Magic graphs with pendant edges

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Abstract

A graph G is edge-magic if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, 3, \cdots, |V(G)| + |E(G)|\}$ such that for any edge uv of G, $f(u) + f(uv) + f(v)$ is constant. Moreover, G is super edge-magic if $V(G)$ receives $|V(G)|$ smallest labels. In this paper, we propose methods for constructing new (super) edge-magic graphs from some old ones by adding some new pendant edges.

1 Introduction

In this paper we consider only finite and simple graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

Let G be a graph with p vertices and q edges. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \cdots, p+q\}$ is called an *edge-magic total labeling* of G if there exists an integer k such that $f(x) + f(xy) + f(y) = k$, independent of the choice of any edge xy of G. If such a labeling exists, then the

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constant k is called the *magic constant* of f, and G is said to be *edge-magic graph.* An edge-magic total labeling f is called *super edge-magic* if $f(V(G)) = \{1, 2, 3, \cdots, p\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic total labeling.

The edge-magic concept was first introduced and studied by Kotzig and Rosa [11, 12], although under a different name, i.e., the magic valuation. The super edge-magic notion was first introduced by Enomoto, Lladó, Nakamigawa and Ringel [2]. The (super) edge-magic graphs have been studied in several papers, see for instance [3, 4, 8, 10, 13], and more complete results on (super) edge-magic graphs can be seen in the survey paper by Gallian [9]. However, the long-standing conjectures that "every tree is edge-magic" and "every tree is super edge-magic", proposed in [11] and [2], respectively, still remain open.

The following lemma presented in [3] gives a necessary and sufficient condition for a graph to be super edge-magic.

Lemma 1 *A graph G with p vertices and q edges is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{f(x) + f(y) | xy \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic total labeling of G with magic constant $k = p + q + s$, where $s = min(S)$ and*

$$\begin{aligned} S &= \{f(x) + f(y) | xy \in E(G)\} \\ &= \{k - (p+1), k - (p+2), \cdots, k - (p+q)\}. \end{aligned}$$

In [11], Kotzig and Rosa introduced the concept of edge-magic deficiency of a graph. They defined the *edge-magic deficiency*, $\mu(G)$, of a graph G as a minimum nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. Kotzig and Rosa [11] gave an upper bound of the edge-magic deficiency of a graph G with p vertices, that is $\mu(G) \leq F_{p+2} - 2 - p - \frac{1}{2}p(p-1)$, where F_p is the p-th Fibonacci number.

Furthermore, Figueroa-Centeno *et al.*[6] defined the concept of the super edge-magic deficiency of a graph similarly. The *super edge-magic deficiency*, $\mu_s(G)$, of a graph G is a minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling or $+\infty$ if there exists no such n. Clearly, for every graph G, $\mu(G) \leq \mu_s(G)$.

Figueroa-Centeno *et al.* in two separate papers [6, 7] provided the exact values of (super) edge-magic deficiency of several classes of graphs, such as cycles, complete graphs, some classes of forests, 2-regular graphs, and complete bipartite graphs $K_{2,m}$. They [7] also proposed the conjecture "if F is a forest with two components, then $\mu_s(F) \leq 1$".

In this paper, we propose some methods for constructing new (super) edge-magic graphs from the old ones. From this construction we can obtain new classes of (super) edge-magic graphs. Some of the resulting graphs give support to the correctness of the conjectures "every tree is (super) edge-magic", and "if F is a forest with two components, then $\mu_s(F) \leq 1$".

2 The Results

Throughout this section, we will present a construction of new (super) edge-magic graphs by adding pendant edges to some (not all) vertices of a (super) edge-magic graph G having a specific property. This construction can be viewed as a weaker version of a corona product of a graph G and nK_1.

The *corona product* $G \odot H$ of two given graphs G and H is defined as a graph obtained by taking one copy of a p-vertex graph G and p copies $H_1, H_2, \ldots, H_p$ of H, and then joining the i-th vertex of G to every vertex in H_i. If $H \cong nK_1$, $G \odot H$ is equal to the graph produced by adding n pendant edges to every vertex of G. The corona product of graphs has been studied in several papers, see for instance [1], [5] and [14].

In the next two theorems, we construct (super) edge-magic graphs by adding n pendant edges to every vertex of particular type of edge-magic graph except some vertices with the largest labels.

