The Basis Number of the Cartesian Product of Different Ladders

Maref Y. M. Alzoubi
Department of Mathematics
Yarmouk University
Irbid-Jordan
maref@yu.edu.jo

Abstract

The basis number of a graph G is defined to be the least positive integer d such that G has a $d-$fold basis for the cycle space of G.
In this paper we prove that the basis number of the Cartesian product of different ladders is exactly 4. However, if we apply Theorem 4.1 of Ali and Marougi [4], which is stated in the introduction down as Theorem 1.1, we find that the basis number of the circular and Möbius ladders with circular ladders and Möbius ladders is less than or equal to 5, and the basis number of ladders with circular ladders and circular ladders with circular ladders is at most 4.

1 INTRODUCTION

The graphs considered in this paper are simple and connected and for the undefined terms we recommend the reader to see [13] or [11]. Also, we believe that [12] is a readable and comprehensive reference. It is well known that any graph G is associated with a q-dimensional vector space over the finite field Z_2, say $(Z_2)^q$, where q is the order of the edge-set, $E(G)$, of the graph G. If $E(G) = \{e_1, e_2, \cdots, e_q\}$, then every subset $S \subset E(G)$ corresponds to a vector $v \in (Z_2)^q$ such that the i-th component is 1 if $e_i \in S$ and 0 if $e_i \notin S$. Since the edges are used to define the vector space, some authors call this space *the edge space.* The subspace of the edge space consisting of all the cycles and the edge-disjoint union of cycles is called *the cycle space of* G and it is denoted by $C(G)$. The cycle space has dimension given by $\dim C(G) = q - p + 1$, where p is the order of the vertex-set of G. A basis B *of* $C(G)$ in which every edge of G occurs in at most k cycles of B is called a *k-fold basis.* The minimum positive integer, $b(G)$, such that $C(G)$ has a $b(G)$*-fold* basis is called *the basis number* of G. In 1937 S. MacLane [19] proved that a graph G is planar if and only if $b(G) \leq 2$. After that, the subject of basis number was left aside until the end of 1979 when E. F. Schemeichel [21] found the bases numbers of the complete graphs K_n and the complete Bipartite graphs $K_{n,m}$. Also he proved the existence of graphs of arbitrary basis numbers.

Then, J. Banks and E. Schemeichel [10] proved a conjecture of E. Schemeichel that the basis number of the n-cube is 4. Since 1979, many papers appeared that focus on finding the basis number of special classes of graphs that obtained from different kinds of operations on graphs (like deletion or addition of a set of edges or vertices), or products on graphs (like the cartesian product, the strong product, the semi-strong product, the Lexicographic (or the composition) product, or the semi-composition product), see the references [1 – 9], [14 – 18], [20], and [21].

Definition 1.2. *The Cartesian product of the graphs* $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ *is the graph* $G = G_1 \times G_2$ *with vertex set* $V(G) = V_1 \times V_2$ *and edge set*

$$E(G) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1v_2 \in E_2 \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E_1\}.$$

The following is Theorem 4.1 of Ali and Marougi [4] in which he finds an upper bound of the basis number of the cartesian product of two disjoint connected graphs.

Theorem 1.1.[4] *If G and H are connected disjoint graphs, then*

$$\mathrm{b}(G \times H) \leq \max\{\mathrm{b}(G) + \triangle(T_H),\ \mathrm{b}(H) + \triangle(T_G)\}$$

where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\triangle(T_G)$ are minimum with respect to all spanning trees of H and G. Also, in [4] cited that they have proved the following result in [20].

Theorem.1.2.[20] $G \times H$ *is nonplanar if* G *and* H *are any graphs with* $\triangle(G) \geq 2$ *and* $\triangle(H) \geq 3$.

The main purpose of this paper is to prove that the basis number of the Cartesian product of different ladders is exactly 4. However, if we apply Theorem 1.1 we find that the basis number of the circular and Mobius ladders with circular ladders and Mobius ladders is less than or equal to 5, and the basis number of circular ladders with ladders and circular ladders is at most 4.

