Further results on super edge magic deficiency of unicyclic graphs*

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Abstract

A graph G is called *edge-magic* if there exists a bijective function $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $\phi(x) + \phi(xy) + \phi(y) = c(\phi)$ is a constant for every edge $xy \in E(G)$, called the valance of ϕ. A graph G is said to be *super edge-magic* if $\phi(V(G)) = \{1, 2, \ldots, |V(G)|\}$. The *super edge-magic deficiency*, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$, has a super edge-magic labelings, if such integer does not exist we define $\mu_s(G)$ to be $+\infty$. In this paper, we study the super edge-magic deficiency of some families of unicyclic graphs.

1 Introduction

In this paper, we consider only finite, simple and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An edge-magic labeling of a graph G is a bijection $\phi : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$, where there exists a constant $c(\phi)$ such that $f(x) + f(xy) + f(y) = c(\phi)$, for every edge $xy \in E(G)$, $c(\phi)$ is called valance of ϕ and a graph with an edge-magic labeling is called *edge-magic.* An edge-magic labeling ϕ is called super edge-magic if $\phi(V(G)) = \{1, 2, ..., p\}$.

In [5], Kotzig and Rosa proved that for any graph G there exists an edge magic-graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of edge-magic deficiency of a graph G, which

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is the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic and it is denoted by $\mu(G)$. In particular,

$$\mu(G) = min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge-magic deficiency of a graph G with n vertices, $\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$, where F_n is the nth Fibonacci number. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno *et al.* [3] defined a similar concept for super edge-magic labelings. The super edge-magic deficiency of a graph G, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic labeling or $+\infty$ if there exists no such n.

Let $M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\}$, *then*

$$\mu_s(G) = \begin{cases} min\ M(G), & \text{if } M(G) \neq \phi; \\ +\infty, & \text{if } M(G) = \phi. \end{cases}$$

As a consequence of the above two definitions, we have that for every graph G, $\mu(G) \leq \mu_s(G)$.

In [3, 4], Figueroa-Centeno *et al.* provided the exact values of super edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. They proved that

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \equiv 0 \pmod 4 \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In [7], Ngurah *et al.* proved some upper bounds for the super edge-magic deficiency of fans, double fans, and wheels. In this paper, we study the super edge-magic deficiency of some families of unicyclic graphs.

In proving the main results, we frequently use the lemma below.

Lemma 1. *[2] A graph G with p vertices and q edges is super edge-magic if and only if there exists a bijective function $\phi : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{\phi(x) + \phi(y) | xy \in E(G)\}$ consists of q consecutive integers. In such a case, ϕ extends to a super edge-magic labeling of G.*

Proposition 1. *[1] Let $G = (n, t)-$kite. If G is super edge-magic then n and t have the same parity.*

In [10] Wallis posed the problem of investigating the edge-magic properties of C_n with the path of length t attached to one vertex. Kim and Park [6] call such a graph an $(n, t)-$kite. They prove that an $(n, 1)-$kite is super edge-magic if and only if n is odd and an $(n,3)-$kite is super edge-magic if and only if n is odd and at least 5. Also from the Proposition 1, $(n,1)-$kite is not super edge-magic if n is even, so in the next theorem, we show the exact value of super edge-magic deficiency of $(n,1)-$kite graph.

Theorem 1. *Let $G=(n,1)-$kite. For n even, $\mu_s(G)=1$.*

Proof. Let $G^*=(n,1)-\text{kite}\cup K_1$, the vertex set of G^* is

$$V(G^*)=\{v_i : 1\le i\le n\}\cup\{u,z\}$$

and edge set of G^* is

$$E(G^*)=\{v_iv_{i+1} : 1\le i\le n-1\}\cup\{v_nv_1, v_wu\}$$

$$\text{where } w=\begin{cases}\frac{n-2}{2}, & \text{if } n\equiv 0 \pmod 4\\ \frac{n-4}{2}, & \text{if } n\equiv 2 \pmod 4\end{cases}$$

By Proposition 1 and in [6] it was showed that an $(n,1)-$kite is super edge-magic if and only if n is odd. As $(n,1)-$kite is not super edge-magic for n even. Therefore

$$\mu_s(G)\ge 1. \tag{1}$$

To prove $\mu_s(G)\le 1$, we define the labeling $\phi : V(G^*)\to\{1,2,\ldots,n+1\}$ of the graph G^* as follows:

$$\phi(v_i)=\begin{cases}\frac{i+1}{2}, & \text{if } 1\le i\le n \text{ and } i \text{ odd}\\ \frac{n+2+i}{2}, & \text{if } 1\le i\le w \text{ and } i \text{ even}\\ \frac{n+6+i}{2}, & \text{if } w+1\le i\le n-1 \text{ and } i \text{ even}\\ \frac{n+2}{2} & \text{if } i=n\end{cases}$$

$$\phi(u)=\begin{cases}3(\frac{n}{4}-1)+5, & \text{if } n\equiv 0 \pmod 4\\ 3(\frac{n-2}{4}-1)+5, & \text{if } n\equiv 2 \pmod 4\end{cases}$$