Theorem 1 *Let G be a graph of even order $p \geq 2$ and size of either $q = p$ or $p-1$ for which there exists an edge-magic total labeling f with the property that all vertices of G receive odd labels such that*

$$\{f(x)+f(y) | xy \in E(G)\} = \{3p-2q, 3p-2q+2, \cdots, 3p-4, 3p-2\}. \quad (1)$$

Then, the graph H formed by adding n pendant edges to each vertex of G except the vertex with the largest label is edge-magic for every positive integer n.

Proof Suppose $V(G) = \{x_i | 1 \leq i \leq p\}$. Let f be an edge-magic total labeling of G satisfying the conditions of Theorem 1. Then, the magic constant of f is $k = 3p$. Since all vertices receive odd labels, we may assume that $f(x_i) = 2i-1$ for every integer $1 \leq i \leq p$. Let H be a graph defined as follows.

$$V(H) = V(G) \cup \{y_i^j | 1 \leq i \leq p-1 \text{ and } 1 \leq j \leq n\},$$

and

$$E(H) = E(G) \cup \{x_i y_i^j | 1 \leq i \leq p-1 \text{ and } 1 \leq j \leq n\}.$$

Now, define a total labeling

$$g : V(H) \cup E(H) \to \{1, 2, 3, \cdots, 2n(p-1) + p + q\}$$

such that $g(x) = f(x)$ for every $x \in V(G)$ and

$$g(y_i^j) = \begin{cases} (2j+1)p + 2(i-j) - 1, & \text{for } 1 \leq i \leq \frac{p}{2} \text{ and } 1 \leq j \leq n, \\ (2j-1)p + 2(i-j) + 1, & \text{for } \frac{p+2}{2} \leq i \leq p-1 \text{ and } 1 \leq j \leq n. \end{cases}$$

It can be verified that all odd labels are assigned to the vertices of H.

Let $S_i^j = \{g(x_i) + g(y_i^j)\}$ for $1 \leq i \leq p-1$, $1 \leq j \leq n$. It can be verified that $m_j = min_{1 \leq j \leq n}\{S_i^j\} = (2j+1)(p-1)+3$ and $M_j = max_{1 \leq j \leq n}\{S_i^j\} = (2j+2)(p-1)+p$. Observe that $m_1 = 3p$, $M_n = (2n+2)(p-1)+p$ and $m_{j+1} = M_j + 2$ for $1 \leq j \leq n-1$. Also, $\bigcup_{i,j} S_i^j = \{3p, 3p+2, \cdots, (2n+2)(p-1)+p-2, (2n+2)(p-1)+p\}$. Thus, the set $\{g(x)+g(y) | xy \in E(H)\}$ forms an arithmetic sequence starting from $3p-2q$ with common difference 2. If we take

$$g(xy) = (2n+3)p - 2n - g(x) - g(y), \quad \text{for every } xy \in E(H)$$

then, g is an edge-magic total labeling of H with the magic constant $(2n+2)(p-1)+p+2 = 2n(p-1)+k$. □

It can be shown that each of the following classes of graphs has an edge-magic total labeling f satisfying the conditions of Theorem 1.

- Paths of an even number of vertices P_{2k} for $k \geq 1$.
- Caterpillars formed by adding $m \geq 1$ pendant edges to every vertex of P_{2k}, $k \geq 1$ (We denote such caterpillars by $P^1_{2k,m}$).
- Caterpillars formed by adding one pendant edge to every vertex of P_{2k+1}, $k \geq 1$ (denoted by $P^2_{2k+1,1}$).
- Caterpillars formed by adding one pendant edge to one vertex of degree one and $m \geq 1$ pendants to other vertices of P_{2k+1}, $k \geq 1$ (denoted by $P^3_{2k+1,m}$).
- Path-like-trees P_T with an even number of vertices.

Additionally, a cycle of odd length with one pendant attached to a vertex also admits the labeling satisfying the conditions of Theorem 1.

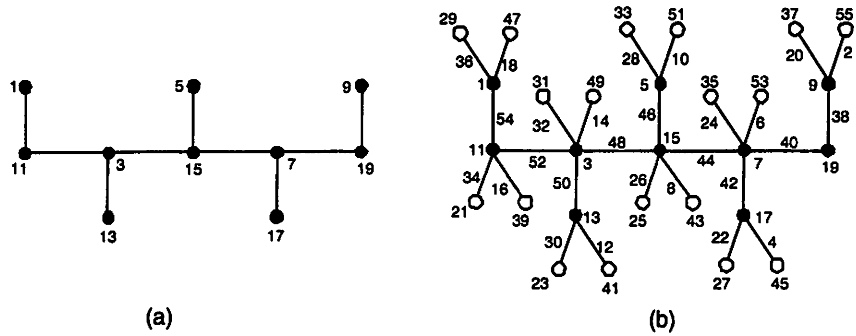

Figure 1: The graph $P^2_{5,1}$ and the new graph resulting by applying Theorem 1.