2 MAIN RESULTS

In this section, we investigate the basis number of the cartesian product of different kinds of ladders. We denote by L_n the graph of a ladder with the sets of vertices and edges given respictively by $V(L_n) = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$, $E(L_n) = \{a_ia_{i+1}, b_ib_{i+1} : 1 \leq i \leq n-1\} \cup \{a_ib_i : 1 \leq i \leq n\}$. Also, we denote by ML_m the Möbius ladder graph with the set of vertices $V(ML_m) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_m\}$, and the set of edges $E(ML_m) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq m-1\} \cup \{u_iv_i : 1 \leq i \leq m\} \cup \{u_mv_1, v_mu_1\}$. In the following proof, we are interested in simplifying our notations for the ladder L_n and the Möbius ladder ML_m by introducing the change of notations for the verticies as follows: $a_{n+i} = b_{n-i+1}$, and $u_{n+i} = v_{n-i+1}$; where $i = 1, 2, \cdots, n$. Following this notation we define the paths $P_{2n}^{(a)} = a_1a_2\cdots a_{2n}$, and $P_{2m}^{(u)} = u_1u_2\cdots u_{2m}$. It is clear that $P_{2n}^{(a)}$ is a subgraph of L_n and $P_{2m}^{(u)}$ is a subgraph of ML_m.

One can easily verify that the cartesian product of the ladder L_n and the Möbius ladder ML_m, denoted by $L_n \times ML_m$, has $4mn$ vertices and $12mn - 4m$ edges, thus the dimension of the cycle space of $L_n \times ML_m$ is $\dim C(L_n \times ML_m) = 8mn - 4m + 1$.

In the following theorem we prove that $\mathrm{b}(L_n \times ML_m) = 4$. However, applying Theorem 4.1 of Ali and Marougi [4], one can see that $\mathrm{b}(L_n \times ML_m) \leq 5$.

Theorem 2.1. *For every integers* $n, m \geq 3$, *we have* $\mathrm{b}(L_n \times ML_m) = 4$.

Proof. Let $n, m \geq 4$ be two integer numbers. To prove that $\mathrm{b}(L_n \times ML_m) \leq 4$, we have to find a *4-fold* basis for the cycle space $\mathcal{C}(L_n \times ML_m)$. The graph $P_{2n}^{(a)} \times P_{2m}^{(u)}$ is a planar subgraph of the graph $L_n \times ML_m$. The set $\mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$, that contains all the cycles obtained from the boundaries of the finite faces of the graph $P_{2n}^{(a)} \times P_{2m}^{(u)}$, is a *2-fold* basis for the cycle subspace $\mathcal{C}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$.

For each $u_i \in V(ML_m)$, the graph $L_n \times \{u_i\} = L_{n,i}$ is a copy of the ladder L_n, so it is planar subgraph of $L_n \times ML_m$ and the set of all the boundaries of the finite faces of $L_{n,i}$, say $\mathcal{B}_{n,i}$, is a basis of the cycle subspace $\mathcal{C}(L_{n,i})$. Thus, $\mathcal{B}^* = \bigcup_{i=1}^{2m} \mathcal{B}_{n,i}$ is a linearly independent set of cycles being each $\mathcal{B}_{n,i}$ is linearly independent and $\mathcal{B}_{n,i} \cap \mathcal{B}_{n,j} = \phi$ for all $i \neq j$; $1 \leq i, j \leq 2m$. Moreover, every cycle in $\mathcal{B}^*$ contains one or two edges of the form $(a_k, u_i)(b_k, u_i)$; $k = 1, 2, \cdots, n-1$ that make it linearly independent with all the cycles of $\mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$ because it cannot occur as a linear combination of cycles from $\mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$. Thus, every cycle in $\mathcal{B}^*$ is linearly independent with all the other cycles in $\mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$. Therefore, $\mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$ is a linearly idependent set of cycles in $\mathcal{C}(L_n \times ML_m)$. Note that $\langle \mathcal{B}^* \rangle \cap \left\langle \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right) \right\rangle = \langle 0_v \rangle$; where 0_v is the zero vector in $\mathcal{C}(L_n \times ML_m)$.

