Combinatorial Sums of Generalized Fibonacci and Lucas Numbers

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Abstract

In this study, we consider generalization of the well-known Fibonacci and Lucas numbers related with combinatorial sums by using finite differences. To write generalized Fibonacci and Lucas sequences in a new direct way, we investigate some new properties of these numbers.

Keywords : Generalized Fibonacci Numbers, Combinatorial Sums
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1 Introduction

In this paper, we mainly interest in whether some new mathematical developments can be applied to Fibonacci and Lucas numbers and their generalization and obtain new results about these numbers. As a reminder, for $a, b \in \mathbb{R}$ and $n > 2$, the well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$ and generalized Fibonacci sequences $\{G_n\}$ are defined by $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$ and $G_n = G_{n-1} + G_{n-2}$ where $F_1 = F_2 = 1$, $L_1 = 2$, $L_2 = 1$ and $G_1 = a$, $G_2 = b$, respectively.

As is well known, there has been a huge interest to linear difference equations related to these numbers in number theory, applied mathematics, physics, computer science etc., recently. For the prettiness and rich applications of these numbers and their relatives one can see science and the nature [1-5] . For instance, the ratio of two consecutive of these numbers converges to the Golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$. Applications of Golden

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ratio appears in many research areas. Physicists gave some examples of the Golden ratio in Theorotical Physics and Physics of High Energy Particles in [9-13]. Falco´n S. and Plaza A. obtained many properties of k-Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$, related with the so-called Pascal 2-triangle and studied the sums of k-Fibonacci numbers with indexes in an arithmetic sequence, say $an+r$ for fixed integers a and r in [7, 8]. Taskara et al., in [6], obtained new properties of Lucas numbers with binomial coefficients and gave some important results related to the Fibonacci numbers. Arthur T. Benjamin et al., in [15], extended the combinatorial approach to understand relationships among generalized Fibonacci numbers. In [16], they gave a new family of k-Fibonacci numbers and established some properties of the relation to the ordinary Fibonacci numbers.

2 Main result

Fibonacci numbers arise in the solution of many combinatorial problems. In this studies, we investigate the generalization of the Fibonacci numbers. Then we give theorems and corollaries with generalization for the fibonacci numbers. In addition to these theorems, we may also obtain in different way generalized Fibonacci numbers as in the following.

The following theorem gives Generalized Fibonacci and Lucas numbers.

Theorem 1 *For* $n > 0$, $G_{n+7} = 13G_{n+1} + 8G_n$.

Proof. Let us use the principle of mathematical induction on n.

For $n = 1$, it is easy to see that

$$G_8 = 13G_2 + 8G_1 = 13b + 8a.$$

Assume that it is true for all positive integers $n = k$. That is,

$$G_{k+7} = 13G_{k+1} + 8G_k. \tag{1}$$

Therefore, we have to show that it is true for $n = k+1$. Adding G_{k+6} to both sides of (1), we have

$$\begin{aligned} G_{k+7} + G_{k+6} &= 13G_{k+1} + 8G_k + G_{k+6} \\ G_{k+8} &= 13G_{k+1} + 8G_k + G_{k+5} + G_{k+4} \\ &= 13G_{k+1} + 8G_k + 2G_{k+4} + G_{k+3}. \end{aligned}$$

By recurring this procedure, we obtain $G_{k+8} = 13G_{k+2} + 8G_{k+1}$ as required.
■

The following theorem gives us denoting in the form of sum for some Generalized Fibonacci and Lucas numbers.

Theorem 2 *For $n \geq 2$ and $b \neq \frac{55}{34}a$, it is hold the equation*

$$\begin{aligned} G_{3n-4} &= \frac{a^2+ab-b^2}{55a-34b}2^{2n+1} - \frac{34a-21b}{55a-34b}G_{3n-5} \\ &+ \frac{1}{55a-34b}\sum_{i=1}^{n-1} 2^{2n-2(i+1)}(aG_{3i+2} - bG_{3i+1}). \end{aligned}$$

Proof. Assume that $c_1 = \frac{a^2+ab-b^2}{55a-34b}$ and $c_2 = \frac{34a-21b}{55a-34b}$. Let us use the induction on n.