As an example of Theorem 1, Figure 1 (a) shows the graph $P^2_{5,1}$ and its vertex labeling, and Figure 1 (b) shows the new graph resulting by applying Theorem 1 to $P^2_{5,1}$.

For simplicity, we denote by P^*_T a tree formed by applying the Theorem 1 to P_T. Similarly, we denote by $L^1_{2k,m}$, $L^2_{2k+1,1}$ and $L^3_{2k+1,m}$ the graphs formed by applying the Theorem 1 to $P^1_{2k,m}$, $P^2_{2k+1,1}$ and $P^3_{2k+1,m}$, respectively. These three graphs are all lobsters.

Therefore, by Theorem 1 we have the following corollary.

Corollary 1 *The tree P^*_T and the lobsters $L^1_{2k,m}$, $L^2_{2k+1,1}$ and $L^3_{2k+1,m}$ are edge-magic graphs.* □

Now, we refer the readers to the following result.

Theorem 2 *[3] Let T be an edge-magic tree of order p with an edge-magic total labeling f whose magic constant is k such that $f(v)$ is odd for any vertex v of $V(T)$. Then, the bijective function $g : V(T) \cup E(T) \rightarrow \{1, 2, 3, \cdots, 2p-1\}$ defined as*

$$g(x) = \begin{cases} \frac{f(x)+1}{2}, & if\ x \in V(T), \\ \frac{f(x)}{2} + p, & if\ x \in E(T), \end{cases}$$

is a super edge-magic labeling. Furthermore, given a super edge-magic labeling of a tree, a labeling can be obtained with all vertices receiving an odd label by reversing the above process.

Note that Theorem 2 can be extended to graphs for which $p = q$.

By Theorem 2, all graphs satisfying the conditions of Theorem 1 are also super edge-magic. Especially, we have the following corollary.

Corollary 2 *The tree P_T^* and the lobsters $L^1_{2k,m}$, $L^2_{2k+1,1}$ and $L^3_{2k+1,m}$ are super edge-magic graphs.* □

These results provides supporting examples of the conjectures proposed by Kotzig and Rosa [11] and by Enomoto, Llado, Nakamigawa and Ringel [2].

If the condition "all vertices of G receive odd labels" in Theorem 1 is removed, then the conclusion is not true. For example, consider graph G in Figure 2(a). If G has an edge-magic total labeling satisfying the condition of Theorem 1, then all vertices of G must receive even labels (since the set in (1) consists of only even numbers and G is connected). Then, there are only two such labelings possible (see Figure 2(b) and 2(c)). Let H be a graph formed by adding n pendant edges to every vertex of G except to the vertex of label 12 (the largest vertex label). Then H is not edge-magic for *any* integer n. In fact, if H is edge-magic then the magic constant is $10n + 19 - \frac{12}{5n+6}$. Therefore, this is not possible for all positive integers n. Consequently, the condition that all vertices of G receive odd labels in Theorem 1 is crucial.

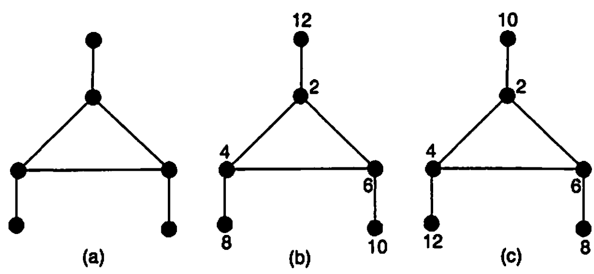

Figure 2: The graph G and its vertex labeling.

Theorem 3 *Let G be a graph with odd order p (≥ 3) for which there exists a super edge-magic total labeling g with the property that*

$$max\{g(x) + g(y) | xy \in E(G)\} = \frac{1}{2}(3p - 1).$$

Then, the graph H formed by adding n pendant edges to every vertex of G except the vertices u and v with $g(u) = p - 1$ and $g(v) = p$ is super edge-magic for every positive integer n.