The isolated vertex, z is labeled such that

$$\phi(z)=\begin{cases}3(\frac{n}{4}-1)+4, & \text{if } n\equiv 0 \pmod 4\\ 3(\frac{n-2}{4}-1)+6, & \text{if } n\equiv 2 \pmod 4\end{cases}$$

(See Figure 1 for an illustration.) The set of all edge-sums generated by the above formula forms a consecutive integer sequence $\frac{n+4}{2}, \frac{n+4}{2}+1, \ldots, \frac{3n+4}{2}$. Therefore by Lemma 1, ϕ can be extended to a super edge-magic labeling. This shows that

$$\mu_s(G)\le 1. \tag{2}$$

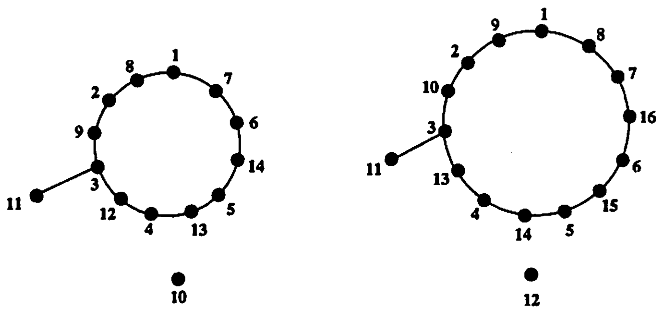

Figure 1: An illustration for the labeling given in the proof of Theorem 1

From equation (1) and (2), we get

$$\mu_s(G) = 1.$$

Which completes the proof.

□

In the next theorem we show the super edge-magic deficiency of $(n, 3)-$kite is exactly 1, for $n \equiv 0 \pmod 2$.

Theorem 2. *For n even, the super edge-magic deficiency of the $(n, 3)-$kite graph is*

$$\mu_s((n, 3) - kite) = 1.$$

Proof. Let $G = (n, 3)-$kite graph, where

$$V(G) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq 3\}$$

and

$$E(G) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} | 1 \leq i \leq 2\} \cup \{v_n v_1, u_3 v_1\}.$$

By the Proposition 1, G is not super edge-magic for n even. Therefore

$$\mu_s(G) \geq 1. \tag{3}$$

To prove $\mu_s(G) \leq 1$ for n even, we label the vertices of $G \cup z$, where z is isolated vertex in the following way.

$$\phi(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i = 1, 3 \\ \frac{n}{2} + 3, & \text{if } i = 2 \end{cases}$$

Case I For $n \equiv 0 \pmod 4$

$$\phi(v_i) = \begin{cases} \frac{i}{2}+3, & \text{if } 1 \le i \le n-1 \text{ and } i \equiv 0 \pmod 2 \\ \frac{n+7+i}{2}, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+9+i}{2}, & \text{if } \frac{n}{2}+1 \le i \le n-1 \text{ and } i \equiv 1 \pmod 2 \\ 3, & \text{if } i = n \end{cases}$$

$\phi(z) = \frac{3n+16}{4}$.

Case II For $n \equiv 2 \pmod 4$

$$\phi(v_i) = \begin{cases} \frac{n+i+9}{2}, & \text{if } 1 \le i \le \frac{n}{2}-1 \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+11+i}{2}, & \text{if } \frac{n}{2} \le i \le \frac{n}{2}+1 \text{ and } i \equiv 1 \pmod 2 \\ \frac{i+5}{2}, & \text{if } \frac{n}{2}+2 \le i \le n-1 \text{ and } i \equiv 1 \pmod 2 \\ \frac{i+4}{2}, & \text{if } 1 \le i \le \frac{n}{2}+1 \text{ and } i \equiv 0 \pmod 2 \\ \frac{n+10+i}{2}, & \text{if } \frac{n}{2}+2 \le i \le n-1 \text{ and } i \equiv 0 \pmod 2 \\ \frac{n+8}{2}, & \text{if } i = n \end{cases}$$

$\phi(z) = \frac{3n+18}{4}$.

The set of all edge-sums generated by the above formula forms a set of $n+1$ consecutive integers. Therefore by Lemma 1, ϕ can be extended to a super edge-magic total labeling. This shows that

$$\mu_s(G) \le 1 \tag{4}$$

from equations (3) and (4), we get

$$\mu_s(G) = 1.$$

Which completes the proof.