Now, for each $a_k \in V(L_n)$, the graph $\{a_k\} \times ML_m = ML_{m,k}$ is a copy of the nonlanar graph of ML_m that appears as a subgraph of $L_n \times ML_m$. For each $k = 1, 2, \cdots, 2n$, we consider the following basis for the cycle subspace $\mathcal{C}(ML_{m,k})$:

$$\begin{aligned} \mathcal{B}_{m,k} &= \{(a_k, u_i)(a_k, u_{i+1})(a_k, v_{i+1})(a_k, v_i)(a_k, u_i) : i = 1, 2, \cdots, m-1\} \\ &\cup \{(a_k, u_1)(a_k, v_n)(a_k, u_n)(a_k, v_1)(a_k, u_1)\} \\ &\cup \{(a_k, u_1)(a_k, v_n)(a_k, v_{n-1}) \cdots (a_k, v_1)(a_k, u_1)\}. \end{aligned}$$

It is clear that $\mathcal{B}_{m,k}$ is linearly independent for each $k = 1, 2, \cdots, 2n$. And since $\mathcal{B}_{m,j} \cap \mathcal{B}_{m,r} = \phi$ for each $j \neq r$ where $1 \leq j, r \leq 2n$, we conclude that $\mathcal{B}^{**} = \bigcup_{k=1}^{2n} \mathcal{B}_{m,k}$ is linearly independent. Moreover, every cycle in $\mathcal{B}^{**}$ contains one or two edges of the form $(a_k, u_1)(a_k, v_n)$, $(a_k, u_n)(a_k, v_1)$ or $(a_k, u_i)(a_k, v_i)$ where $k = 1, 2, \cdots, 2n$, and each of such edges cannot occur in any cycle of the set $\mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$, so any cycle in $\mathcal{B}^{**}$ cannot be obtained as a linear combination of cycles from $\mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$. Thus every cycle in $\mathcal{B}^{**}$ is linearly independent with $\mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$. Therefore, $\mathcal{B}^{**} \cup \mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$ is linearly indrependent. Now, define $\mathcal{B}(L_n \times ML_m) = \mathcal{B}^{**} \cup \mathcal{B}^* \cup \mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)$. Then,

$$
\begin{aligned}
|\mathcal{B}(L_n \times ML_m)| &= |\mathcal{B}^{**}| + |\mathcal{B}^{*}| + \left|\mathcal{B}\left(P_{2n}^{(a)} \times P_{2m}^{(u)}\right)\right| \\
&= (2n-1)(2m-1) + 2m(n-1) + 2n(m+1) \\
&= 4nm - 2n - 2m + 1 + 2nm - 2m + 2nm + 2n \\
&= 8nm - 4m + 1 = \dim \mathcal{C}(L_n \times ML_m).
\end{aligned}
$$

Hence, $\mathcal{B}(L_n \times ML_m)$ is a basis for $\mathcal{C}(L_n \times ML_m)$ being it is linearly independent and $|\mathcal{B}(L_n \times ML_m)| = \dim \mathcal{C}(L_n \times ML_m)$.

We define now tha following sets of edges:

$$E_1 = \{(a_1, u_i)(a_{2n}, u_i) : 1 \le i \le 2m\} \cup \{(a_j, u_m)(a_j, u_{2m}) : 1 \le j \le 2n\}$$

$$E_4 = \{(a_i, u_j)(a_i, u_{j+1}) : 2 \le i \le 2n-1,\ m \le j \le 2m-1\}$$

$$
\begin{aligned}
E_3 = \ & \{(a_1, u_j)(a_1, u_{j+1}) : m \le j \le 2m-1\} \cup \{(a_1, u_{2m})(a_1, u_1)\} \cup \\
& \{(a_{2n}, u_j)(a_1, u_{j+1}) : m \le j \le 2m-1\} \cup \{(a_{2n}, u_{2m})(a_{2n}, u_1)\} \cup \\
& \{(a_i, u_j)(a_i, u_{j+1}) : 2 \le i \le 2n-1,\ 1 \le j \le m-1\} \cup \\
& \{(a_i, u_j)(a_{i+1}, u_j) : 1 \le i \le 2n-1,\ 2 \le j \le 2m-1\}
\end{aligned}
$$