Proof Let G be a graph of odd order p satisfying the conditions of Theorem 3, and g be a super edge-magic total labeling of G with the magic constant k. Assume $g(x_i) = i$ for every $1 \leq i \leq p$, where $V(G) = \{x_i | 1 \leq i \leq p\}$.

Now, define H as a graph with the vertex and edge sets

$$V(H) = V(G) \cup \{y_i^j : 1 \leq i \leq p-2, 1 \leq j \leq n\}$$

and

$$E(H) = E(G) \cup \{xy_i^j : 1 \leq i \leq p-2, 1 \leq j \leq n\},$$

respectively.

Next, consider the vertex labeling $h : V(H) \rightarrow \{1, 2, 3, \cdots, (n+1)p-2n\}$ defined as follows.

$$h(x) = g(x), \text{ for every } x \in V(G),$$

and

$$h(y_i^j) = \begin{cases} (j+1)p - 2(j+i-1), & \text{for } 1 \leq i \leq \frac{p-1}{2} \text{ and } 1 \leq j \leq n, \\ (j+2)p - 2(j+i), & \text{for } \frac{p+1}{2} \leq i \leq p-2 \text{ and } 1 \leq j \leq n. \end{cases}$$

To show that h can be extended to a super edge-magic total labeling of H, let $S_i^j = \{h(x_i) + h(y_i^j)\}$ for $1 \leq i \leq p-2$, $1 \leq j \leq n$. It can be verified that $m_j = min_{1 \leq j \leq n}\{S_i^j\} = j(p-2) + \frac{1}{2}(p+5)$ and $M_j = max_{1 \leq j \leq n}\{S_i^j\} = j(p-2) + \frac{1}{2}(3p-1)$. Note that $m_1 = \frac{1}{2}(3p+1)$, $M_n = n(p-2) + \frac{1}{2}(3p-1)$ and $m_{j+1} = M_j + 1$ for $1 \leq j \leq n-1$. Also, $\bigcup_{i,j} S_i^j$ is a set of consecutive integers. By Lemma 1, h extends to a super edge-magic total labeling of H with the magic constant $\frac{1}{2}(5p+1) + 2n(p-2) = k + 2n(p-2)$. □

There are some classes of super edge-magic graphs satisfying the conditions of Theorem 3, such as $P_m \cup K_{1,1}$ for $4 \leq m \equiv 1, 3 \ (mod\ 4)$ [4] (see Theorem 10), $P_2 \cup K_{1,n}$ for $n \equiv 0 \ (mod\ 2)$ [4] (see Theorem 5), $P_m \cup K_{1,2}$ for $4 \leq m \equiv 2 \ (mod\ 4)$ [4] (see Theorem 10), and $C_n \cup K_1$ for $n \equiv 0 \ (mod\ 4)$ [6] (see Theorem 9). It can be verified that a path of odd number of vertices also admits such a labeling required in Theorem 3.

By applying Theorem 3 to $P_m \cup K_{1,1}$, $P_2 \cup K_{1,n}$, and $P_m \cup K_{1,2}$, respectively, we obtain new classes of forests with two components which are super edge-magic. For short, we denote them by F_1, F_2, and F_3, respectively.

Corollary 3 $\mu_s(F_1) = \mu_s(F_2) = \mu_S(F_3) = 0$. □

This result gives support to the correctness of the conjecture proposed in [7].

In the next theorems we present a construction of new super edge-magic graphs by adding n pendant edges to every vertex of a specific super edge-magic graph with the exception of some vertices receiving the smallest labels.

Theorem 4 *Let G be a graph of even order $p > 2$, for which there exists a super edge-magic total labeling f with the property that*

$$max\{f(x) + f(y)|xy \in E(G)\} = \frac{1}{2}(3p + 2).$$

Then, the graph H formed by adding n pendant edges to every vertex of G except the vertex u with $f(u) = 1$ is super edge-magic for every positive integer n.

Proof Let G be a super edge-magic graph with the magic constant k, and let $V(G) = \{x_i : 1 \leq i \leq p\}$. We may assume that $f(x_i) = i$ for every i, $1 \leq i \leq p$.