□

Theorem 3. *For $t > 3$ odd and $n \equiv 0 \pmod 4$, the super edge-magic deficiency of $(n,t)-$kite graph is*

$$\mu_s((n,t) - kite) = 1.$$

Proof. Let $G = (n,t)-$kite graph, the vertex set of G is

$$\{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le t\}$$

and the edge set of G is

$$\{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{u_i u_{i+1} | 1 \le i \le t-1\} \cup \{v_n v_1, u_t v_1\}$$

By the Proposition 1, $G = (n,t)-$kite graph is not super edge-magic for n even and t odd. Therefore

$$\mu_s(G) \geq 1. \tag{5}$$

To prove $\mu_s(G) \leq 1$, we label the vertices of $(n,t)-$kite in the following way.

$$\phi(u_i) = \begin{cases} \lceil \frac{i+1}{2} \rceil, & \text{if } 1 \leq i \leq t \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+t+1+i}{2}, & \text{if } 1 \leq i \leq t \text{ and } i \equiv 0 \pmod 2 \end{cases}$$

$$\phi(v_i) = \begin{cases} \frac{t+1+i}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \equiv 0 \pmod 2 \\ \frac{n+2t+1+i}{2}, & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+2t+3+i}{2}, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \text{ and } i \equiv 1 \pmod 2 \end{cases}$$

The isolated vertex z is labeled as $\phi(z) = \frac{3n+4t+4}{4}$.

The set of all edge-sums generated by the above formula forms a set of $n+1$ consecutive integers. Therefore by Lemma 1, ϕ can be extended to a super edge-magic total labeling. This shows that

$$\mu_s(G) \leq 1 \tag{6}$$

from equations (5) and (6), we get

$$\mu_s(G) = 1.$$

Which completes the proof.

□

Theorem 4. *For $t > 3$ even and $n \equiv 0 \pmod 4$, the upper bound of super edge-magic deficiency of $(n,t)-$kite graph is*

$$\mu_s((n,t) - kite) \leq 1$$

Proof. Let $G = (n,t)-$kite graph, the vertex set of G is

$$\{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq t\}$$

and the edge set of G is

$$\{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} | 1 \leq i \leq t-1\} \cup \{v_n v_1, u_t v_1\}$$

Let $n \equiv 0 \pmod 4$ be a nonnegative integer. According to Lemma 1 it is sufficient to prove that there exists a vertex labeling with the property that the edge-sums under this labeling are consecutive integers. It is easy to see that the following labeling $\phi : V(G \cup K_1) \to \{1, 2, \ldots, n+1\}$ has the

desired property, for $n \equiv 0 \pmod 4$.
Here, we label $G \cup K_1$ where $V(K_1) = \{z\}$.

$$\phi(u_i) = \begin{cases} \lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le t \text{ and } i \equiv 0 \pmod 2 \\ \frac{n+t+1+i}{2}, & \text{if } 1 \le i \le t \text{ and } i \equiv 1 \pmod 2 \end{cases}$$

$$\phi(v_i) = \begin{cases} \frac{t+i}{2}, & \text{if } 1 \le i \le n \text{ and } i \equiv 0 \pmod 2 \\ \frac{n+2t+1+i}{2}, & \text{if } 1 \le i \le \lceil \frac{n}{2} \rceil \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+2t+3+i}{2}, & \text{if } \lceil \frac{n}{2} \rceil + 1 \le i \le n \text{ and } i \equiv 1 \pmod 2 \end{cases}$$

The isolated vertex z under the labeling ϕ is labeled as $\phi(z) = \frac{3n+4t+4}{4}$.

It is easy to see that the edge-sums forms a set of $n+1$ consecutive integers. This shows that

$$\mu_s(G) \le 1 \tag{7}$$

Which completes the proof.

□

Wallis [10] also posed the problem of determining when $K_2 \cup C_n$ is super edge-magic. Park et al. [8] and [6] showed that $K_2 \cup C_n$ is super edge-magic if and only if n is even.

In the next theorem we show the super edge-magic deficiency of $K_2 \cup C_n$ is at most 1 for $n \equiv 1 \pmod 4$.

Theorem 5. *For $n \equiv 1 \pmod 4$, the super edge-magic deficiency of $K_2 \cup C_n$ is*

$$\mu_s(K_2 \cup C_n) = 1.$$

Proof. Let $G = K_2 \cup C_n$ The vertex set of G is

$$\{v_i | 1 \le i \le n\} \cup \{u, w\}$$

and the edge set of G is

$$\{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{v_n v_1, uw\}$$

Park et al. [8] and [6] proved that $K_2 \cup C_n$ is super edge-magic if and only if n is even. So, $\mu_s(G) \ge 1$ for n odd. To prove $\mu_s(G) \le 1$ for $n \equiv 1 \pmod 4$, we label the vertices of $K_2 \cup C_n \cup z$, where z is isolated vertex in the following way.