$$E_2 = E(L_n \times ML_m) \backslash (E_1 \cup E_3 \cup E_4)$$

From counting the fold of every edge $e \in L_n \times ML_m$, we notice that $f_{\mathcal{B}(L_n \times ML_m)}(e) = i$, if $e \in E_i$. Thus, $\mathcal{B}(L_n \times ML_m)$ is a *4-fold* basis for $\mathcal{C}(L_n \times ML_m)$.

On the other hand, since $L_n \times ML_m$ is nonplanar, then by MacLane's Theorem we have $b(L_n \times ML_m) \ge 3$. So, to prove that $b(L_n \times ML_m) \ge 4$, we eleminate any possibility for $\mathcal{C}(L_n \times ML_m)$ to have a *3-fold* basis. Suppose that $\mathcal{B}$ is a *3-fold* basis of $\mathcal{C}(L_n \times ML_m)$ and note that $L_n \times ML_m$ is a graph of girth 4. Since the number of 4-cycles in $L_n \times ML_m$ is $8nm - 4m - 2n + 1$, which is less than $|\mathcal{B}|$, then $\mathcal{B}$ cannot contains only 4-cycles. Also, $\mathcal{B}$ cannot contain only cycles of length greater than or equal to 5 because if $\mathcal{B}$ contains only cycles of length greater than or equal to 5 then we have

$$5 \dim \mathcal{C}(L_n \times ML_m) \le 3\,|E(L_n \times ML_m)|$$

and this implies that $4m(n-2) + 5 \le 0$, which cannot hold for all $n \ge 3$ and $m \ge 1$. This shows that $\mathcal{B}$ must contain a mixture of cycles of length 4 and cycles of length greater than 4. Since $\mathcal{B}$ is a *3-fold* basis, its cycles cannot contain any edge of $L_n \times ML_m$ more than three times. Thus, $\mathcal{B}$ must contain as much as possible of linearly independent 4-cycles. In fact, all the 4-cycles are linearly independent and we have used them in building $\mathcal{B}(L_n \times ML_m)$ above. Without loss of generality, we assume that $\mathcal{B}$ contains all the 4-cycles. Then every edge in the set of edges E^* has fold 3, where E^* is defined as follows:

$$
\begin{aligned}
E^* = \ & \{(a_i, u_j)(a_i, u_{j+1}) : 2 \le i \le 2n-1, 1 \le j \le 2m-1\} \\
& \cup \{(a_i, u_j)(a_{i+1}, u_j) : 1 \le i \le 2n-1,\ 2 \le j \le 2m-1\}.
\end{aligned}
$$

But, to add any cycle to the set of 4-cycles we must use edges from E^* which implies to a contradiction being $\mathcal{B}$ is a *3-fold* basis. Hence, $\mathcal{B}$ cannot exist. Therefore, b($L_n \times ML_m$) ≥ 4. This completes the proof.

We consider the graph of CL_n as a graph obtained from the graph of L_n by adding the set of edges $\{a_na_1, b_nb_1\}$ and L_n will be considered as it is defined above. It is easy to verify that $|V(CL_n \times ML_m)| = 4mn$, $|E(CL_n \times ML_m)| = 12mn$ and dim$\mathcal{C}(CL_n \times ML_m) = 8mn + 1$. Also, we notice that the graph of $CL_n \times ML_m$ is obtained from the grapg of $L_n \times ML_m$ by adding the set of edges

$$E^{**} = \{(a_n, u_j)(a_1, u_j), (b_n, u_j)(b_1, u_j) : 1 \leq j \leq 2m\}.$$

Applying Theorem 4.1 of Ali and Marougi [4], we get b($CL_n \times ML_m$) ≤ 5. However, we prove in the following theorem that b($CL_n \times ML_m$) $= 4$.