Next, let

$$V(H) = V(G) \cup \{y_i^j : 2 \leq i \leq p, 1 \leq j \leq n\}$$

and

$$E(H) = E(G) \cup \{x_i y_i^j : 2 \leq i \leq p, 1 \leq j \leq n\}.$$

Now, define a vertex labeling $g : V(H) \rightarrow \{1, 2, 3, \cdots, p+(p-1)n\}$ such that

$$g(x) = f(x) \text{ for every vertex } x \in V(G),$$

and

$$h(y_i^j) = \begin{cases} \frac{1}{2}(2i + 2 - p) + j(p - 1), & \text{for } \frac{p}{2} + 1 \leq i \leq p \text{ and } 1 \leq j \leq n, \\ \frac{1}{2}(2i + p) + j(p - 1), & \text{for } 2 \leq i \leq \frac{p}{2} \text{ and } 1 \leq j \leq n. \end{cases}$$

We can see that the labels of pendant vertices are consecutive and greater than p. By a similar argument used in the proof of Theorem 3, it can be shown that h extends to a super edge-magic total labeling of H with the magic constant $k + 2n(p - 1)$. □

As an illustration of Theorem 4, see Figure 3.

Theorem 5 *Let G be a graph of order $p = (c+1)(m+1)+1$, where $m \geq 2$, $c \geq 1$ for which there exists a super edge-magic total labeling f with the property that*

$$max\{f(x)+f(y)|xy \in E(G)\} = (2m+1)(c+1)+1.$$

Then, the graph H formed by adding n pendant edges to every vertex of G except the vertices with labels $1,2,3,\cdots,m(c+1)-c-3$ is super edge-magic for every positive integer n.

Proof Let G be a graph satisfying the conditions of Theorem 5 with $V(G) = \{x_i : 1 \leq i \leq p\}$. Take a super edge-magic total labeling g with the magic constant k such that $g(x_i) = i$ for $1 \leq i \leq p$. Now, define the graph H as follows:

$$V(H) = V(G) \cup \{y_i^j : m(c+1)-c-2 \leq i \leq p, 1 \leq j \leq n\}$$

and

$$E(H) = E(G) \cup \{x_i y_i^j : m(c+1)-c-2 \leq i \leq p, 1 \leq j \leq n\}.$$

It is easy to verify that the vertex labeling $h : V(H) \to \{1,2,3,\cdots,p+(2c+5)n\}$ defined by

$$h(x) = g(x), \text{ for every } x \in V(G),$$

and

$$h(y_i^j) = \begin{cases} a+i+c+3, & \text{for } b-c-2 \leq i \leq b-1 \text{ and } 1 \leq j \leq n, \\ a+i-c-2, & \text{for } b \leq i \leq p \text{ and } 1 \leq j \leq n, \end{cases}$$

where $a = j(2c+5)$ and $b = m(c+1)$, extends to a super edge-magic total labeling of H with the magic constant $k + 2n(2c+5)$. □

Graphs $P_m \cup K_{1,3}$ for $m \equiv 0 \bmod 4$, and $K_m \cup K_n$ for n is a multiple of $m+1$ satisfy the conditions required in Theorem 4 and Theorem 5, respectively, see [6]. Hence, by applying the algorithm in the proof of Theorems 4 and 5, respectively, we have new classes of forests with two components, denoted by F_4 and F_5, respectively, which are super edge-magic. Consequently, we have the following results.

Corollary 4 $\mu_s(F_4) = \mu_s(F_5) = 0$. □

Again, this result gives more examples of the correctness of the conjecture proposed in [7].

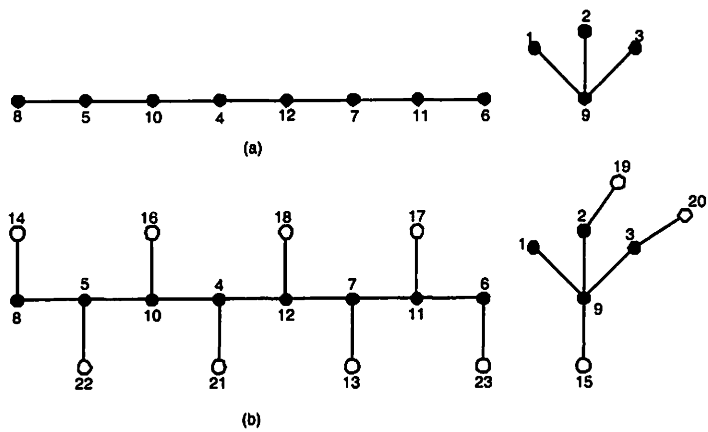

Figure 3: The super edge-magic $P_8 \cup K_{1,3}$ and the new graph resulting by applying Theorem 4.