For $n = 2$, it is easy to see that

$$G_2 = c_1 2^5 + \frac{1}{55a-34b}\sum_{i=1}^{1} 2^{4-2(i+1)}(aG_5 - bG_4) - c_2 G_1 = b.$$

Assume it is true for all positive integers $n = m-1$. That is,

$$G_{3m-7} = c_1 2^{2m-1} + \frac{1}{55a-34b}\sum_{i=1}^{m-2} 2^{2m-2-2(i+1)}(aG_{3i+2} - bG_{3i+1}) - c_2 G_{3m-8}. \quad (2)$$

Therefore, we have to show that it is true for $n = m$. In other words,

$$G_{3m-4} = c_1 2^{2m+1} + \frac{1}{55a-34b}\sum_{i=1}^{m-1} 2^{2m-2(i+1)}(aG_{3i+2} - bG_{3i+1}) - c_2 G_{3m-5}.$$

If we multiply both sides of (2) with 4, then we have

$$\begin{aligned} 4G_{3m-7} + \frac{136a-84b}{55a-34b}G_{3m-8} &= 4\sum_{i=1}^{m-2} \frac{2^{2m-2-2(i+1)}(aG_{3i+2} - bG_{3i+1})}{55a-34b} \\ &+ 4c_1 2^{2m-1}. \end{aligned} \quad (3)$$

By considering the equality $G_n = G_{n+2} - G_{n+1}$, we can rewrite on the left hand side of above equality as

$$\begin{aligned} &\frac{1}{55a-34b}[(220a-136b)G_{3m-7} + (136a-84b)G_{3m-8}] \\ &= 4c_1 2^{2m-1} + \frac{4}{55a-34b}\sum_{i=1}^{m-2} 2^{2m-2-2(i+1)}(aG_{3i+2} - bG_{3i+1}). \\ &\frac{1}{55a-34b}[(136a-84b)G_{3m-6} + (84a-52b)G_{3m-7}] \\ &= 4c_1 2^{2m-1} + \frac{4}{55a-34b}\sum_{i=1}^{m-2} 2^{2m-2-2(i+1)}(aG_{3i+2} - bG_{3i+1}). \end{aligned}$$

Hence, by recurring this procedure, we have

$$G_{3m-4}+\frac{1}{55a-34b}((34a-21b)G_{3m-5}-aG_{3m-1}+bG_{3m-2})$$
$$= 4c_1 2^{2m-1}+\frac{4}{55a-34b}\sum_{i=1}^{m-2}2^{2m-2-2(i+1)}(aG_{3i+2}-bG_{3i+1}).$$
$$G_{3m-4}+c_2G_{3m-5}+\frac{1}{55a-34b}(-aG_{3m-1}+bG_{3m-2})$$
$$= 4c_1 2^{2m-1}+\frac{4}{55a-34b}\sum_{i=1}^{m-2}2^{2m-2-2(i+1)}(aG_{3i+2}-bG_{3i+1}).$$

Consequently, if the last equation is rearranged, then we obtain

$$\begin{aligned} G_{3m-4}+c_2G_{3m-5} &= c_1 2^{2m+1}+\frac{aG_{3m-1}-bG_{3m-2}}{55a-34b} \\ &\quad +\frac{1}{55a-34b}\sum_{i=1}^{m-2}2^{2m-2(i+1)}(aG_{3i+2}-bG_{3i+1}) \\ G_{3m-4} &= \frac{1}{55a-34b}\sum_{i=1}^{m-1}2^{2m-2(i+1)}(aG_{3i+2}-bG_{3i+1}) \\ &\quad +c_1 2^{2m+1}-c_2G_{3m-5} \end{aligned}$$

which ends up the induction. Therefore we have the required formulate on G_{3n-4}. ■

Corollary 3 *In the above theorem, for $n \geq 2$, it is obvious that the following results hold:*

i. For $a=b=1$, it is obvious $F_n = G_n$. Then it gives the Fibonacci sequence with binomial coefficients, in [5], as

$$F_{3n+3}=2^{2n+1}+\sum_{i=1}^{n-1}2^{2n-2(i+1)}F_{3i},$$

ii. For $a=2$ and $b=1$, we can clearly see that $L_n=G_n$ and we obtain the Lucas sequences with Binomial coefficients given in [14] as

$$L_{3n-4}=\frac{5}{76}2^{2n+1}+\frac{1}{76}\sum_{i=1}^{n-1}2^{2n-2(i+1)}(L_{3i}+L_{3i+2})-\frac{47}{76}L_{3n-5}.$$

Now, the next theorems give us Generalized Fibonacci and Lucas numbers that are obtained by using binomial sums.