$\phi(u) = 1, \phi(w) = n + 3$

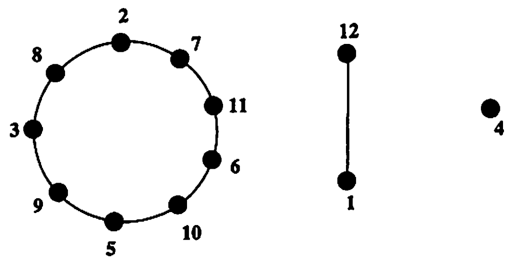

Figure 2: An illustration for the labeling given in the proof of Theorem 5

$$\phi(v_i) = \begin{cases} \frac{i+3}{2}, & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \text{ and } i \equiv 1 \pmod 2 \\ \frac{i+5}{2}, & \text{if } \lfloor \frac{n}{2} \rfloor \le i \le n \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+5+i}{2}, & \text{if } 1 \le i \le n \text{ and } i \equiv 0 \pmod 2 \end{cases}$$

$\phi(z) = \lfloor \frac{n}{4} \rfloor + 2.$

(See Figure 2 for illustration.) The set of all edge-sums generated by the above formula forms a set of $n+1$ consecutive integers. Therefore by Lemma 1, ϕ can be extended to a super edge-magic total labeling. This shows that $\mu_s(G) \le 1$. Therefore we get $\mu_s(G) = 1$. Which completes the proof.

□

In the next theorem we show that $1 \le \mu_s(K_2 \cup C_n) \le 2$ when $n \equiv 3 \pmod 4$.

Theorem 6. *For $n \equiv 3 \pmod 4$, the super edge-magic deficiency of $K_2 \cup C_n$ is*

$$1 \le \mu_s(K_2 \cup C_n) \le 2.$$

Proof. Let $n \equiv 3 \pmod 4$ be a nonnegative integer. Let $G = K_2 \cup C_n$ The vertex set of G is

$$\{v_i | 1 \le i \le n\} \cup \{u, w\}$$

and the edge set of G is

$$\{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{v_n v_1, uw\}$$

Park et al. [8] and [6] prove that $K_2 \cup C_n$ is super edge-magic if and only if n is even. So, for n odd

$$\mu_s(G) \ge 1. \tag{8}$$

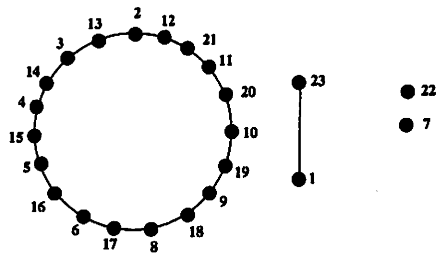

Figure 3: An illustration for the labeling given in the proof of Theorem 6

To prove $\mu_s(G) \leq 2$ for $n \equiv 3 \pmod 4$, according to Lemma 1 it is sufficient to prove that there exists a vertex labeling with the property that the edge-sums under this labeling are consecutive integers. It is easy to see that the following labeling $\phi : V(G \cup 2K_1) \rightarrow \{1, 2, \ldots, |V(G)| + 2\}$ has the desired property.
Here, we label $G \cup 2K_1$ where $V(2K_1) = \{z_1, z_2\}$ as follows:

$\phi(u) = 1, \phi(w) = n + 4$

$$\phi(v_i) = \begin{cases} \lceil \frac{i+1}{2} \rceil + 1, & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil \text{ and } i \equiv 1 \pmod 2 \\ \lceil \frac{i+1}{2} \rceil + 2, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \text{ and } i \equiv 1 \pmod 2 \\ \frac{n+5+i}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \equiv 0 \pmod 2 \end{cases}$$

The isolated vertex z_i are labeled as

$$\phi(z_i) = \begin{cases} \lceil \frac{n}{4} \rceil + 2, & \text{if } i = 1 \\ n + 3, & \text{if if } i = 2 \end{cases}$$

(See Figure 2 for illustration.) It is easy to see that the edge-sums forms a set of $n + 1$ consecutive integers. This shows that

$$\mu_s(G) \leq 2. \tag{9}$$

From equations (8) and (9), we get

$$1 \leq \mu_s(K_2 \cup C_n) \leq 2.$$

Which completes the proof.

□

2 Closing remarks

We have shown that the $(n,t)-$kite graph has super edge-magic deficiency at most 1, for n even and t odd. we also determined the upper bound for the super edge-magic deficiency of $(n,t)-$kite graph for n even and $t > 3$ even. Also we have found the exact value and upper bound for the super edge-magic deficiency of $C_n \cup K_2$, we encourage researchers to try to determine the super edge-magic deficiency of other graphs for further research. In fact, it seems to be a very challenging problem to find the exact value for the super edge-magic deficiency of families of graphs.

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