In the next theorem, we consider a particular type of forest with two components, namely the forest $H \cong K_{1,m} \cup P_n^m$, where P_n^m is a caterpillar with vertex and edge sets

$$V(P_n^m) = \{u_i : 1 \leq i \leq n\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and

$$E(P_n^m) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\},$$

respectively.

Theorem 6 *For every positive integers m and n where $n \geq 2$ is even, the forest $H \cong K_{1,m} \cup P_n^m$ is super edge-magic.*

Proof We can consider that H is a forest with

$$V(H) = V(P_n^m) \cup \{c, w_j : 1 \leq j \leq m\}$$

and

$$E(H) = E(P_n^m) \cup \{cw_j : 1 \leq j \leq m\}.$$

Let the vertex labeling $g : V(H) \to \{1, 2, 3, \cdots, |V(H)|\}$ defined as follows.

$$g(y) = \begin{cases} 1, & \text{if } y = c, \\ j(n+1) + \frac{1}{2}(n+4), & \text{if } y = w_j, \text{ for } 1 \le j \le m, \\ \frac{1}{2}(i+2), & \text{if } y = u_i \text{ for even } i, \\ \frac{1}{2}(n+i+3), & \text{if } y = u_i \text{ for odd } i, \\ \frac{1}{2}(2j(n+1)+i+3), & \text{if } y = v_i^j \text{ for odd } i \text{ and } 1 \le j \le m, \\ \frac{1}{2}(2j(n+1)+n+i+4), & \text{if } y = v_i^j \text{ for even } i \ne n \text{ and } 1 \le j \le m, \\ j(n+1)+1, & \text{if } y = v_i^j \text{ for } i = n \text{ and } 1 \le j \le m. \end{cases}$$

It is not difficult to verify that $\{g(x) + g(y) : xy \in E(H)\}$ is a set of consecutive integers starting from $\frac{1}{2}(n+8)$. By Lemma 1, g extends to a super edge-magic total labeling of F_2 with the magic constant $k = \frac{1}{2}(4nm + 5n + 4m + 8)$. □

The last theorem gives support to the correctness of the conjecture "if F is a forest with two components, then $\mu_s(F) \le 1$".

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References

[1] E.T. Baskoro, I.W. Sudarsana and Y.M. Cholily, How to construct new super edge-magic graphs from some old ones, *J. Indones. Math. Soc. (MIHMI)*, Vol. **11**, No. 2 (2005), 155 - 162.

[2] H. Enomoto, A. Llado, T. Nakamigawa, and G. Ringel, Super edge magic graphs, *SUT J. Math.*, **34** (1998), 105 - 109.

[3] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001), 153 - 168.

[4] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On edge-magic labelings of certain disjoint union graphs, *Australas. J. Combin.*, **32** (2005), 225 - 242.

[5] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, Magical coronations of graphs, *Australas. J. Combin.*, **26** (2002), 199 - 208.

[6] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, Elsevier, Amsterdam, 2002.

[7] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55**, (2005), 17 - 31.

[8] Y. Fukuchi, Edge-magic labeling of generalized Petersen graphs $P(n, 2)$, *Ars Combin.*, **59** (2001), 253 - 257.

[9] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, (2005).

[10] J. Ivančo and I. Lučkaničová, On edge-magic disconnected graphs, *SUT Journal of Math.*, Vol. **38**, No. 2, (2002), 175 - 184.

[11] A. Kotzig and A. Rosa, Magic valuation of finite graphs, *Canad. Math. Bull.*, Vol. **13** (4), 1970, 451 - 461.

[12] A. Kotzig and A. Rosa, Magic valuation of complete graphs, *Publications du Centre de Recherches mathematiques Universite de Montreal*, **175** (1972).

[13] A.A. G. Ngurah, E.T. Baskoro, On magic and antimagic total labeling of generalized Petersen graph, *Util. Math.*, **63** (2003), 97 - 107.

[14] A.A. G. Ngurah, E.T. Baskoro, On the new families of (super) edge-magic graphs, *Util. Math.*, to appear.

[15] Slamin, M. Baca, Y. Lin, M. Miller and R. Simanjuntak, Edge magic total labeling of wheels, fans and friendship graphs, *Bull. Inst. Combin. Appl.*, **35** (2002), 89 - 98.