Theorem 4 *Suppose that n is odd positive integer and $b \neq \frac{8}{5}a$. Then we have*

$$G_{3n} = \frac{a^2+ab-b^2}{8a-5b}\sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i} - \frac{5a-3b}{8a-5b}G_{3n-1}.$$

Proof. Let $\alpha = \frac{a^2+ab-b^2}{8a-5b}$ be a real number and let us consider the sequence $\{S_k\}$. From [5], elements of this sequence can be written as

$$\begin{aligned} S_6 &= \alpha\left[\binom{1}{1}2^3\right] = \alpha F_6 = 8\alpha, \\ S_{12} &= \alpha\left[2^7\binom{3}{3} + 2^3\binom{2}{1}\right] = \alpha F_{12} = 144\alpha, \\ &\vdots \end{aligned} \tag{4}$$

To obtain the elements of sequence $\{S_k\}$, we can use the following method:

By forming a $\frac{k}{6} \times \frac{k}{6}$ square matrix with the rows of $\{S_k\}$ sequence and with the columns of the coefficients $\alpha 2^3$, $\alpha 2^7$, $\alpha 2^{11}, \ldots, \alpha 2^{2n+1}$, we see that the elements in the main diagonal and those elements above the main diagonal come from the binomial expansion as follows:

$$\begin{bmatrix} \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} & \cdots & \binom{\frac{n+1}{2}}{1} \\ 0 & \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \cdots & \binom{\frac{n+3}{2}}{3} \\ 0 & 0 & \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \cdots & \binom{\frac{n+5}{2}}{5} \\ 0 & 0 & 0 & \binom{7}{7} & \binom{8}{7} & \binom{9}{7} & \cdots & \binom{\frac{n+7}{2}}{7} \\ 0 & 0 & 0 & 0 & \binom{9}{9} & \binom{10}{9} & \cdots & \binom{\frac{n+9}{2}}{9} \\ 0 & 0 & 0 & 0 & 0 & \binom{11}{11} & \cdots & \binom{\frac{n+11}{2}}{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n}{n} \end{bmatrix}.$$

It is clear that kth element of sequence $\{S_k\}$ can be written as the sums of multiply with $\alpha 2^{2n+1}$ of the elements on $\frac{k}{6}$th column of above matrix. That is,

$$S_k = \alpha\left[2^{2n+1}\binom{n}{n} + 2^{2n-3}\binom{n-1}{n-2} + 2^{2n-7}\binom{n-2}{n-4} + \cdots + 2^3\binom{\frac{n+1}{2}}{1}\right], \tag{5}$$

where $k = 3n+3$. If the equation (5) can be written as a sum, then we have

$$S_k = \alpha\sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i}. \tag{6}$$

We can obtain the elements of sequence $\{S_k\}$ from (4) as

$$S_k = \{8\alpha,\ 144\alpha,\ ...,\ \alpha F_{3n+3}\}.$$

Hence, by using $F_n = \frac{aG_{n+2}-bG_{n+1}}{a^2+ab-b^2}$, it is obvious that the equality

$$S_k = G_{3n} + \frac{5a-3b}{8a-5b} G_{3n-1} \tag{7}$$

holds. As a result of (6) and (7), the proof is completed. ■

Corollary 5 *From the above theorem, the following results can be clearly seen.*

i. For $a = b = 1$, it is obvious $F_n = G_n$. Then it gives the Fibonacci sequence with binomial coefficients, in [5], as

$$F_{3n+3} = \sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i} \binom{n-i}{n-2i},$$

ii. For $a = 2$ and $b = 1$, we can clearly see that $L_n = G_n$ and we obtain the Lucas sequences with Binomial coefficients given in [14] as

$$\begin{aligned}
L_{3n+3} &= \sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i} \binom{n-i}{n-2i} + F_{3n+1}, \\
L_{3n+2} &= \frac{5}{4} \sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i} \binom{n-i}{n-2i} - \frac{3}{4} L_{3n+1}, \\
L_{3n+4} &= \left[5 \left(\sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i} \binom{n-i}{n-2i} \right)^2 + 4 \right]^{1/2}.
\end{aligned}$$

