11726, 13584, 15, 114(10)6, 10(10)1, 16(10)35, 11294, 11(10)89, 14768, 14329, 1581(10)
[70,5]	57	11865, 12413, 12416, 12494, 1584(10), 12734, 12154, 1(10)8, 1064, 12393, 1016, 115(10)7, 1423(10), 13134
[75,5]	61	125, 112, 1145(10), 1452(10), 104, 13768, 11(10)5, 1521, 1246, 11879, 192, 11967, 1454, 14638, 14768
[80,5]	66	17, 11343, 16565, 12345, 11878, 11(10)69, 11627, 12875, 11827, 13745, 11649, 10(10), 11579, 11784, 125, 16578
[85,5]	70	12(10)57, 1556, 15, 117(10)6, 1(10)6, 1015, 12764, 1035, 1138(10), 1472(10), 1358, 114(10)9, 199, 1718, 16425, 118(10)7, 12184
[90,5]	74	1233, 1963, 158, 1873, 1366, 163, 127(10)4, 11728, 1174, 11834, 12139, 1383, 13289, 14768, 1899, 1098, 1719, 1174(10)
[95,5]	79	165, 1912, 12476, 11917, 134, 1026, 14(10)(10), 1077, 1682, 1316, 11, 1485, 16565, 13759, 12376, 11837, 11117, 112(10)2, 11438
[100,5]	83	16(10), 11354, 1955, 1261(10), 1745, 136, 1(10)83, 1983, 12537, 12565, 1949, 128(10)3, 1129(10), 118(10)3, 11879, 12984, 11515, 11796, 1253, 11817
[105,5]	88	1045, 11, 15(10)6, 11217, 1494(10), 1651, 14(10)58, 12823, 1568, 1088, 1542, 1354, 191, 1289(10), 13253, 12(10)84, 1292, 1534, 1139, 16925, 11978
[110,5]	92	126, 12(10)47, 11979, 13798, 1681, 136, 16947, 11475, 11184, 11589, 13(10)45, 11674, 12352, 14(10)42, 14219, 11653, 13837, 11438, 1(10)(10)1, 123(10)2, 1441, 11488
[115,5]	96	104, 18, 14532, 12925, 1175, 1(10)76, 113(10)5, 1841, 1472, 1272, 11577, 1236, 1838, 11339, 1351, 1831, 116(10)6, 1055, 1196(10), 131(10)2, 15(10)7, 1111(10), 1186
[120,5]	101	19, 1328(10), 117(10)9, 17198, 12319, 1(10)47, 1494, 196, 12868, 14968, 11737, 13543, 134(10)3, 12, 14(10)2, 13547, 1916, 14798, 1674, 1179, 132(10)8,13529, 12(10)7(10), 128(10)3