Theorem 2.2. *For every ineger* $n \geq 3$ and $m \geq 3$, *we have* b($CL_n \times ML_m$) $= 4$.

Proof. To prove that b($CL_n \times ML_m$) ≤ 4 we exhibit a *4-fold* basis for $\mathcal{C}(CL_n \times ML_m)$. Define $\mathcal{B}(CL_n \times ML_m) = \mathcal{B}(L_n \times ML_m) \cup \mathcal{B}_{a,b}$ where $\mathcal{B}(L_n \times ML_m)$ is the *4-fold* basis of the subspace $\mathcal{C}(L_n \times ML_m)$ that constructed in Theorem2.1 and $\mathcal{B}_{a,b} = \mathcal{B}_a \cup \mathcal{B}_b$ where

$$\mathcal{B}_a = \{(a_1, u_j)(a_2, u_j) \cdots (a_n, u_j)(a_1, u_j) : 1 \leq j \leq 2m\}$$

$$\mathcal{B}_b = \{(b_1, u_j)(b_2, u_j) \cdots (b_n, u_j)(b_1, u_j) : 1 \leq j \leq 2m\}.$$

Then every cycle in $\mathcal{B}_{a,b}$ contains an edge either of the form $(a_n, u_j)(a_1, u_j)$ or of thre form $(b_n, u_j)(b_1, u_j)$ where $1 \leq j \leq 2m$ and this edge doesnot occur in any other cycle of $\mathcal{B}(CL_n \times ML_m)$. Thus, every cycle in $\mathcal{B}_{a,b}$ is linearly independent with all the other cycles in $\mathcal{B}(CL_n \times ML_m)$. Hence, $\mathcal{B}(CL_n \times ML_m)$ is linearly independent. Furthermore, $\mathcal{B}(CL_n \times ML_m)$ is a basis of $\mathcal{C}(CL_n \times ML_m)$ being $|\mathcal{B}(CL_n \times ML_m)| =$ dim$\mathcal{C}(CL_n \times ML_m)$. The fold of every edge from E^{**} is one. If $e \in E^{***}$, then its fold in $\mathcal{B}(L_n \times ML_m)$ is at most 3 and its fold in $\mathcal{B}_{a,b}$ is 1, then its fold in $\mathcal{B}(CL_n \times ML_m)$ is at most 4. The fold of any other edge of $CL_n \times ML_m$ in $\mathcal{B}(CL_n \times ML_m)$ is the same as it is in $\mathcal{B}(L_n \times ML_m)$. Thus $\mathcal{B}(CL_n \times ML_m)$ is a *4-fold* basis for $\mathcal{C}(CL_n \times ML_m)$.

On the other hand, we want to prove that $\mathcal{C}(CL_n \times ML_m)$ cannot have a *3-fold* basis. Suppose the contrary, that is $\mathcal{B}$ is a *3-fold* basis for $\mathcal{C}(CL_n \times ML_m)$. If $n = 3$, the girth of $CL_n \times ML_m$ is 3 and number of 3-cycles is $4m$, and so the number of cycles of length 3 or 4 in $CL_n \times ML_m$ is $8mn - 2n + 1$. If $n \geq 4$, the girth of $CL_n \times ML_m$ is 4 and the number of 4-cycles is $8mn - 2n + 1$. In both cases number of cycles of length less than or equal to 4 is not enough to form a basis for $\mathcal{C}(CL_n \times ML_m)$ because $|\mathcal{B}| = 8mn + 1$. Thus $\mathcal{B}$ cannot contain only cycles of length less than or equal to 4. Also, $\mathcal{B}$ cannot consists only of cycles of length greater than or equal to 5, if so then we have $5(8mn + 1) \leq 3(12mn)$, or $4mn + 5 \leq 0$, which is impossible for all $n, m \geq 3$. Since $\mathcal{B}$ is a *3-fold* basis it is not allowed to any edge of $CL_n \times ML_m$ to appear in more than 3 cycles of $\mathcal{B}$. And so, $\mathcal{B}$ must contain as much as possible of independent cycles of minimum length. Since all the cycles of length less than or equal to 4 are linearly independent, being we used them in building $\mathcal{B}(CL_n \times ML_m)$ that constructed above, $\mathcal{B}$ must contain $2n$ more cycles. Now, the fold of every edge in E^* is 3 and choosing new $2n$ cycles forces us to pass throuhg these edges which will icrease the fold of them and this ensures that such *3-fold* basis $\mathcal{B}$ cannot exist.