In the following theorem, it is given odd Generalized Fibonacci and Lucas numbers.

Theorem 6 *For $n \geq 0$ and $a \neq 0$, we have*

$$G_{2n+3} = \frac{a^2+ab-b^2}{a} \sum_{i=0}^{n} \binom{n+i}{2i} + \frac{b}{a} G_{2n+2}.$$

Proof. Let us $k = \frac{a^2+ab-b^2}{a}$. For $n \geq 0$, we have the following iteration:

$$\begin{aligned} G_3 &= k\binom{0}{0} + \frac{b}{a}b \\ G_5 &= k\left[\binom{1}{0} + \binom{2}{2}\right] + \frac{b}{a}(a+2b) \\ G_7 &= k\left[\binom{2}{0} + \binom{3}{2} + \binom{4}{4}\right] + \frac{b}{a}(3a+5b) \\ &\vdots \end{aligned} \tag{8}$$

The coefficients of a and b are Fibonacci numbers in (8). Hence, by iterating this procedure, we have

$$G_{2n+3} = k\left[\binom{n}{0} + \binom{n+1}{2} + \cdots + \binom{2n}{2n}\right] + \frac{b}{a}(aF_{2n} + bF_{2n+1}). \tag{9}$$

Furthermore, considering the equality $G_{2n+2} = aF_{2n} + bF_{2n+1}$, we rewrite the equation in (9)

$$G_{2n+3} = k\left[\binom{n}{0} + \binom{n+1}{2} + \cdots + \binom{2n}{2n}\right] + \frac{b}{a}G_{2n+2}$$

or using the summation symbol, we have

$$G_{2n+3} = \frac{a^2+ab-b^2}{a}\sum_{i=0}^{n}\binom{n+i}{2i} + \frac{b}{a}G_{2n+2}.$$

■

Corollary 7 *From the above theorem, the following results can be clearly seen.*

i. For $a = b = 1$, it is obvious $F_n = G_n$. Then it gives the Fibonacci sequence with binomial coefficients, in [5], as $F_{2n+1} = \sum_{i=0}^{n}\binom{n+i}{2i}$,

ii. For $a = 2$ and $b = 1$, we can clearly see that $L_n = G_n$ and we obtain the Lucas sequences with Binomial coefficients given in [6] as

$$L_{2n+2} = \left[5\left(\sum_{i=0}^{n}\binom{n+i}{2i}\right)^2 - 4\right]^{1/2}.$$

Similarly, the following theorem gives odd Generalized Fibonacci and Lucas numbers with different point of view.

Theorem 8 *Suppose that $n \geq 0$ is an even integer and $a \neq 0$. Then we have*

$$G_{n+3} = \frac{a^2 + ab - b^2}{a} \sum_{i=0}^{\frac{n}{2}} \binom{n-i}{n-2i} + \frac{b}{a} G_{n+2}.$$

Proof. The proof can also be seen by using the iteration method as in Theorem 6. ■

Corollary 9 *From the above theorem, the following results can be clearly seen.*

i. For $a = b = 1$, it is obvious $F_n = G_n$. Then it gives the Fibonacci sequence with binomial coefficients, in [5], as $F_{n+1} = \sum_{i=0}^{\frac{n}{2}} \binom{n-i}{n-2i}$,

ii. For $a = 2$ and $b = 1$, we can clearly see that $L_n = G_n$ and we obtain the Lucas sequences with Binomial coefficients given in [6] as

$$L_{2n+3} = 5 \left(\sum_{i=0}^{n/2} \binom{n-i}{n-2i} \right)^2 - 2.$$

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