Table 4: Best QC Codes over $\mathbb{F}_{11}$ with $p = 6$

code	d	$r_i(x)$
[12,6]	6	11, 1855(10)
[18,6]	11	152, 14895, 139525
[24,6]	16	108, 1323, 116639, 12647(10)
[30,6]	21	17, 1658(10), 17688, 112946, 1318
[36,6]	26	1915, 123, 112726, 112174, 19387, 12161
[42,6]	31	103, 1014, 10(10)77, 114(10)9, 11(10)568, 12145, 1159(10)2
[48,6]	36	14, 127(10)7(10), 11523, 143(10)63, 126827, 19(10)(10)2, 113516, 123514
[54,6]	41	104, 11, 11(10)152, 11626, 112383, 11432(10), 1(10)312, 19516, 1466
[60,6]	46	13, 165(10), 10305, 114696, 13973, 118762, 123187, 16913, 123652, 16313
[66,6]	52	1151, 147965, 1, 1116, 13465, 13889, 138634, 127(10)45, 125958, 1134, 11225(10)
[72,6]	57	15, 149218, 133, 146578, 15938, 147272, 111187, 131819, 19737, 135653, 1302(10), 13674
[78,6]	62	1045, 1111, 1(10)552, 115162, 114539, 10559, 12(10)86(10), 12317, 10125, 10647, 115437, 10(10)36, 12242
[84,6]	67	108, 117164, 113956, 116(10)82, 11(10)85, 12(10)187, 17526, 11(10)63, 151942, 1297(10), 119159, 10959, 11(10)297, 156(10)7
[90,6]	72	15, 13151, 123187, 134(10)38, 141, 19793, 1156(10)9, 18133, 15271(10), 123279, 1(10)237, 1289(10)2, 171(10)5, 115119, 17242
[96,6]	77	17, 111, 131(10)82, 14721, 19682, 1977, 12149, 16415, 111497, 1166(10)3, 137692, 118258, 112(10)94, 124534, 12(10)6(10)7, 14266
[102,6]	82	1022, 1, 13, 13223, 13928, 1(10)584, 114878, 151868, 151729, 121973, 121676, 12979, 1443, 118869, 112139, 1343(10)2, 117132
[108,6]	87	114, 1, 13, 17237, 112739, 117(10)36, 123953, 10358, 11415, 142(10)72, 16472, 1314(10), 11969(10), 1179, 1239(10)4, 19416, 125952, 1128(10)6
[114,6]	93	13(10)1, 19877, 1(10)8, 14353, 16848, 1778(10), 12245, 115786, 11863, 1(10)(10)(10)8, 1128(10)5, 147832, 14251, 149219, 1917, 114697, 195(10)7, 187(10)9, 111277
[120,6]	98	128, 1, 113715, 149657, 11(10)74, 154, 112274, 129847, 10542, 124138, 19179, 132983, 11357, 1468(10), 137159, 10372, 131672, 17922, 116392, 15671
[126,6]	103	15871, 1867(10), 11, 14, 112757, 17889, 1588, 108, 117324, 146468, 10653, 128398, 131(10)62, 12558, 115523, 17915, 18254, 1156(10)7, 112483, 11(10)4(10)2, 16368
[132,6]	108	11(10), 1258(10)3, 115975, 132342, 1112(10)2, 15838, 13(10)793, 111779, 146947, 1(10)199, 116757, 123834, 146942, 135234, 11146, 1337, 117153, 12124(10), 188(10)6, 12546, 111678, 126393
[138,6]	114	165, 1, 1494, 1(10)6(10)9, 11(10)657, 12(10)175, 15499, 1(10)896, 11955, 142837, 137582, 12(10)584, 1(10)593, 1(10)786, 17677, 10584, 14023, 123139, 127438, 111175, 1456, 13592(10), 13(10)41
[144,6]	119	1066, 1879, 119257, 1(10), 19133, 116918, 15483, 194(10), 145768, 124359, 1292(10)2, 112768, 11931, 1491(10), 162(10)5, 10(10)8, 19815, 10596, 12522, 131(10)42, 131534, 1768(10), 124514, 163818