This completes the proof.

For the following result, we consider the graph ML_n as a graph obtained from the graph CL_n by deleting the set of edges $\{a_n a_1, b_n b_1\}$ and replacing it by the set of edges $\{a_n b_1, b_n a_1\}$. Thus, $ML_n \times ML_m$ is obtained from $CL_n \times ML_m$ by deleting the set of edges E^{**} and repacing it by the following set

$$E^{***} = \{(a_n, u_j)(a_1, u_j), (b_n, u_j)(b_1, u_j) : 1 \leq j \leq 2m\}.$$

Clearly, $|V(CL_n \times ML_m)| = 4mn$, $|E(CL_n \times ML_m)| = 12mn$ and dim$C(CL_n \times ML_m) = 8mn + 1$.

One can easily verify that applying Theorem 4.1 of Ali and Marougi [4], implie b($ML_n \times ML_m$) ≤ 5. However, we prove in the following theorem that b($CL_n \times ML_m$) $= 4$.

Theorem 2.3. *For every ineger* $n \geq 3$ and $m \geq 3$, *we have* b($ML_n \times ML_m$) $= 4$.

Proof. To prove b($ML_n \times ML_m$) ≤ 4, we define $\mathcal{B}(ML_n \times ML_m) = \mathcal{B}(L_n \times ML_m) \cup \mathcal{B}^{a,b}$ where $\mathcal{B}(L_n \times ML_m)$ is the same *4-fold* basis obtained in Theorem2.1 and $\mathcal{B}^{a,b} = \mathcal{B}^a \cup \mathcal{B}^b$ where $\mathcal{B}^a$ and $\mathcal{B}^b$ are sets of cycles defined as follows:

$$\mathcal{B}^a = \{(a_1, u_j)(b_n, u_j)(b_{n-1}, u_j)\cdots(b_1, u_j)(a_1, u_j) : 1 \leq j \leq 2m\},$$

$$\mathcal{B}^b = \{(b_1, u_j)(a_n, u_j)(a_{n-1}, u_j)\cdots(a_1, u_j)(b_1, u_j) : 1 \leq j \leq 2m\}.$$

Since $\mathcal{B}(L_n \times ML_m)$ is linearly independent and every cycle in $\mathcal{B}^{a,b}$ contains an edge from E^{***} that doesnot occur in any other cycle of $\mathcal{B}(ML_n \times ML_m)$ then each of these cycles is linearly independent with all the cycles in $\mathcal{B}(ML_n \times ML_m)$, thus $\mathcal{B}(ML_n \times ML_m)$ is linearly independent. Moreover, $|\mathcal{B}(ML_n \times ML_m)|$ =dim$C(ML_n \times ML_m)$. Hence, $\mathcal{B}(ML_n \times ML_m)$ is a basis of $C(ML_n \times ML_m)$. It is an easy matter to verify that $\mathcal{B}(ML_n \times ML_m)$ is a *4-fold* basis for $C(ML_n \times ML_m)$.

On the other hand, to prove that b($ML_n \times ML_m$) ≥ 4, we can use almost the same arguments to those used in the proof of Theorem 2.2 to guarantee that the cycle space $C(ML_n \times ML_m)$ cannot have a *3-fold* basis. This completes the proof.

Following Theorem 4.1 of Ali and Marougi [4], we can verify that the basis number of the graphs $L_n \times CL_m$ and $CL_n \times CL_m$ is less than or equal to 4. Also, we can use similar techniques to those used in the proof of Theorems 2.1 and 2.2 to prove that the basis number of these graphs is greater than or equal to 4. Thus, we have the following two results.

Theorem 2.4. *For every integers* $n \geq 3$ and $m \geq 3$, *we have* b($L_n \times CL_m$) $= 4$.

Theorem 2.5. *For every integers* $n \geq 3$ and $m \geq 3$, *we have* b($CL_n \times CL_m$) $= 4$.

[1] A. A. Ali, *The basis number of the direct product of paths and cycles*, Ars Combin. 27 (1989), 155-163.

[2] A. A. Ali, *The basis number of complete multipartite graphs*, Ars Combin. 28, 41-49 (1989).

[3] A. A. Ali and G.T. Marougi, *The basis number of the strong product of graphs*, Mu'tah Lil-Buhooth Wa Al-Dirasat 7, no.1, 211-222 (1992).

[4] A. A. Ali and G. T. Marougi. The basis number of the Cartesian product of some graphs. The J. of Indian Math. Soc. 58(1992), no. 1-4: 123-134.

[5] A. A. Ali and G. T. Marougi. The basis number of the Cartesian product of some

special graphs. Mu'tah Lil-Buhooth Wa Al-Dirasat (Series B: Natural and Applied Sciences and Physics). 8(2)(1993): 83-100.

[6] A. S. Alsardary and J. Wojciechowski. *The basis number of the powers of the complete graph*, Discrete Math. 188, no. 1-3, 13-25 (1998).

[7] M.Y. Alzoubi and M.M.M. Jaradat. *The basis number of the composition of theta graph with stars and wheels*, Acta Math. Hungar. 103, no. 3, 201-209 (2004).

[8] M.Y. Alzoubi and M.M.M. Jaradat, On the basis number of composition of different ladders with some graphs. International Journal of Mathematics and Mathematical Sciences 2005: 12 (2005) 1861-1868.

[9] M.Y. Alzoubi and M.M.M. Jaradat, The basis number of the composition of theta graphs with some graphs. To appear in "Ars. Combinatoria".

[10] J. A. Banks and Schmeichel. The basis number of the N-cube, J. Comb. Theory, Ser. B, 33(1982): 95-100.

[11] J. A. Bondy and U. S. Murty, Graph Theory with Applications, American Elsiever, New York, 1976.

[12] J. Gross and J. Yellen, Graph Theory and Its Applications, CRC Press Newyork, 1999.

[13] F. Harary, Graph Theory, 3rd. printing, Addison Wisely, Reading, Massachusetts, 1972.

[14] M.Q. Hailat and M.Y. Alzoubi, *The basis number of the composition of graphs*, Istanbul Univ. Fen Fak. Mat. Der., 53 (1994), 43-60.

[15] M. M. M. Jaradat, *The basis number of the direct product of a theta graph and a path*, Ars combin. 75 (2005) 105-111.

[16] M. M. M. Jaradat and M.Y. Alzoubi, *On the basis number of the semi-strong product of bipartite graphs with cycles*, Kyungpook Math. J. 45, no. 1, 45-53 (2005).

[17] M. M. M. Jaradat and M.Y. Alzoubi, *An upper bound of the basis number of the lexicographic product of graphs*, Australas. J. Combin. 32, 305-312.

[18] M. M. M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, *The basis number of the Lexicographic product of different ladders*, SUT Journal of Mathematics, 40, 91-1001 (2005)

[19] S. MacLane. Acombinatorial condition for planar graphs. Fund. Math., 28 (1937): 22-32.

[20] G. T. Marougi, The Basis Number of the Cartesian product of Graphs, M.Sc. Thesis, Mosul University,1988.

[21] E. F. Schmeichel. The basis number of a graph. J. Comb. Theory, Ser. B, 30(1981): 123-129.