Table 5: Best QC Codes over $\mathbb{F}_{11}$ with $p = 7$

code	d	$r_i(x)$
[14,7]	7	12, 105115
[21,7]	12	113, 140(10)(10), 17144
[28,7]	18	11113, 103509, 11(10)04, 108815
[35,7]	24	11(10), 112762, 16204, 16421, 1139(10)9
[42,7]	29	1123, 103837, 112722, 108624, 107893, 10561
[49,7]	35	1111, 111191, 19845, 18776, 18(10)76, 10(10)311, 104459
[56,7]	41	111111, 11124, 16902, 1266(10), 13451, 12969, 111292, 104582
[63,7]	47	111112, 111117, 113346, 114062, 14468, 103546, 13595, 19672, 10739
[70,7]	53	111223, 111126, 111111, 106798, 107822, 114532, 103331, 102398, 1(10)92, 107297
[77,7]	59	111122, 115728, 111112, 111114, 113867, 10(10)4(10)4, 12036, 105944, 1145(10)9, 108098, 1023(10)6
[84,7]	64	111111, 111112, 114758, 111121, 15058, 116026, 107305, 112135, 111253, 1029(10)8, 1506(10), 115437
[91,7]	70	111112, 111111, 14274, 112125, 12(10)36, 12184, 105133, 113935, 1(10)291, 1471(10), 104993, 115575, 18958
[98,7]	75	111112, 112726, 111111, 112213, 11952, 111223, 112595, 101454, 1083(10)1,11214, 101536, 16083, 10999(10), 104653
[105,7]	81	111111, 111212, 11326(10), 112123, 111466, 12124, 111162, 13035, 111296, 105(10)97, 111693, 18693, 104(10)13, 16502, 1(10)923
[112,7]	87	111111, 111113, 111116, 113594, 112(10)61, 107952, 1079(10)1, 113(10)61, 1144(10)8, 1766(10), 16196, 113134, 107(10)19, 1834(10), 18(10)9(10), 13509
[119,7]	94	111111, 111112, 112121, 106508, 112123, 103293, 14275, 111539, 127(10)6, 112138, 107585, 112414, 113522, 1717(10), 111524, 11328(10), 1(10)265
[126,7]	100	111114, 111111, 112121, 113524, 112074, 113499, 109117, 108898, 115169, 111533, 16856, 101687, 111237, 161(10)6, 109123, 114916, 101537, 114671
[133,7]	106	111139, 111111, 114745, 111112, 113(10)46, 101567, 114124, 111232, 101245, 106492, 11312(10), 111375, 16417, 11287(10), 115456, 101322, 104574, 11(10)22, 108691
[140,7]	111	111115, 111111, 112731, 112411, 114873, 111112, 111212, 112123, 12134, 102125, 103519, 102795, 19(10)43, 18(10)13, 11161(10), 13486, 14(10)91, 106486, 11392(10), 114174
[147,7]	117	111214, 114978, 111111, 111121, 1856(10), 112121, 105272, 115479, 16958, 1379(10), 113718, 107676, 18692, 114885, 111212, 12946, 113556, 111236, 15634, 13961, 11324(10)
[154,7]	122	112123, 111111, 113(10)46, 111111, 114124, 111112, 112121, 106492, 15(10)42, 112125, 16417, 11287(10), 115456, 101322, 104574, 11(10)22, 108691, 113214, 115(10)47, 118(10)3, 106(10)23, 12127
[161,7]	128	111113, 111111, 113274, 115174, 111111, 112121, 101212, 12816, 113(10)32,110174, 112312, 111489, 112314, 12438, 10636(10), 112316, 12(10)27, 113125, 14393, 18(10)81, 1132(10)7, 110813, 10(10)933
[168,7]	133	111112, 111111, 111111, 112113, 112132, 105877, 101154, 109576, 113(10)59, 17498, 105587, 113093, 16549, 107798, 107168, 16723, 11487(10), 112121, 112123, 112741, 112132, 1136(10)2, 102125, 111886

Table 6: Best QT Codes over $\mathbb{F}_{11}$ with $p = 4$ and $\alpha = -1$

code	d	$r_i(x)$
[12,4]	9	12, 194, 1264
[44,4]	37	1125, 198, 1291, 15, 1217, 132, 1127, 1786, 1529, 1282, 13(10)2
[96,4]	83	1571, 1219, 1194, 1215, 195, 1175, 166, 1181, 143, 1296, 191, 1159, 122, 1237, 155, 163, 16, 131, 1184, 1(10)1, 1469, 1118, 1359, 1(10)3

Table 7: Best QT Codes over $\mathbb{F}_{11}$ with $p = 6$ and $\alpha = -1$

code	d	$r_i(x)$
[12,6]	7	111,165(10)
[108,6]	88	11937, 137(10)9, 11839 ,11711, 11111, 11984, 125(10)2, 1388(10), 121(10)7, 1231(10), 11112, 13937, 1215, 1136(10), 114(10)3, 1267(10), 13986, 1486